Yutaka Shoukaku

Oscillation criteria for nonlinear differential equations with $p(t)$-Laplacian

OSCILLATION CRITERIA FOR NONLINEAR DIFFERENTIAL EQUATIONS WITH $p(t)$-LAPLACIAN

YUTAKA SHOUKAKU, Kanazawa

Dedicated to Professor Norio Yoshida on the occasion of his 65th birthday

Abstract. Recently there has been an increasing interest in studying $p(t)$-Laplacian equations, an example of which is given in the following form

$$(|u'(t)|^{p(t)-2}u'(t))' + c(t)|u(t)|^{q(t)-2}u(t) = 0, \quad t > 0.$$

In particular, the first study of sufficient conditions for oscillatory solution of $p(t)$-Laplacian equations was made by Zhang (2007), but to our knowledge, there has not been a paper which gives the oscillatory conditions by utilizing Riccati inequality. Therefore, we establish sufficient conditions for oscillatory solution of nonlinear differential equations with $p(t)$-Laplacian via Riccati method. The results obtained are new and rare, except for a work of Zhang (2007). We present more detailed results than Zhang (2007).

Keywords: $p(t)$-Laplacian; oscillation theory; Riccati inequality

MSC 2010: 34K11, 35B05, 35R10

1. INTRODUCTION

The purpose of this paper is to generalize p-Laplacian equations to the case of $p(t)$-Laplacian equations of the form

(E) $$(|u'(t)|^{p(t)-2}u'(t))' + c(t)|u(t)|^{q(t)-2}u(t) = 0, \quad t > 0,$$

where $c(t) \in C((0,\infty); (0,\infty))$.

We assume throughout this paper that: $p(t), q(t) \in C^1(\mathbb{R}; (1,\infty))$ and satisfy

$$1 < \inf_{t \in \mathbb{R}} p(t), \quad \sup_{t \in \mathbb{R}} p(t) < \infty, \quad 1 < \inf_{t \in \mathbb{R}} q(t), \quad \sup_{t \in \mathbb{R}} q(t) < \infty.$$

Definition 1.1. A function $u \in C^1(0, \infty)$ with the property that $|u'|^{p(t)-2}u' \in C^1(0, \infty)$ is said to be a *solution* of (E) if u satisfies (E) at every point in $(0, \infty)$.

Definition 1.2. A nontrivial solution of (E) is said to be *oscillatory* if it has arbitrarily large zeros ((E) is an *oscillatory equation* if its solution is oscillatory), otherwise it is nonoscillatory.

The first interest in $p(\cdot)$-type Laplacian was in function spaces called variable exponent spaces. Variable exponent space, which appeared in W. Orlicz's article of 1931, was studied afterwards by many authors (see, e.g., [1]).

In 1999, Růžička studied equations with non-standard $p(x)$-growth in the modeling of the so-called electrorheological fluids (see [2]). After this article, the importance of $p(\cdot)$-type Laplacian was recognized [3].

In recent years, Zhang in [9] investigated the oscillation problem for the $p(t)$-Laplacian equation, and obtained the following:

Theorem A (Zhang [9]). *Assume that $c(t) = t^{-\theta(t)}$ and*

(A1) $\lim\limits_{t \to \infty} f(t) = f_\infty$, $t^{|f(t)-f_\infty|} < M$ *($f(t)$ satifies the log-Hölder decay condition);*

(A2) $g(t, \cdot) \in C((0, \infty) \times \mathbb{R})$ *is increasing for any fixed $t > 0$ and $0 < \liminf\limits_{t \to \infty} g(t, u)u \leqslant$*
$\limsup\limits_{t \to \infty} g(t, u)u < \infty$, $u \in \mathbb{R} \setminus \{0\}$.

If $p(t)$ possesses (A1) and $\limsup\limits_{t \to \infty} \theta(t) < \liminf\limits_{t \to \infty} q(t)$, where

$$1 < \limsup_{t \to \infty} q(t) < \liminf_{t \to \infty} p(t)$$

or

$$\lim_{t \to \infty} q(t) = \lim_{t \to \infty} p(t), \quad q(t) \text{ is possesses (A1),}$$

then every solution of (E) is oscillatory.

