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BOUNDARY AUGMENTED LAGRANGIAN METHOD FOR THE SIGNORINI PROBLEM

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Abstract. An augmented Lagrangian method, based on boundary variational formulations and fixed point method, is designed and analyzed for the Signorini problem of the Laplacian. Using the equivalence between Signorini boundary conditions and a fixed-point problem, we develop a new iterative algorithm that formulates the Signorini problem as a sequence of corresponding variational equations with the Steklov-Poincaré operator. Both theoretical results and numerical experiments show that the method presented is efficient.

Keywords: Signorini problem; augmented Lagrangian; fixed point; Steklov-Poincaré operator; boundary integral equation

MSC 2010: 35J58, 35J05, 65N38

1. Introduction

As we know, Signorini problems are very important for a wide range of applications in mechanics and engineering [1], [4], [7], and these problems are very complicated, because their boundary conditions involve inequality constraints, which make them strong nonlinear. On a part of the boundary, the zone of the classical Dirichlet and Neumann boundary conditions is unknown in advance. Therefore, the main challenge in such problems is how to identify the boundary conditions. Usually Signorini problems have been transformed into variational inequalities, which can be solved with the finite element method (FEM) [1], [3], [4], [6], [21], [24] or the boundary element method (BEM) [2], [8], [12], [20]. The development of new fast

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convergent, accurate and efficient methods for the numerical simulation of Signorini problems is still a very active field of research, and we mention selected contributions [5], [11], [14], [16], [18], [23], [25], [26].

Recently, the projection method has been successfully applied to constrained problems such as complementary problems and variational inequalities in finite-dimensional space [9], [17]. The main idea of this method is to transform the problem into the fixed-point problem by using projection, which is very useful in developing various iterative methods for solving the original problem. During the last twenty years, a number of projection methods have been studied extensively [9], [17]. In these methods, the problem has been formulated only by equality with a projection operator, and no inequality constraint is needed. In comparison to other methods, the projection method is much easier to implement in both theory and application.

BEM has turned out to be an accurate and effective method for many partial differential equations, especially elliptic boundary value problems. The advantage of BEM is the significant reduction of expense mesh generation because of discretization only on the boundary of the domain. In the case of Signorini problems, the unknown boundary values are the potential and its derivative on the boundary, which are considered primary variables in BEM and can be obtained directly [10], [22]. Therefore, BEM is more appropriate for Signorini problems [14], [25], [26]. However, little research has been done on the Signorini problem using the fixed-point method and BEM up to now.

The focus of this paper is to develop a boundary augmented Lagrangian method (BALM) for the solution of Signorini problems, which is inspired by the classical augmented Lagrangian methods (ALM). Although ALM needs to solve a nonlinear problem in every iteration step, the semismooth Newton method can be applied for the solution [11], [21]. For the Signorini problem of Laplace equation, we first use the projection technique to deal with the Signorini boundary conditions by an equality which is based on the fixed-point method. Next, we deduce a boundary weak formulation with Steklov-Poincaré operator [13], [15], [19], [22]. Although the new problem is still strong nonlinear on the boundary, this problem no longer has the inequality constraint and is useful from a numerical point of view. Using transformations, we then propose a BALM for the Signorini problem which needs only the iteration for boundary values and the computing of the boundary variational problem. We can use the properties of projection and boundary integral operators to analyse the convergence of the method. Numerical results show that our method is accurate and efficient.

The paper is organized as follows. In Section 2, we start with the classical Signorini problem of the Laplacian and establish equivalent formulations of the nonlinear boundary conditions and a fixed-point problem. We use the Steklov-Poincaré operator to introduce the boundary variational formulation in Section 3. In Section 4 we propose a new ALM for the Signorini problem and obtain monotone convergence of the method, which shows unconditional convergence for all positive parameters. In Section 5, we present some numerical examples to investigate the performance of our method, and finally a brief conclusion is given in Section 6.

2. The Signorini problem and the fixed-point method

For the sake of simplicity, we consider the Signorini problem for the Laplace equation in an open and bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\Gamma = \partial\Omega$. This boundary Γ consists of three disjoint parts Γ_D, Γ_N, and $\Gamma_S \neq \emptyset$, where Dirichlet, Neumann, and Signorini conditions are prescribed. For a given $g \in H^{1/2}(\Gamma \setminus \overline{\Gamma}_N)$, $f \in H^{-1/2}(\Gamma \setminus \overline{\Gamma}_D)$, find $u \in H^1(\Omega)$ and $\lambda \in H^{-1/2}(\Gamma_S)$ such that

$$\Delta u = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$u = g \quad \text{on } \Gamma_D, \tag{2.2}$$

$$\lambda = f \quad \text{on } \Gamma_N, \tag{2.3}$$

$$u \geqslant g, \quad \lambda \geqslant f, \quad (u-g)(\lambda - f) = 0 \quad \text{on } \Gamma_S, \tag{2.4}$$

where $\lambda := \frac{\partial u}{\partial n}$. It can be proved in the theory of variational inequalities that this problem has a unique solution if $\Gamma_S \neq \emptyset$ or $\int_{(\Gamma \setminus \overline{\Gamma}_D)} f \, \mathrm{d}s < 0$, see [7], [8], [20].

