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EXTENDING BABUŠKA-AZIZ'S THEOREM TO HIGHER-ORDER LAGRANGE INTERPOLATION

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Cordially dedicated to Prof. Ivo Babuška on the occasion of his 90th birthday.

Abstract. We consider the error analysis of Lagrange interpolation on triangles and tetrahedrons. For Lagrange interpolation of order one, Babuška and Aziz showed that squeezing a right isosceles triangle perpendicularly does not deteriorate the optimal approximation order. We extend their technique and result to higher-order Lagrange interpolation on both triangles and tetrahedrons. To this end, we make use of difference quotients of functions with two or three variables. Then, the error estimates on squeezed triangles and tetrahedrons are proved by a method that is a straightforward extension of the original one given by Babuška-Aziz.

Keywords: Lagrange interpolation; Babuška-Aziz's technique; difference quotients

MSC 2010: 65D05, 65N30

1. Introduction

Lagrange interpolation on triangles and tetrahedrons and the associated error estimates are important subjects in numerical analysis. In particular, they are crucial in the error analysis of finite element methods. Let $d = 2$ or 3. Throughout this paper, $K \subset \mathbb{R}^d$ denotes a triangle or tetrahedron with vertices $\mathbf{x}_i$, $i = 1, \ldots, d+1$. We always suppose that triangles and tetrahedrons are closed sets in this paper. Let λ_i be its barycentric coordinates with respect to $\mathbf{x}_i$. By definition, we have $0 \leqslant \lambda_i \leqslant 1$, $\sum_{i=1}^{d+1} \lambda_i = 1$. Let $\mathbb{N}_0$ be the set of nonnegative integers, and $\gamma =$

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$(a_1, \ldots, a_{d+1}) \in \mathbb{N}_0^{d+1}$ a multi-index. Let k be a positive integer. If $|\gamma| := \sum_{i=1}^{d+1} a_i = k$, then $\gamma/k := (a_1/k, \ldots, a_{d+1}/k)$ can be regarded as a barycentric coordinate in K. The set $\Sigma^k(K)$ of points on K is defined by

$$\Sigma^k(K) := \left\{ \frac{\gamma}{k} \in K \;\middle|\; |\gamma| = k,\ \gamma \in \mathbb{N}_0^{d+1} \right\}.$$

Let $1 < p \leqslant \infty$. From Sobolev's imbedding theorem and Morry's inequality, we have the continuous imbeddings

$$\begin{aligned} W^{2,p}(K) &\subset C^{1,1-d/p}(K), \quad p > d, \\ W^{2,d}(K) &\subset W^{1,q}(K) \subset C^{0,1-d/q}(K) \quad \forall\, q > d, \\ W^{2,p}(K) &\subset W^{1,dp/(d-p)}(K) \subset C^{0,2-d/p}(K), \quad \frac{d}{2} < p < d. \end{aligned}$$

If $d = 3$, we also have the continuous imbeddings

$$\begin{aligned} W^{3,3/2}(K) &\subset W^{2,3}(K) \subset W^{1,q}(K) \subset C^{0,1-3/q}(K) \quad \forall\, q > 3, \\ W^{3,p}(K) &\subset W^{2,3p/(3-p)}(K) \subset W^{1,3p/(3-2p)}(K) \subset C^{0,3-3/p}(K), \quad 1 < p < \frac{3}{2}. \end{aligned}$$

Although Morry's inequality may not be applied, the continuous imbedding $W^{d,1}(K) \subset C^0(K)$ $(d = 2, 3)$ still holds. For the imbedding theorem, see [1], [7], and [16]. In the sequel we always suppose that p is taken such that the imbedding $W^{k+1,p}(K) \subset C^0(K)$ holds, that is,

$$\begin{aligned} 1 \leqslant p \leqslant \infty, &\quad \text{if } d = 2,\ k+1 \geqslant 2 \text{ or } d = 3,\ k+1 \geqslant 3, \quad \text{and} \\ \frac{3}{2} < p \leqslant \infty, &\quad \text{if } d = 3,\ k+1 = 2. \end{aligned}$$

Note that our discussion includes the case $d = k+1$, $p = 1$ that is sometimes ignored in literature.