Motivated by this article [9], Yoshida established oscillation theorems, Picone identities and Sturmian comparison theorems for half-linear elliptic inequalities with $p(x)$-Laplacians (see, for example, [7] and [8]). Recently, Şahiner and Zafer [4], [5] also studied forced oscillation of half-linear elliptic inequlities with $p(x)$-Laplacians under the condition $q(t) > p(t) > 1$. However, there is a few part having to study the results of Zhang [9] in detail. Therefore, we provide new oscillation criteria for the solution of (E).

2. Main results

In order to discuss our main results, we need the following lemma, which is due to Usami [6].

Lemma 2.1. *If there exists a function $\varphi(t) \in C^1([T_0, \infty); (0, \infty))$ such that*

$$(2.1) \qquad \int_{T_1}^{\infty} \left(\frac{\bar{p}(t) |\varphi'(t)|^\beta}{\varphi(t)} \right)^{1/(\beta-1)} \mathrm{d}t < \infty, \quad \int_{T_1}^{\infty} \varphi(t) \bar{q}(t) \, \mathrm{d}t = \infty,$$

and

$$\int_{T_1}^{\infty} \frac{1}{\bar{p}(t) (\varphi(t))^{\beta-1}} \, \mathrm{d}t = \infty$$

for some $T_1 \geqslant T_0 > 0$, then the Riccati inequality

$$(2.2) \qquad x'(t) + \frac{1}{\beta} \frac{1}{\bar{p}(t)} |x(t)|^\beta \leqslant -\bar{q}(t),$$

where $\beta > 1$, $\bar{p}(t) \in C([T_0, \infty); (0, \infty))$ and $\bar{q}(t) \in C([T_0, \infty); \mathbb{R})$, has no solution on $[T, \infty)$ for all large T.

Theorem 2.1. *Let*

$$p^- \equiv \inf_{t \geqslant T} p(t), \quad p^+ \equiv \sup_{t \geqslant T} p(t), \quad q^- \equiv \inf_{t \geqslant T} q(t), \quad q^+ \equiv \sup_{t \geqslant T} q(t)$$

for some $T > 0$. If one of the following cases holds:

(i) *$p(t)$ is increasing, $1 < p(t) = q(t)$ or $1 < p(t) < q(t)$, there exists $\varphi(t) \in C^1((0, \infty); (0, \infty))$ such that*

$$(2.3) \qquad \int^{\infty} \left\{ \frac{|\varphi'(t)|^{p^+/(p^+ -1)}}{(p(t) - 1 + p'(t)t)\varphi(t)} \right\}^{p^+ -1} \mathrm{d}t < \infty,$$

$$(2.4) \qquad \int^{\infty} \left\{ \frac{|\varphi'(t)|^{p^-/(p^- -1)}}{(p(t) - 1 + p'(t)t)\varphi(t)} \right\}^{p^- -1} \mathrm{d}t < \infty,$$

$$\int^{\infty} \frac{p(t) - 1 + p'(t)t}{\varphi(t)^{1/(p^+ -1)}} \, \mathrm{d}t = \infty, \quad \int^{\infty} \frac{p(t) - 1 + p'(t)t}{\varphi(t)^{1/(p^- -1)}} \, \mathrm{d}t = \infty,$$

$$(2.5) \qquad \int^{\infty} \varphi(t) c(t) \, \mathrm{d}t = \infty;$$

(ii) *$q(t)$ is increasing, $1 < q(t) < p(t)$, there exists $\varphi(t) \in C^1((0, \infty); (0, \infty))$ such that*

$$(2.6) \qquad \int^{\infty} \left\{ \frac{|\varphi'(t)|^{q^+/(q^+ -1)}}{(q(t) - 1 + q'(t)t)\varphi(t)} \right\}^{q^+ -1} \mathrm{d}t < \infty,$$