Since the main difficulty of the problem arises from the nonlinear boundary conditions (2.4), in this paper we transfer them to a fixed-point problem [9], [14], [17], [26]. Let us introduce the projection notation for $a \in \mathbb{R}$

$$[a]_+ = \begin{cases} a & \text{if} \quad a > 0, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, we obtain the following result.

Lemma 2.1. *For all $\varrho > 0$, the boundary conditions (2.4) on Γ_S are equivalent to*

$$\lambda - f - [\lambda - f - \varrho(u-g)]_+ = 0 \quad \text{on } \Gamma_S. \tag{2.5}$$

Proof. Let u and λ be such that (2.4) holds. From the condition $\lambda \geqslant f$ we have either $\lambda > f$ or $\lambda = f$. Suppose first that $\lambda > f$. Then the condition $(u-g)(\lambda - f) = 0$ implies that $u = g$. In this case, it holds that

$$[\lambda - f - \varrho(u-g)]_+ = [\lambda - f]_+ = \lambda - f.$$

Then, suppose that $\lambda = f$. The condition $u \geqslant g$ can also be expressed as $[-\varrho(u-g)]_+ = 0$, so

$$[\lambda - f - \varrho(u-g)]_+ = [-\varrho(u-g)]_+ = \lambda - f.$$

On the other hand, let u and λ be such that (2.5) holds. Note first that it implies $\lambda \geqslant f$. If $\lambda = f$, then (2.5) can be rewritten as

$$[-\varrho(u-g)]_+ = 0,$$

which is equivalent to the condition $u \geqslant g$. Since $\lambda = f$, then the condition

$$(u-g)(\lambda - f) = 0$$

also holds. We now consider the case $\lambda > f$. From (2.5), $[\lambda - f - \varrho(u-g)]_+ > 0$, so in this case

$$\lambda - f = [\lambda - f - \varrho(u-g)]_+ = \lambda - f - \varrho(u-g),$$

which implies $u = g$, so all conditions (2.4) hold. □

3. Boundary weak formulation of the Signorini problem

To develop a boundary variational formulation that is suitable for the Signorini problem we start with the space of functions defined as

$$H^1_D(\Omega) := \{v \in H^1(\Omega),\ v = g \text{ on } \Gamma_D\}.$$

From Green's formula and (2.1) we obtain the following variational problem: find $u \in H^1(\Omega)$ and $\lambda \in H^{-1/2}(\Gamma_S)$ such that

$$\int_\Omega \nabla u \nabla v \,\mathrm{d}x = \int_{\Gamma_N \cup \Gamma_S} \lambda v \,\mathrm{d}s \quad \forall v \in H^1_D(\Omega), \tag{3.1}$$

with boundary conditions (2.3) and (2.5).

As in [15], [19], [22], we introduce the single layer potential V, the double layer potential K, the adjoint double layer potential K' and the hypersingular integral operator D by

$$\begin{aligned}
(V\lambda)(x) &= \int_\Gamma U(x,y)\lambda(y)\,\mathrm{d}s_y, && V\colon\ H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),\\
(Ku)(x) &= \int_\Gamma \frac{\partial}{\partial n_y} U(x,y)u(y)\,\mathrm{d}s_y, && K\colon\ H^{1/2}(\Gamma) \to H^{1/2}(\Gamma),\\
(K'\lambda)(x) &= \int_\Gamma \frac{\partial}{\partial n_x} U(x,y)\lambda(y)\,\mathrm{d}s_y, && K'\colon\ H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma),\\
(Du)(x) &= -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y} U(x,y)u(y)\,\mathrm{d}s_y, && D\colon\ H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),
\end{aligned}$$

where $U(x,y)$ is the fundamental solution of the two-dimensional Laplace equation

$$U(x,y) = -\frac{1}{2\pi} \ln |x-y|.$$

Next, we introduce the Dirichlet-to-Neumann mapping on Γ

$$\begin{aligned} S\colon\ H^{1/2}(\Gamma) &\to H^{-1/2}(\Gamma), \\ u|_\Gamma &\mapsto \lambda|_\Gamma. \end{aligned}$$

Note that $S(u|_\Gamma) = \lambda|_\Gamma$, so

$$\int_\Omega \nabla u \nabla v \,\mathrm{d}x = \int_\Gamma S(u|_\Gamma) v \,\mathrm{d}s \quad \forall v \in H^1_D(\Omega),$$

where the Steklov-Poincaré operator S, see [15], [22] is defined by

$$(Su)(x) = \Big[D + \Big(\frac{1}{2}I + K'\Big)V^{-1}\Big(\frac{1}{2}I + K\Big)\Big]u(x). \tag{3.2}$$