We define the subset $\mathcal{T}_p^k(K) \subset W^{k+1,p}(K)$ by

$$\mathcal{T}_p^k(K) := \{ v \in W^{k+1,p}(K) \mid v(\mathbf{x}) = 0 \ \forall\, \mathbf{x} \in \Sigma^k(K) \}.$$

Let $\mathcal{P}_k$ be the set of polynomials with two or three variables for which the degree is at most k. For a continuous function $v \in C^0(K)$, the kth-order Lagrange interpolation $\mathcal{I}_K^k v \in \mathcal{P}_k$ is defined by

$$v(\mathbf{x}) = (\mathcal{I}_K^k v)(\mathbf{x}) \quad \forall\, \mathbf{x} \in \Sigma^k(K).$$

From this definition, it is clear that

$$v - \mathcal{I}_K^k v \in \mathcal{T}_p^k(K) \quad \forall\, v \in W^{k+1,p}(K).$$

For an integer m such that $0 \leqslant m \leqslant k$, $B_p^{m,k}(K)$ is defined by

$$B_p^{m,k}(K) := \sup_{v \in \mathcal{T}_p^k(K)} \frac{|v|_{m,p,K}}{|v|_{k+1,p,K}}.$$

Note that we have

$$B_p^{m,k}(K) = \inf\{C \in \mathbb{R}^1 \mid |v - \mathcal{I}_K^k v|_{m,p,K} \leqslant C|v|_{k+1,p,K} \quad \forall\, v \in W^{k+1,p}(K)\},$$

that is, $B_p^{m,k}(K)$ is the *best* constant C for the error estimation

$$|v - \mathcal{I}_K^k v|_{m,p,K} \leqslant C|v|_{k+1,p,K} \quad \forall\, v \in W^{k+1,p}(K).$$

To establish the mathematical foundation of the finite element methods, we must show that $B_p^{m,p}(K)$ is bounded. Many textbooks on finite element methods, such as those by Ciarlet [8], Brenner-Scott [6], and Ern-Guermond [10], present the following theorem. Let h_K be the diameter of K and ϱ_K the radius of the inscribed ball of K.

Shape-regularity. Let $\sigma > 2$ be a constant. If $h_K/\varrho_K \leqslant \sigma$ and $h_K \leqslant 1$, then there exists a constant $C = C(\sigma)$ independent of h_K such that

$$\|v - \mathcal{I}_K^1 v\|_{1,2,K} \leqslant Ch_K|v|_{2,2,K} \quad \forall\, v \in H^2(K).$$

The maximum of the ratio h_K/ϱ_K in a triangulation is called the *chunkiness parameter* [6]. The shape regularity, however, is not necessarily needed to obtain an error estimate for triangles and tetrahedrons. For triangles, the following estimations are well-known [4], [5], [11].

The maximum angle condition. Let θ_1 ($\pi/3 \leqslant \theta_1 < \pi$) be a constant. If any angle θ of K satisfies $\theta \leqslant \theta_1$ and $h_K \leqslant 1$, then there exists a constant $C = C(\theta_1)$ independent of h_K such that

$$\|v - \mathcal{I}_K^1 v\|_{1,2,K} \leqslant Ch_K|v|_{2,2,K} \quad \forall\, v \in H^2(K). \tag{1.1}$$

Later, Křížek [14] introduced the *semiregularity condition* for triangles, which is equivalent to the maximum angle condition. Let R_K be the circumradius of K.

The semiregularity condition. Let $p > 1$ and $\sigma > 0$ be constants. If $R_K/h_K \leqslant \sigma$ and $h_K \leqslant 1$, then there exists a constant $C = C(\sigma)$ independent of h_K such that

$$\|v - \mathcal{I}_K^1 v\|_{1,p,K} \leqslant C h_K |v|_{2,p,K} \quad \forall v \in W^{2,p}(K). \tag{1.2}$$

For tetrahedrons, the following estimation is well-known [15], [9].