$$(2.7) \qquad \int^{\infty} \left\{ \frac{|\varphi'(t)|^{q^-/(q^- - 1)}}{(q(t) - 1 + q'(t)t)\varphi(t)} \right\}^{q^- - 1} dt < \infty,$$

$$\int^{\infty} \frac{q(t) - 1 + q'(t)t}{\varphi(t)^{1/(q^+ - 1)}} \, dt = \infty, \qquad \int^{\infty} \frac{q(t) - 1 + q'(t)t}{\varphi(t)^{1/(q^- - 1)}} \, dt = \infty,$$

$$(2.8) \qquad \int^{\infty} \varphi(t)c(t) \, dt = \infty,$$

then every solution $u(t)$ of (E) is oscillatory.

P r o o f. Suppose that u is a nonoscillatory solution. We prove only the case $u > 0$, $t \geqslant t_0$ for some $t_0 > 0$, as the proof of the case $u < 0$ is similar. It follows that

$$(2.9) \qquad (|u'|^{p(t)-2}u')' = -c(t)|u|^{q(t)-2}u < 0, \quad t \geqslant t_0.$$

Now we claim that $u'(t) > 0$, $t > t_1$ for some $t_1 > t_0$. In fact, if $u'(t) > 0$ does not hold, then $u'(t) \leqslant 0$, $t > t_1$. Hence we show that

$$|u'(t)|^{p(t)-2}u'(t) < |u'(t_1)|^{p(t_1)-2}u'(t_1) \leqslant 0.$$

Thus we can find a $t_2 > t_1$ such that $u'(t_2) < 0$. Integrating (2.9) over $[t_2, t]$ yields

$$|u'(t)|^{p(t)-2}u'(t) \leqslant |u'(t_2)|^{p(t_2)-2}u'(t_2) < 0$$

for $t > t_2$, and therefore,

$$-(-u'(t))^{p(t)-1} \leqslant -|u'(t_2)|^{p(t_2)-1}, \quad t > t_2.$$

This shows that

$$u'(t) \leqslant -|u'(t_2)|^{(p(t_2)-1)/(p(t)-1)}$$
$$\leqslant -\min_{t \geqslant t_2} |u'(t_2)|^{(p(t_2)-1)/(p(t)-1)} := -a < 0.$$

Integrate the above inequality to obtain

$$u(t) \leqslant -a(t - t_2) + u(t_2) \to -\infty$$

as $t \to \infty$. This contradicts the assumption. Hence, we have $u'(t) > 0$, $t > t_3$ for some $t_3 > t_0$.

(i) If $p(t) = q(t)$ or $1 < p(t) < q(t)$, then we define the function $w_1(t)$ such that

(2.10)
$$w_1(t) = \left(\frac{u'(t)}{u(t)}\right)^{p(t)-1} > 0,$$

which is led by

(2.11)
$$u(t) = \exp\left(\int_0^t w_1^{1/(p(s)-1)}(s)\,ds\right).$$

This means that $u(t) > 0$, $t > 0$. Making use of the above argument we easily see that $u'(t) > 0$. Differentiating both sides of (2.10), we see that

(2.12)
$$w_1'(t) = -c(t)u(t)^{q(t)-p(t)}$$
$$-w_1(t)\{(p(t)-1)w_1^{1/(p(t)-1)}(t) + p'(t)\log u(t)\}$$

for $t > t_3$. In view of (2.10) and (2.11), we have $\log u > 0$. Accordingly, we see that $u(t) > 1$ and
$$w_1'(t) \leqslant 0, \quad t \geqslant t_4$$

for some $t_4 > t_3$. Therefore, we see that $\lim_{t\to\infty} w_1(t) := w_1(\infty)$ exists, and we can separate the two case of $0 < w_1(\infty) < 1$ and $1 \leqslant w_1(\infty) < \infty$. First, we take the case when $0 < w_1(\infty) < 1$. Then it follows from (2.11) that

$$\log u(t) = \int_0^t w_1^{1/(p(s)-1)}(s)\,ds,$$

and consequently

(2.13)
$$\log u(t) \geqslant t w_1^{1/(p^- -1)}(t), \quad t \geqslant t_5$$

for $t_5 > t_4$. Combining (2.12) with (2.13), we have

$$w_1'(t) \leqslant -c(t) - ((p(t)-1) + p'(t)t)w_1^{p^-/(p^- -1)}(t), \quad t \geqslant t_5.$$