Let us define

$$\begin{aligned} H^{1/2}_D(\Gamma) &:= \{v \in H^{1/2}(\Gamma),\ v = g \text{ on } \Gamma_D\}, \\ H^{1/2}_0(\Gamma, \Gamma_D) &:= \{v \in H^{1/2}(\Gamma),\ v = 0 \text{ on } \Gamma_D\}, \\ \langle Su, v\rangle_\Gamma &:= \int_\Gamma Su(x)v(x)\,\mathrm{d}s_x, \\ \langle \lambda, v\rangle_{\Gamma_S} &:= \int_{\Gamma_S} \lambda(x)v(x)\,\mathrm{d}s_x, \\ L(v) &:= \int_{\Gamma_N} f(x)v(x)\,\mathrm{d}s_x. \end{aligned}$$

We then can obtain a pure boundary weak formulation of the original problem (2.1)–(2.4) as follows: find $u \in H^{1/2}(\Gamma_S)$ and $\lambda \in H^{-1/2}(\Gamma_S)$ such that

$$\langle Su, v\rangle_\Gamma - \langle \lambda, v\rangle_{\Gamma_S} = L(v) \quad \forall v \in H^{1/2}_D(\Gamma), \tag{3.3}$$

with boundary condition (2.5). From the properties of the boundary integral operators it follows that the Steklov-Poincaré operator S is linear, bounded, symmetric, and semielliptic on $H^{1/2}(\Gamma)$. Moreover, the operator S has the following characterization [22].

Lemma 3.1. *The Steklov-Poincaré operator* S *defined by* (3.2) *is elliptic on* $H_0^{1/2}(\Gamma,\Gamma_D)$, *i.e., there exists a constant* $\alpha>0$ *such that for any* $v\in H_0^{1/2}(\Gamma,\Gamma_D)$

$$\langle Sv,v\rangle_{L^2(\Gamma)}\geqslant\alpha\|v\|^2_{H^{1/2}(\Gamma)}. \tag{3.4}$$

Now, we obtain the boundary weak formulation (3.3) for the problem (2.1)–(2.4) via the Steklov-Poincaré operator S (3.2) and the fixed-point problem (2.5) for the nonlinear boundary conditions (2.4), which has only boundary integral operators and avoids inequality constraints. Both alternative equivalent formulations are also convenient for the numerical and theoretical analysis.

4. Boundary augmented Lagrangian method for the Signorini problem

With the above preparations, we can now present our boundary augmented Lagrangian method (BALM) for the Signorini problem as follows.

Algorithm BALM

Step 0: Choose $\lambda^{(0)}\in L^2(\Gamma_S)$, $\varrho\in\mathbb{R}^+$ and set $k:=0$.

Step 1: Solve

$$\langle Su^{(k+1)},v\rangle_\Gamma-\langle\lambda^{(k+1)},v\rangle_{\Gamma_S}=L(v)\quad\forall v\in H_D^{1/2}(\Gamma), \tag{4.1}$$

with

$$\lambda^{(k+1)}-f-[\lambda^{(k)}-f-\varrho(u^{(k+1)}-g)]_+=0\quad\text{on }\Gamma_S, \tag{4.2}$$

for $u^{(k+1)}$ and $\lambda^{(k+1)}$ on Γ_S.

Step 2: Update $k:=k+1$ and go to Step 1.

Let u^* and λ^* denote the solution of the Signorini problem and the corresponding derivative on the boundary Γ, respectively. In order to analyse the convergence of the BALM, we define

$$B_\varrho(u,\lambda):=B(u,\lambda;g,f,\varrho):=\lambda-f-\varrho(u-g),$$

and introduce the following projection property on Γ_S [9], [17], [21].

Lemma 4.1. *For all $u^{(k)}, \lambda^{(k)} \in L^2(\Gamma_S)$ generated by* (4.2), *we have*

$$\langle \lambda^{(k+1)} - \lambda^*, B_\varrho(u^{(k+1)}, \lambda^{(k)}) - B_\varrho(u^*, \lambda^*) \rangle_{\Gamma_S} \geqslant \|\lambda^{(k+1)} - \lambda^*\|^2_{\Gamma_S}, \tag{4.3}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_S}$ stands for the extension of the usual scalar product.

Proof. Let us separate Γ_S into four subparts Γ_{S1}, Γ_{S2}, Γ_{S3}, and Γ_{S4}, where

$$\begin{aligned}
&B_\varrho(u^{(k+1)}, \lambda^{(k)}) \geqslant 0, \ \ B_\varrho(u^*, \lambda^*) \geqslant 0 \quad \text{on } \Gamma_{S1},\\
&B_\varrho(u^{(k+1)}, \lambda^{(k)}) \geqslant 0, \ \ B_\varrho(u^*, \lambda^*) < 0 \quad \text{on } \Gamma_{S2},\\
&B_\varrho(u^{(k+1)}, \lambda^{(k)}) < 0, \ \ B_\varrho(u^*, \lambda^*) \geqslant 0 \quad \text{on } \Gamma_{S3},\\
&B_\varrho(u^{(k+1)}, \lambda^{(k)}) < 0, \ \ B_\varrho(u^*, \lambda^*) < 0 \quad \text{on } \Gamma_{S4}.
\end{aligned}$$