Křížek's maximum angle condition. Let θ_2 ($\pi/3 \leqslant \theta_2 < \pi$) be a constant. Let γ_k be the maximum angle of faces of a tetrahedron K and φ_K the maximum angle between faces of K. If $\gamma_k \leqslant \theta_2$, $\varphi_K \leqslant \theta_2$, and $h_K \leqslant 1$, then there exists a constant $C = C(\theta_2)$ independent of h_K such that

$$\|v - \mathcal{I}_K^1 v\|_{1,p,K} \leqslant C h_K |v|_{2,p,K} \quad \forall v \in W^{2,p}(K),\ 2 < p \leqslant \infty. \tag{1.3}$$

Jamet [11] presented a general result which covers both the triangles and tetrahedrons. Let $E_d := \{e_s\}_{s=1}^d \subset \mathbb{R}^d$ be a set of unit vectors which are linearly independent. Let $\xi \in \mathbb{R}^d$ be a unit vector and θ_s, $0 \leqslant \theta_s \leqslant \pi/2$ the angle between ξ and the line which is defined by e_s. Define

$$\theta(E_d) := \max_{\xi \in \mathbb{R}^d} \min_{e_s \in E_d} \{\theta_s\}.$$

Let $K \subset \mathbb{R}^d$ be a d-simplex. Let $N := d(d+1)/2$ and let E_N be the set of N unit vectors that are parallel to the edges of K. Define $\theta_K := \min_{E_d \subset E_N} \{\theta(E_d)\}$. Note that if $d = 2$ and K is an obtuse triangle, then $2\theta_K$ is the maximum angle of K.

Theorem 1.1 (Jamet). *Let $1 \leqslant p \leqslant \infty$. Let $m \geqslant 0$, $k \geqslant 1$ be integers such that $k + 1 - m > 2/p$ ($1 < p \leqslant \infty$) or $k - m \geqslant 1$ ($p = 1$) if $d = 2$, or $k + 1 - m > 3/p$ if $d = 3$. Then the following estimate holds:*

$$|v - \mathcal{I}_K^k v|_{m,p,K} \leqslant C \frac{h_K^{k+1-m}}{(\cos\theta_K)^m} |v|_{k+1,p,K} \quad \forall\, v \in W^{k+1,p}(K), \tag{1.4}$$

where C depends only on k, p.

Remark. Note that in [11], Théorème 3.1, the case $d = 2$ and $p = 1$ is not mentioned explicitly but clearly holds for triangles.

For further results of error estimations on "skinny elements", readers are referred to the monograph by Apel [2].

The common idea of the above mentined estimations is that (i) show an error estimate for a particular type of elements, then (ii) extend it for general elements by affine transformation. To prove the maximum angle condition for triangles, for example, Babuška and Aziz showed the following theorem (see [4], Lemma 2.2, Lemma 2.4). Let $\widehat{K}$ be the right triangle with vertices $(0,0)^\top$, $(1,0)^\top$, and $(0,1)^\top$, and K_α the right triangle with vertices $(0,0)^\top$, $(1,0)^\top$, and $(0,\alpha)^\top$ $(0 < \alpha \leqslant 1)$. That is, K_α is obtained by squeezing $\widehat{K}$.

Theorem 1.2 (Babuška-Aziz). *There exists a constant independent of α $(0 < \alpha \leqslant 1)$ such that $B_2^{m,1}(K_\alpha) \leqslant C$, $m = 0$, 1. As an immediate consequence, we obtain the error estimation of Lagrange interpolation $\mathcal{I}_K^1$ on a right triangle K; for $m = 0, 1$,*

$$|v - \mathcal{I}_K^1 v|_{m,2,K} \leqslant C h_K^{2-m} |v|_{2,2,K}.$$

Theorem 1.2 claims that squeezing a right isosceles triangle perpendicularly does not deteriorate the optimal approximation order of $\mathcal{I}_K^1$. Babuška and Aziz then claim that the estimate (1.1) for general triangular elements is obtained by affine transformations. Kobayashi and Tsuchiya [12] extended Theorem 1.2 to any p $(1 \leqslant p \leqslant \infty)$.

Now, let $\widehat{K}$ denote also the reference tetrahedron with vertices $(0,0,0)^\top$, $(1,0,0)^\top$, $(0,1,0)^\top$, and $(0,0,1)^\top$. Let $K_{\alpha\beta}$ be the "right" tetrahedron with vertices $(0,0,0)^\top$, $(1,0,0)^\top$, $(0,\alpha,0)^\top$, and $(0,0,\beta)^\top$ $(0 < \alpha, \beta \leqslant 1)$.

The aim of this paper is to extend Theorem 1.2 and establish the following theorem.