Next, for the case when $1 \leqslant w_1(\infty) < \infty$, it can be shown by using a similar method that
$$w_1'(t) \leqslant -c(t) - (p(t)-1 + p'(t)t)w_1^{p^+/(p^+ -1)}(t), \quad t \geqslant t_5.$$

By applying Lemma 2.1, we see that (2.3)–(2.5) imply that the above Riccati inequalities cannot have a solution. This is a contradiction.

(ii) If $1 < q(t) < p(t)$, then we define the function $w_2(t)$ such that

$$(2.14) \qquad w_2(t) = \frac{u'(t)^{p(t)-1}}{u(t)^{q(t)-1}} = \left(\frac{u'(t)}{u(t)}\right)^{q(t)-1} u'(t)^{p(t)-q(t)} > 0,$$

which is led by

$$(2.15) \qquad u(t) = \exp\left(\int_0^t \left(\frac{w_2(s)}{u'(t)^{p(s)-q(s)}}\right)^{1/(q(s)-1)} ds\right).$$

Differentiating both sides of (2.14), we see that

$$(2.16) \qquad w_2'(t) = \frac{(|u'(t)|^{p(t)-2}u'(t))'}{u(t)^{q(t)-1}} - w_2(t)\frac{(u(t)^{q(t)-1})'}{u(t)^{q(t)-1}}.$$

This implies that

$$(2.17) \qquad w_2'(t) = -c(t) - w_2(t)\left\{(q(t)-1)\frac{u'(t)}{u(t)} + q'(t)\log u(t)\right\}, \quad t \geqslant t_3,$$

which together with (2.14) and (2.15) ensures that $\log u > 0$ and

$$w_2'(t) \leqslant 0, \quad t \geqslant t_6$$

for some $t_6 > t_3$. At this point, it is clear that $\lim_{t\to\infty} w_2(t) := w_2(\infty)$ exists, and we can separate the two cases of $0 < w_2(\infty) < 1$ and $1 \leqslant w_2(\infty) < \infty$. First, we take the case when $0 < w_2(\infty) < 1$. From (2.9) it follows that

$$u'(t)^{p(t)-1} \leqslant u'(t_2)^{p(t_2)-1} \equiv k_0$$

for some constant $k_0 > 0$. From (2.14) we see that

$$w_2(t) \leqslant \left(\frac{u'(t)}{u(t)}\right)^{q(t)-1} k_0^{(p(t)-q(t))/(p(t)-1)} \leqslant \left(\frac{u'(t)}{u(t)}\right)^{q(t)-1} k_1,$$

and so

$$(2.18) \qquad \frac{u'(t)}{u(t)} \geqslant \left(\frac{w_2(t)}{k_1}\right)^{1/(q(t)-1)} \geqslant k_2 w_2^{1/(q^--1)}(t), \quad t \geqslant t_7$$

for some $t_7 > t_6$. On the other hand, by (2.15), we also obtain

$$(2.19) \qquad \log u(t) \geqslant \int_0^t \left(\frac{w_2(s)}{k_0^{(p(s)-q(s))/(p(s)-1)}}\right)^{1/(q(s)-1)} ds$$

$$\geqslant k_3 t w_2^{1/(q^--1)}(t), \quad t \geqslant t_8$$

for some constant $k_i > 0$, $i = 1, 2, 3$, and some $t_8 > t_7$. Combining (2.17), (2.18) with (2.19), we have

$$w_2'(t) \leqslant -c(t) - K(q(t) - 1 + q'(t)t)w_2^{q^-/(q^- - 1)}(t), \quad t \geqslant t_8.$$