From (2.5) and (4.2) we then have

$$\begin{aligned}
&\lambda^{(k+1)} - \lambda^* = B_\varrho(u^{(k+1)}, \lambda^{(k)}) - B_\varrho(u^*, \lambda^*) \quad \text{on } \Gamma_{S1},\\
0 \leqslant\ &\lambda^{(k+1)} - \lambda^* = B_\varrho(u^{(k+1)}, \lambda^{(k)}) - 0 < B_\varrho(u^{(k+1)}, \lambda^{(k)}) - B_\varrho(u^*, \lambda^*) \quad \text{on } \Gamma_{S2},\\
0 \geqslant\ &\lambda^{(k+1)} - \lambda^* = 0 - B_\varrho(u^*, \lambda^*) > B_\varrho(u^{(k+1)}, \lambda^{(k)}) - B_\varrho(u^*, \lambda^*) \quad \text{on } \Gamma_{S3},\\
&\lambda^{(k+1)} - \lambda^* = 0 - 0 = 0 \quad \text{on } \Gamma_{S4}.
\end{aligned}$$

It follows that

$$\langle \lambda^{(k+1)} - \lambda^*, B_\varrho(u^{(k+1)}, \lambda^{(k)}) - B_\varrho(u^*, \lambda^*) \rangle_{\Gamma_S} \geqslant \|\lambda^{(k+1)} - \lambda^*\|^2_{\Gamma_S}.$$

□

Theorem 4.1. *Let $\{(u^{(k)}, \lambda^{(k)})\}$ be the sequence generated by the* BALM. *Then for all k, $u^{(k)}$ converges to u^* in $H^{1/2}(\Gamma)$ and $\lambda^{(k)}$ converges to λ^* in $L^2(\Gamma_S)$ as $k \to \infty$.*

Proof. Let $\delta_u^{(k)} := u^{(k)} - u^*$ and $\delta_\lambda^{(k)} := \lambda^{(k)} - \lambda^*$. Then $\delta_u^{(k)} \in H_0^{1/2}(\Gamma, \Gamma_D)$ and $\delta_\lambda^{(k)} \in L^2(\Gamma_S)$. Considering that (u^*, λ^*) satisfies (3.3), we have

$$\langle S u^*, \delta_u^{(k+1)} \rangle_\Gamma - \langle \lambda^*, \delta_u^{(k+1)} \rangle_{\Gamma_S} = L(\delta_u^{(k+1)}). \tag{4.4}$$

From (4.1) of BALM we get

$$\langle S u^{(k+1)}, \delta_u^{(k+1)} \rangle_\Gamma - \langle \lambda^{(k+1)}, \delta_u^{(k+1)} \rangle_{\Gamma_S} = L(\delta_u^{(k+1)}). \tag{4.5}$$

Subtracting (4.4) from (4.5) results in

$$\langle S \delta_u^{(k+1)}, \delta_u^{(k+1)} \rangle_\Gamma = \langle \delta_\lambda^{(k+1)}, \delta_u^{(k+1)} \rangle_{\Gamma_S}. \tag{4.6}$$

Using Lemma 4.1 and Young's inequality, we obtain

$$\begin{aligned}\langle \delta_\lambda^{(k+1)}, \delta_u^{(k+1)} \rangle_{\Gamma_S} &= \varrho^{-1} \langle \delta_\lambda^{(k)}, \delta_\lambda^{(k+1)} \rangle_{\Gamma_S} - \varrho^{-1} \langle \delta_\lambda^{(k+1)}, B_\varrho(u^{(k+1)}, \lambda^{(k)}) - B_\varrho(u^*, \lambda^*) \rangle_{\Gamma_S} \\ &\leqslant \varrho^{-1} \langle \delta_\lambda^{(k)}, \delta_\lambda^{(k+1)} \rangle_{\Gamma_S} - \varrho^{-1} \| \delta_u^{(k+1)} \|_{\Gamma_S}^2 \\ &\leqslant (2\varrho)^{-1} \| \delta_\lambda^{(k)} \|_{\Gamma_S}^2 - (2\varrho)^{-1} \| \delta_\lambda^{(k+1)} \|_{\Gamma_S}^2.\end{aligned}$$

From (4.6) and Lemma 3.1, we then have

$$\langle S \delta_u^{(k+1)}, \delta_u^{(k+1)} \rangle_\Gamma \leqslant (2\varrho)^{-1} \| \delta_\lambda^{(k)} \|_{\Gamma_S}^2 - (2\varrho)^{-1} \| \delta_\lambda^{(k+1)} \|_{\Gamma_S}^2 \tag{4.7}$$

and

$$\langle S \delta_u^{(k+1)}, \delta_u^{(k+1)} \rangle_\Gamma \geqslant \alpha \| \delta_u^{(k+1)} \|_{H^{1/2}(\Gamma)}^2. \tag{4.8}$$

It follows from (4.7) and (4.8) that

$$\alpha \| \delta_u^{(k+1)} \|_{H^{1/2}(\Gamma)}^2 \leqslant (2\varrho)^{-1} \| \delta_\lambda^{(k)} \|_{\Gamma_S}^2 - (2\varrho)^{-1} \| \delta_\lambda^{(k+1)} \|_{\Gamma_S}^2. \tag{4.9}$$