Theorem 1.3. *If $d = 2$, there exists a constant $C_{k,m,p}$ such that, for $m = 0, \dots, k$,*

$$B_p^{m,k}(K_\alpha) := \sup_{v \in \mathcal{T}_p^k(K_\alpha)} \frac{|v|_{m,p,K_\alpha}}{|v|_{k+1,p,K_\alpha}} \leqslant C_{k,m,p}, \quad k \geqslant 1,\ 1 \leqslant p \leqslant \infty. \tag{1.5}$$

If $d = 3$, there exists a constant $C_{k,m,p}$ such that, for $m = 0, \dots, k$,

$$B_p^{m,k}(K_{\alpha\beta}) := \sup_{v \in \mathcal{T}_p^k(K_{\alpha\beta})} \frac{|v|_{m,p,K_{\alpha\beta}}}{|v|_{k+1,p,K_{\alpha\beta}}} \leqslant C_{k,m,p}, \ \begin{cases} k - m = 0, & 2 < p \leqslant \infty, \\ k = 1,\ m = 0, & \frac{3}{2} < p \leqslant \infty, \\ k \geqslant 2,\ k - m \geqslant 1, & 1 \leqslant p \leqslant \infty. \end{cases} \tag{1.6}$$

Using Theorem 1.3 and affine transformations, we can derive an error estimation on general triangles. See Section 4.

The above mentioned estimations (1.1), (1.2), (1.3), (1.4) cover Theorem 1.3 partially. We also mention that Shenk [18] showed (1.5) for $p = 2$, $k \geqslant 1$, $m = 0$, 1, and (1.6) for $p = 2$, $k \geqslant 2$, $m = 0$, 1.

Because of the restrictions for m, k, and p in the above mentioned estimations, it seems that (1.5) with $k = m \geqslant 2$, $1 \leqslant p \leqslant 2$, and (1.6) with $k = m \geqslant 2$,

$1 \leqslant p \leqslant 3$ have not yet been proved. To prove Theorem 1.3, we introduce the difference quotients of functions with two or three variables in Section 2. Then, Theorem 1.3 is proved in Section 3 by a method that is a straightforward extension of Babuška-Aziz's original argument. The notation of functional spaces used in this paper are exactly the same as those in [13].

2. Difference quotients for multi-variable functions

In this section, we define the difference quotients for two- and three-variable functions. Our treatment is based on the theory of difference quotients for one-variable functions given in standard textbooks such as [3] and [19]. All statements in this section can be readily proved.

For a positive integer k, the set $\widehat{\Sigma}^k \subset \widehat{K}$ is defined by

$$\widehat{\Sigma}^k := \left\{ \mathbf{x}_\gamma := \frac{\gamma}{k} \in \widehat{K} \;\middle|\; \gamma \in \mathbb{N}_0^d,\ 0 \leqslant |\gamma| \leqslant k \right\},$$

where $\gamma/k = (a_1/k, \ldots, a_d/k)$ is understood as the coordinate of a point in $\widehat{\Sigma}^k$.

For $\mathbf{x}_\gamma \in \widehat{\Sigma}^k$ and a multi-index $\delta \in \mathbb{N}_0^d$ with $|\gamma| \leqslant k - |\delta|$, we define the correspondence Δ^δ between nodes by

$$\Delta^\delta \mathbf{x}_\gamma := \mathbf{x}_{\gamma+\delta} \in \widehat{\Sigma}^k.$$

For two multi-indexes $\eta = (m_1, \ldots, m_d)$, $\delta = (n_1, \ldots, n_d)$, $\eta \leqslant \delta$ means that $m_i \leqslant n_i$ $(i = 1, \ldots, d)$. Also, $\delta \cdot \eta$ and $\delta!$ are defined by $\delta \cdot \eta := \sum_{i=1}^{d} m_i n_i$ and $\delta! := n_1! \ldots n_d!$, respectively. Using Δ^δ, we define the *difference quotients* on $\widehat{\Sigma}^k$ for $f \in C^0(\widehat{K})$ by

$$f^{|\delta|}[\mathbf{x}_\gamma, \Delta^\delta \mathbf{x}_\gamma] := k^{|\delta|} \sum_{\eta \leqslant \delta} \frac{(-1)^{|\delta|-|\eta|}}{\eta!(\delta-\eta)!} f(\Delta^\eta \mathbf{x}_\gamma).$$