Finally, for the case when $1 \leqslant w_2(\infty) < \infty$, it is easy to verify that

$$w_2'(t) \leqslant -c(t) - K(q(t) - 1 + q'(t)t)w_2^{q^+/(q^+ - 1)}(t), \quad t \geqslant t_8$$

for some constant $K > 0$. By applying Lemma 2.1, we see that (2.6)–(2.8) imply that the above Riccati inequalities cannot have a solution. This contradiction completes the proof of the theorem. $\qquad\square$

Corollary 2.1. *Assume that $c(t) = t^{-\theta(t)}$, where $\theta(t) \in C((0, \infty); \mathbb{R})$, and that*

$$1 + \limsup_{t \to \infty} \theta(t) < \liminf_{t \to \infty} q(t).$$

If one of the following cases holds:

(i) *$p(t)$ is increasing, $1 < p(t) < q(t)$ or $\lim\limits_{t \to \infty} p(t) = \lim\limits_{t \to \infty} q(t)$;*

(ii) *$q(t)$ is increasing, $1 < q(t) < p(t)$,*

then every solution $u(t)$ of (E) is oscillatory.

P r o o f. If $1 < p(t) < q(t)$ or $\lim\limits_{t \to \infty} p(t) = \lim\limits_{t \to \infty} q(t)$, then we can derive by applying Theorem 2.1 with $\varphi(t) = t^{q^- - k}$ for some $1 + q^- - p^+ < k < q^- - \theta^+$ that

$$\int^\infty \left\{ \frac{((q^- - k)t^{q^- - k - 1})^{p^+/(p^+ - 1)}}{(p(t) - 1 + p'(t)t)t^{q^- - k}} \right\}^{p^+ - 1} dt < c_0 \int^\infty t^{-p^+ + (q^- - p^+ - k + 1)} dt < \infty,$$

$$\int^\infty \left\{ \frac{((q^- - k)t^{q^- - k - 1})^{p^-/(p^- - 1)}}{(p(t) - 1 + p'(t)t)t^{q^- - k}} \right\}^{p^- - 1} dt < \infty,$$

$$\int^\infty \frac{p(t) - 1 + p'(t)t}{(t^{p^- - k})^{1/(p^+ - 1)}} dt > c_1 \int^\infty t^{1 - (q^- - k)/(p^+ - 1)} dt = \infty,$$

$$\int^\infty \frac{p(t) - 1 + p'(t)t}{(t^{p^- - k})^{1/(p^- - 1)}} dt > \infty,$$

(2.20)
$$\int^\infty t^{-\theta(t) + q^- - k} dt > \int^\infty t^{q^- - \theta^+ - k} dt.$$

On the other hand, if $1 < q(t) < p(t)$, then we choose $1 + q^- - q^+ < k < q^- - \theta^+$ such that

$$\int^{\infty} \left\{ \frac{((q^- - k)t^{q^- - k - 1})^{q^+/(q^+ - 1)}}{(q(t) - 1 + q'(t)t)t^{q^- - k}} \right\}^{q^+ - 1} \mathrm{d}t < c_2 \int^{\infty} t^{-q^+ + (q^- - q^+ - k + 1)} \, \mathrm{d}t < \infty,$$

$$\int^{\infty} \left\{ \frac{((q^- - k)t^{q^- - k - 1})^{q^-/(q^- - 1)}}{(q(t) - 1 + q'(t)t)t^{q^- - k}} \right\}^{q^- - 1} \mathrm{d}t < \infty,$$

$$\int^{\infty} \frac{q(t) - 1 + q'(t)t}{(t^{q^- - k})^{1/(q^+ - 1)}} \, \mathrm{d}t > c_3 \int^{\infty} t^{1 - (q^- - k)/(q^+ - 1)} \, \mathrm{d}t = \infty,$$

$$\int^{\infty} \frac{q(t) - 1 + p'(t)t}{(t^{q^- - k})^{1/(q^- - 1)}} \, \mathrm{d}t > \infty,$$