Consequently,

$$\sum_{k=0}^{\infty} \alpha \| \delta_u^{(k+1)} \|_{H^{1/2}(\Gamma)}^2 \leqslant (2\varrho)^{-1} \| \delta_\lambda^{(0)} \|_{\Gamma_S}^2 < \infty,$$

which means that

$$\lim_{k \to \infty} \| \delta_u^{(k+1)} \|_{H^{1/2}(\Gamma)}^2 = 0.$$

Thus $u^{(k)}$ converges to u^* in $H^{1/2}(\Gamma)$ and from (4.2) of BALM, $\lambda^{(k)}$ converges to λ^* in $L^2(\Gamma_S)$ as $k \to \infty$. □

From (4.9), it is easy to verify that the sequence $\{\lambda^{(k)}\}$ is bounded and the sequence $\{\|\delta_\lambda^{(k)}\|_{\Gamma_S}\}$ is monotonically decreasing. Furthermore, larger values of parameter ϱ result in faster convergence of the algorithm. Therefore, we can use this method to identify the boundary condition on Γ_S.

5. Numerical examples

In order to demonstrate the efficiency and accuracy of the proposed method, we present three numerical examples of Signorini problems in this section. An analytic solution is available for the first example, and the analytic solution for the other two examples is unknown. In order to simplify the numerical process, we apply the constant BEM to the problem (4.1) with iteration (4.2) and solve the corresponding linear systems [14], [26]. Let N and N_S denote the total number of boundary elements on Γ and Γ_S, respectively. We choose $\|u_h^{(k+1)} - u_h^{(k)}\|_{\infty,\Gamma_S} \leqslant 10^{-10}\|u_h^{(k+1)}\|_{\infty,\Gamma_S}$ as the stopping criterion, where $\|u_h^{(k)}\|_{\infty,\Gamma_S} := \max\limits_{1\leqslant i\leqslant N_S} |u_h^{(k)}(x_i)|$ and $u_h^{(k)}(x_i)$ denotes numerical solution for the mesh step h.

5.1. Dirichlet-Signorini problem. First we consider a Signorini problem for the Laplacian $\Delta u = 0$ in the annular domain $\Omega = \{(x,y)\colon\ a < \sqrt{x^2+y^2} < b\}$ $(a,b \in \mathbb{R}^+)$ with a Dirichlet boundary condition on the boundary $\Gamma_D = \{(x,y)\colon \sqrt{x^2+y^2} = b\} \cup \{(x,y)\colon\ \sqrt{x^2+y^2} = a,\ y \geqslant 0\}$ and the following Signorini boundary conditions on the $\Gamma_S = \{(x,y)\colon\ \sqrt{x^2+y^2} = a,\ y < 0\}$:

$$u \geqslant 0, \quad \lambda \geqslant 0, \quad u\lambda = 0 \quad \text{on } \Gamma_S.$$

For this problem, the analytic solution in the domain Ω is given by the function

$$u(x,y) = \operatorname{Im}\omega^3(x+\mathrm{i}y),$$

with

$$\begin{aligned}\omega(x+\mathrm{i}y) = &\sqrt{\frac{1}{2}\sqrt{\Big(\frac{x^2-y^2}{r^2}\Big)^2 + \frac{1}{4}\Big(\frac{r^2}{a^2} - \frac{a^2}{r^2}\Big)^2} + \frac{1}{4}\frac{x^2-y^2}{r^2}\Big(\frac{r^2}{a^2}+\frac{a^2}{r^2}\Big)}\operatorname{sgn} x \\ &+ \mathrm{i}\sqrt{\frac{1}{2}\sqrt{\Big(\frac{x^2-y^2}{r^2}\Big)^2 + \frac{1}{4}\Big(\frac{r^2}{a^2} - \frac{a^2}{r^2}\Big)^2} - \frac{1}{4}\frac{x^2-y^2}{r^2}\Big(\frac{r^2}{a^2}+\frac{a^2}{r^2}\Big)}\operatorname{sgn} y,\end{aligned}$$

where $r = \sqrt{x^2+y^2} \geqslant a$. From the analytic solution, we can easily obtain the Dirichlet boundary condition on Γ_D.

The analytic solution and its normal derivative on the Signorini boundary Γ_S are

$$u(x,y) = -\sqrt{\max\Big(0, \frac{y^2-x^2}{a^2}\Big)^3}\operatorname{sgn} y, \tag{5.1}$$

$$\lambda(x,y) = -\frac{6}{a^3}\sqrt{\max\Big(0,\frac{x^2-y^2}{a^2}\Big)}|x|y. \tag{5.2}$$

This problem has been solved by the BEM with different methods, such as the decomposition-coordination method [20], and the projection iterative algorithm [25], [26].

For the case $a = 0.1$ and $b = 0.25$, we introduce the parameterizations $t \to (a\cos \pi t, -a\sin \pi t)$ and $t \to (b\cos \pi t, b\sin \pi t)$. First we apply our method to this problem on a uniform grid for t with $\varrho = 10000$ and $N = 160$. Here, the discretization includes 40 boundary elements on Γ_S and 120 boundary elements on Γ_D. The numerical and exact solutions for the potential u and the normal derivative λ are shown in Figures 1–2, respectively. It can be seen that our results are in a good agreement with the exact solution (5.1) and (5.2).

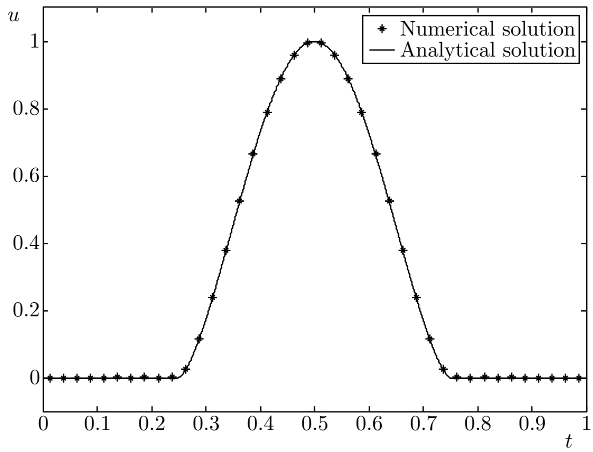

Figure 1. Analytic and approximate solutions for u on Γ_S.

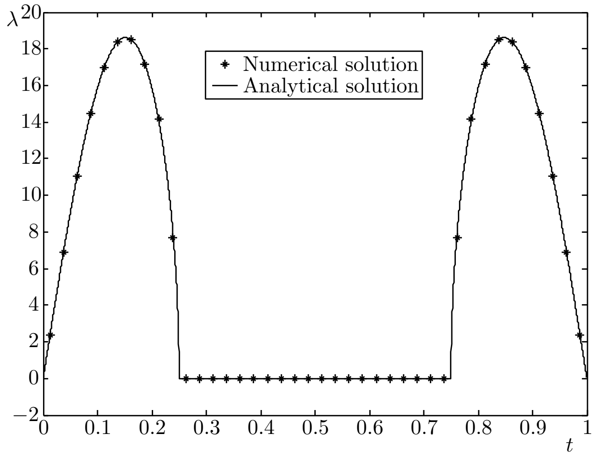

Figure 2. Analytic and approximate solutions for λ on Γ_S.

In order to investigate the convergence behavior of our method, we solve the problem by choosing different numbers of boundary elements N and various parameters ϱ. Table 1 gives the number of iterations for $N = 40$, 80, 160, 320, and 640 on Γ and

	$N = 40$	$N = 80$	$N = 160$	$N = 320$	$N = 640$
$\varrho = 10^2$	23	34	55	93	155
$\varrho = 10^3$	11	11	14	19	26
$\varrho = 10^4$	7	7	8	9	10
$\varrho = 10^5$	6	5	6	7	7
$\varrho = 10^6$	6	5	6	6	6

Table 1. Number of iterations for different values of N and ϱ.

$\varrho = 10^2$, 10^3, 10^4, 10^5, and 10^6. We note that numerical results converge quickly as the parameter ϱ increases. In addition, the number of iterations increases slowly as N increases. Besides, we define the error

$$e(u) = \frac{1}{N_S}\sqrt{\sum_{i=1}^{N_S}(u(x_i) - u_h(x_i))^2},$$

where $u(x_i)$ denotes the exact solution. We draw the error in the logarithmic scale depending on the step h. Figure 3 gives the change trend of the error for u. The results for λ are presented in Figure 4. It can be seen that our method yields very accurate results and converges superlinearly.

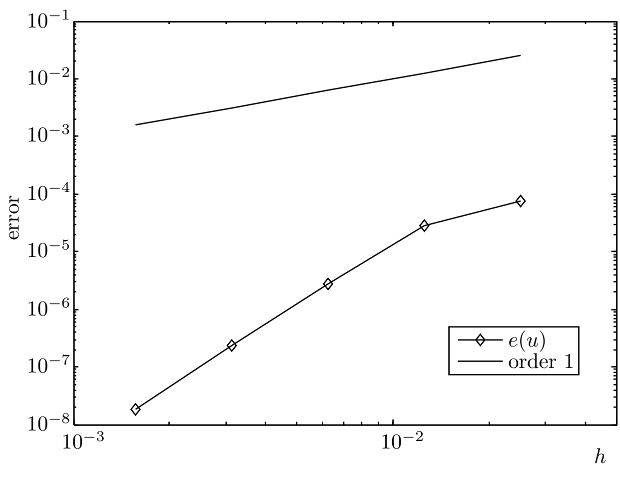

Figure 3. Log-log plot of convergence for approximate solutions u_h on Γ_S.

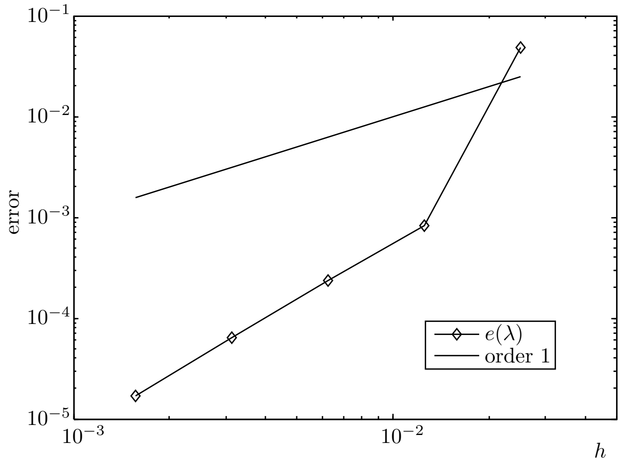

Figure 4. Log-log plot of convergence for approximate solutions λ_h on Γ_S.