(2.21) $$\int^{\infty} t^{-\theta(t) + q^- - k} \, \mathrm{d}t > \int^{\infty} t^{q^- - \theta^+ - k} \, \mathrm{d}t$$

for some positive constants c_i, $i = 0, 1, 2, 3$. Now we assume that $1 + \theta^+ < q^-$ holds, then integral calculus conditions (2.20) and (2.21) become infinite. Clearly, we see that the conditions of Theorem 2.1 hold. Therefore the conclusion follows from Theorem 2.1. □

Evidently, Theorem 2.1 does not apply to Theorem A. Hence we will improve the Lemma 2.1 as follows.

Lemma 2.2. *If there exists a function $\varphi(t) \in C^1([T_0, \infty); (0, \infty))$ such that (2.1) holds for some $T_1 \geqslant T_0 > 0$, then the Riccati inequality (2.2) has no positive solution on $[T, \infty)$ for all large T.*

P r o o f. Let $x(t)$ be a positive solution of (2.2). We assume that $\varphi(t)$ is defined for $t \geqslant T_0$. Multiplying (2.2) by $\varphi(t)$ and integrating over $[T_0, t]$, we obtain

(2.22) $$x(T_0)\varphi(T_0) \geqslant - \int_{T_0}^{t} x(s)\varphi'(s) \, \mathrm{d}s + \int_{T_0}^{t} \frac{\varphi(s)x(s)^{\beta}}{\bar{p}(s)} \, \mathrm{d}s + \int_{T_0}^{t} \varphi(s)\bar{q}(s) \, \mathrm{d}s.$$

By using Young's inequality we have

(2.23) $$x(s)|\varphi'(s)| = x(s)\left(\frac{\varphi(s)}{\bar{p}(s)} \right)^{1/\beta} \left(\frac{\bar{p}(s)}{\varphi(s)} \right)^{1/\beta} |\varphi'(s)|$$

$$\leqslant \frac{1}{\beta} \frac{\varphi(s)}{\bar{p}(s)} x(s)^{\beta} + \frac{\beta - 1}{\beta} \left(\left(\frac{\bar{p}(s)}{\varphi(s)} \right)^{1/\beta} |\varphi'(s)| \right)^{\beta/(\beta - 1)}.$$

Combining (2.22) with (2.23) yields

$$x(T_0)\varphi(T_0) \geqslant \left(1 - \frac{1}{\beta}\right) \int_{T_0}^t \frac{\varphi(s)}{\bar{p}(s)} x(s)^\beta \, \mathrm{d}s$$
$$- \frac{\beta - 1}{\beta} \int_{T_0}^t \left(\frac{\bar{p}(s)}{\varphi(s)} |\varphi'(s)|^\beta\right)^{1/(\beta-1)} \mathrm{d}s + \int_{T_0}^t \varphi(s)\bar{q}(s) \, \mathrm{d}s,$$

which contradicts the condition (2.1). The proof is complete. $\qquad\square$

Theorem 2.2. *If one of the following cases holds:*

(i) *$p(t)$ is increasing, $1 < p(t) = q(t)$ or $1 < p(t) < q(t)$, there exists $\varphi(t) \in C^1((0,\infty);(0,\infty))$ satisfying (2.3)–(2.5);*

(ii) *$q(t)$ is increasing, $1 < q(t) < p(t)$, there exists $\varphi(t) \in C^1((0,\infty);(0,\infty))$ satisfying (2.6)–(2.8),*

then every solution $u(t)$ of (E) is oscillatory.