5.2. Dirichlet-Neumann-Signorini problem. For the second numerical experiment, we consider the following Signorini problem, known as the steady-state shallow dam problem:

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega = (0,1)\times(0,1), \\ \lambda &= 0 && \text{on } \Gamma_N = \{(x,y)\colon\ 0 \leqslant x \leqslant 1,\ y = 0\}, \\ u &= G(1) && \text{on } \Gamma_{D_1} = \{(x,y)\colon\ x = 1,\ 0 \leqslant y \leqslant 1\}, \\ u &= 0 && \text{on } \Gamma_{D_2} = \{(x,y) : x = 0,\ 0 \leqslant y \leqslant 1\}, \end{aligned}$$

with the following Signorini boundary conditions on $\Gamma_S = \{(x,y)\colon\ 0 \leqslant x \leqslant 1, y = 1\}$:

$$u \leqslant G(x), \quad \lambda \leqslant 0, \quad (u - G(x))\lambda = 0.$$

Let $u' := -u$ and $\lambda' := -\partial u/\partial n = -\lambda$. Then the new problem for u' and λ' is same as (2.1)–(2.4). In this problem the function $G(x)$, which describes the surface profile, is known. The Signorini boundary conditions describe the location of the saturated and unsaturated parts of the upper surface, and the solution of the problem depends on the surface profile $G(x)$. This problem has been solved by the FEM [1], BEM [12], method of fundamental solutions [18], switching algorithm [2], and projection iterative algorithm [25].

We now apply our method to this problem, and three cases with different surface profiles are considered. We choose $N = 160$ and $\varrho = 10000$ again, and the numerical results corresponding to the surface profile $G_1(x) = (\frac{1}{2} - x)(1 - x) - x$, $G_2(x) = (\frac{1}{2} - \frac{5}{2}x)(1 - \frac{5}{3}x)(1 - x) - x$ and $G_3(x) = \sin 12x - 2$ are presented in Figures 5–7,

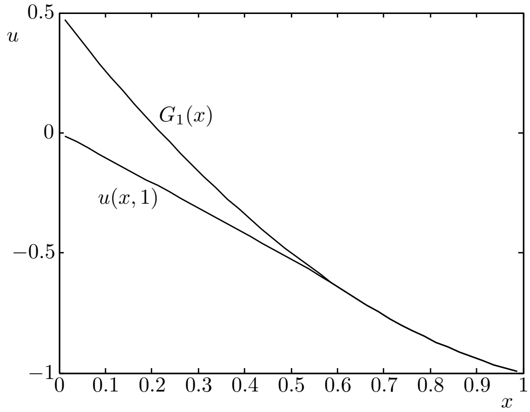

Figure 5. Approximate solution for the first profile on Γ_S.

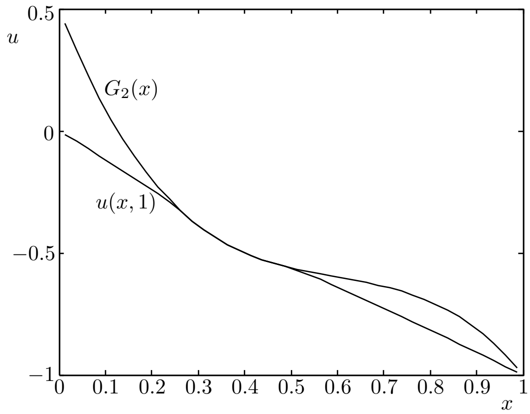

Figure 6. Approximate solution for the second profile on Γ_S.

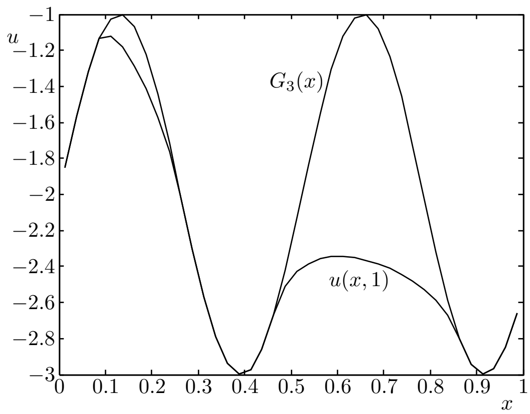

Figure 7. Approximate solution for the third profile on Γ_S.

respectively. Although it is difficult to identify accurately where $u = G(x)$ or $u < G(x)$, it can be observed that our results are in a good agreement with the numerical results of [2], [12], [18], [26].

We also investigate the convergence behavior of our method in this example. Tables 2–4 display the number of iterations for the three cases with different numbers of boundary elements N and various values of ϱ. As can be seen from our tests, our method converges quickly when ϱ is sufficiently large and the number of iterations depends only weakly on N.

	N=40	$N = 80$	$N = 160$	$N = 320$	$N = 640$
$\varrho = 10^2$	11	15	19	26	37
$\varrho = 10^3$	7	8	8	9	11
$\varrho = 10^4$	6	6	6	6	7
$\varrho = 10^5$	5	5	5	5	6
$\varrho = 10^6$	5	5	5	5	6

Table 2. Number of iterations for the first case.

	N=40	$N = 80$	$N = 160$	$N = 320$	$N = 640$
$\varrho = 10^2$	14	16	22	31	49
$\varrho = 10^3$	9	10	10	11	15
$\varrho = 10^4$	7	8	7	8	9
$\varrho = 10^5$	6	7	6	7	8
$\varrho = 10^6$	6	6	6	6	7

Table 3. Number of iterations for the second case.

	$N = 40$	$N = 80$	$N = 160$	$N = 320$	$N = 640$
$\varrho = 10^2$	14	18	23	34	49
$\varrho = 10^3$	8	10	11	13	13
$\varrho = 10^4$	6	7	8	9	9
$\varrho = 10^5$	6	7	6	7	8
$\varrho = 10^6$	5	6	6	7	6

Table 4. Number of iterations for the third case.

5.3. Dirichlet-Signorini problem. Finally, the presented algorithm is applied to a Signorini problem in a domain defined by two ellipses [20]. Let $E(a,b)$ denote the ellipse $\{(x,y)\colon\ (x/a)^2 + (y/b)^2 < 1\}$, and consider the Signorini problem

$$\Delta u = 0 \quad \text{in } \Omega = E(0.4, 0.2) \setminus \overline{E(0.1, 0.15)},$$
$$u = 1 \quad \text{on } \Gamma_D = \partial E(0.4, 0.2),$$

with boundary conditions

$$u \geqslant 0, \quad \lambda \geqslant -12.5, \quad u(\lambda + 12.5) = 0 \quad \text{on } \Gamma_S = \partial E(0.1, 0.15).$$

Following Spann [20], we use the same parameterizations $t \to (0.1\cos 2\pi t, -0.15\sin 2\pi t)$ and $t \to (0.4\cos 2\pi t,\ 0.2\sin 2\pi t)$. The numerical results with $N = 160$ and parameter $\varrho = 10000$ are given in Figures 8–9. It can be seen that our results are again in excellent agreement with those in [20]. Table 5 shows the number of iterations for different ϱ and various N. Similarly, we observe that the algorithm converges quickly and the number of iterations depends only weakly on N as ϱ increases.

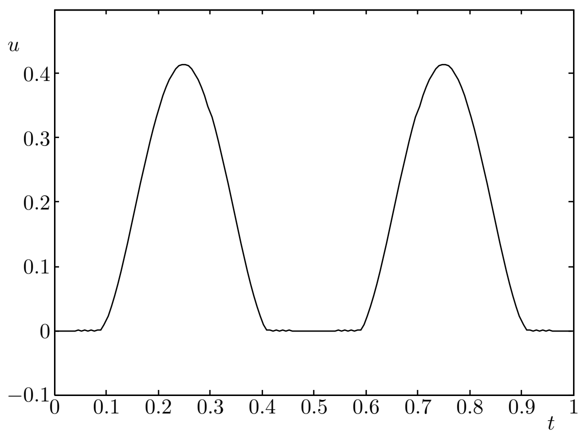

Figure 8. Approximate solutions for u on Γ_S.

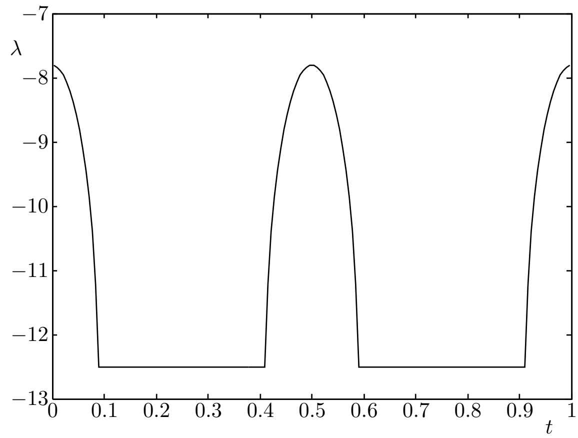

Figure 9. Approximate solutions for λ on Γ_S.

	N=40	N=80	N=160	N=320	N=640
$\varrho = 10^2$	17	28	41	66	98
$\varrho = 10^3$	8	12	12	15	20
$\varrho = 10^4$	5	8	8	8	9
$\varrho = 10^5$	4	7	6	6	6
$\varrho = 10^6$	3	7	5	6	6

Table 5. Number of iterations for different values of N and ϱ.

6. Conclusion

In this paper, we have studied a BALM for the solution of Signorini problems and its convergence analysis. The advantage of this method is that it only needs to solve a simple elliptic variational problem for each iteration. For different boundary elements, the method converges quickly when the parameter ϱ is sufficiently large. Moreover, this method can be easily applied to the Signorini problems defined in domains of arbitrary shape. The numerical examples demonstrate the perfect convergence and effectiveness of the method.

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