E x a m p l e 2.1. We consider the equation

$$(2.24) \qquad (|u'(t)|^{2-4/(3t)} u'(t))' + t^{-1/3+1/t} |u(t)|^{1-1/t} u(t) = 0, \quad t \geqslant 1,$$

where
$$p(t) = 4 - \frac{4}{3t}, \quad q(t) = 3 - \frac{1}{t}, \quad c(t) = t^{-1/3+1/t}.$$

Letting $\varphi(t) = t^{2/3}$, we see that

$$\int^\infty \left\{\frac{(\frac{2}{3}t^{-1/3})^{3/2}}{(1 - 1/t + 1/t)t^{2/3}}\right\}^2 \mathrm{d}t = \int^\infty \left(\frac{2}{3}\right)^3 t^{-7/3} \, \mathrm{d}t < \infty,$$
$$\int^\infty \frac{(\frac{2}{3}t^{-1/3})^2}{(1 - 1/t + 1/t)t^{2/3}} \, \mathrm{d}t = \int^\infty \frac{2}{3} t^{-4/3} \, \mathrm{d}t < \infty,$$
$$\int^\infty t^{2/3} t^{-1/3+1/t} \, \mathrm{d}t = \infty.$$

Hence, it follows from (2.3)–(2.5) that all conditions of Theorem 2.2 (ii) are satisfied. However, Theorem 2.1 (ii) is not applicable, since

$$\int^\infty \frac{1 - 1/t + 1/t}{(t^{2/3})^{3/2}} \, \mathrm{d}t = \int^\infty t^{-1} \, \mathrm{d}t = \infty,$$
$$\int^\infty \frac{1 - 1/t + 1/t}{(t^{2/3})^2} \, \mathrm{d}t = \int^\infty t^{-4/3} \, \mathrm{d}t < \infty.$$

Therefore, from Theorem 2.2 (ii), every solution of (2.24) oscillates.

Corollary 2.2. *Assume that $c(t) = t^{-\theta(t)}$, where $\theta(t) \in C((0, \infty); \mathbb{R})$, and that*

$$\limsup_{t\to\infty} \theta(t) < \liminf_{t\to\infty} q(t).$$

If one of the following cases holds:

(i) *$p(t)$ is increasing, $1 < p(t) < q(t)$ or $\displaystyle\lim_{t\to\infty} p(t) = \lim_{t\to\infty} q(t)$;*

(ii) *$q(t)$ is increasing, $1 < q(t) < p(t)$,*

then every solution $u(t)$ of (E) is oscillatory.

E x a m p l e 2.2. Consider the equation

$$(2.25) \qquad (|u'(t)|^{1-2^{-\log t}} u'(t))' + t^{-1/2+\sin t}|u(t)|^{3+\sin t}u(t) = 0, \quad t \geqslant 1,$$

where

$$p(t) = 3 - 2^{-\log t}, \quad q(t) = 5 + \sin t, \quad c(t) = t^{-1/2+\sin t}.$$

We choose $\varphi(t) = t^{1/2}$ to find that

$$\int^{\infty} \left\{ \frac{(\frac{1}{2}t^{-1/2})^{3/2}}{(2 - 2^{-\log t} + 2^{-\log t})t^{1/2}} \right\}^2 dt = \int^{\infty} \left(\frac{1}{2}\right)^5 t^{-5/2}\, dt < \infty,$$

$$\int^{\infty} \frac{(\frac{1}{2}t^{-1/2})^2}{(2 - 2^{-\log t} + 2^{-\log t})t^{1/2}}\, dt = \int^{\infty} \left(\frac{1}{2}\right)^3 t^{-3/2}\, dt < \infty,$$

$$\int^{\infty} t^{1/2} t^{-1/2+\sin t}\, dt = \int^{\infty} t^{\sin t}\, dt = \infty.$$

Furthermore, it is easy to check that

$$\limsup_{t\to\infty} \theta(t) = \frac{3}{2} < 4 = \liminf_{t\to\infty} q(t).$$

Since all conditions of Theorem 2.2 (i) and Corollary 2.2 (i) are satisfied, every solution of (2.25) oscillates.

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Author's address: *Yutaka Shoukaku*, Institute of Science and Engineering, Kanazawa University, Kakuma-machi, Kanazawa 920-1192, Ishikawa, Japan, e-mail: `shoukaku@se.kanazawa-u.ac.jp`.