Let $\mathbf{0} := (0, \ldots, 0) \in \mathbb{N}_0^d$. For simplicity, we denote $f^{|\delta|}[\mathbf{x_0}, \Delta^\delta \mathbf{x_0}]$ by $f^{|\delta|}[\Delta^\delta \mathbf{x_0}]$. The following are examples of $f^{|\delta|}[\Delta^\delta \mathbf{x_0}]$: if $d = 2$,

$$\begin{aligned}
f^2[\Delta^{(2,0)} \mathbf{x}_{(0,0)}] &= \frac{k^2}{2}(f(\mathbf{x}_{(2,0)}) - 2f(\mathbf{x}_{(1,0)}) + f(\mathbf{x}_{(0,0)})), \\
f^2[\Delta^{(1,1)} \mathbf{x}_{(0,0)}] &= k^2(f(\mathbf{x}_{(1,1)}) - f(\mathbf{x}_{(1,0)}) - f(\mathbf{x}_{(0,1)}) + f(\mathbf{x}_{(0,0)})), \\
f^3[\Delta^{(2,1)} \mathbf{x}_{(0,0)}] &= \frac{k^3}{2}(f(\mathbf{x}_{(2,1)}) - 2f(\mathbf{x}_{(1,1)}) + f(\mathbf{x}_{(0,1)}) - f(\mathbf{x}_{(2,0)}) \\
&\qquad + 2f(\mathbf{x}_{(1,0)}) - f(\mathbf{x}_{(0,0)})),
\end{aligned}$$

and if $d = 3$,

$$\begin{aligned} f^4[\Delta^{(2,1,1)}\mathbf{x}_{(0,0,0)}] = \frac{k^4}{2}(&f(\mathbf{x}_{(2,1,1)}) - 2f(\mathbf{x}_{(1,1,1)}) + f(\mathbf{x}_{(0,1,1)}) \\ &- f(\mathbf{x}_{(2,0,1)}) + 2f(\mathbf{x}_{(1,0,1)}) - f(\mathbf{x}_{(0,0,1)}) \\ &- f(\mathbf{x}_{(2,1,0)}) + 2f(\mathbf{x}_{(1,1,0)}) - f(\mathbf{x}_{(0,1,0)}) \\ &+ f(\mathbf{x}_{(2,0,0)}) - 2f(\mathbf{x}_{(1,0,0)}) + f(\mathbf{x}_{(0,0,0)})). \end{aligned}$$

Let $\eta \in \mathbb{N}_0^d$ be such that $|\eta| = 1$. The difference quotients clearly satisfy the recursive relations

$$f^{|\delta|}[\mathbf{x}_\gamma, \Delta^\delta \mathbf{x}_\gamma] = \frac{k}{\delta \cdot \eta}(f^{|\delta|-1}[\mathbf{x}_{\gamma+\eta}, \Delta^{\delta-\eta}\mathbf{x}_{\gamma+\eta}] - f^{|\delta|-1}[\mathbf{x}_\gamma, \Delta^{\delta-\eta}\mathbf{x}_\gamma]).$$

If $f \in C^k(\widehat{K})$, the difference quotient $f^{|\delta|}[\mathbf{x}_\gamma, \Delta^\delta \mathbf{x}_\gamma]$ is written as an integral of f. Setting $d = 2$ and $\delta = (0, s)$, for example, we have

$$\begin{aligned} &f^1[\mathbf{x}_{(l,q)}, \Delta^{(0,1)}\mathbf{x}_{(l,q)}] = k(f(\mathbf{x}_{l,q+1}) - f(\mathbf{x}_{lq})) = \int_0^1 \partial_{x_2} f\Big(\frac{l}{k}, \frac{q}{k} + \frac{w_1}{k}\Big)\,\mathrm{d}w_1, \\ &f^s[\mathbf{x}_{(l,p)}, \Delta^{(0,s)}\mathbf{x}_{(l,q)}] \\ &\quad = \int_0^1\int_0^{w_1}\cdots\int_0^{w_{s-1}} \partial^{(0,s)} f\Big(\frac{l}{k}, \frac{q}{k} + \frac{1}{k}(w_1 + \ldots + w_s)\Big)\,\mathrm{d}w_s \ldots \mathrm{d}w_2\,\mathrm{d}w_1. \end{aligned}$$

To provide a concise expression for the above integral, we introduce the s-simplex

$$\mathbb{S}_s := \{(t_1, t_2, \ldots, t_s) \in \mathbb{R}^s \mid t_i \geqslant 0,\ 0 \leqslant t_1 + \ldots + t_s \leqslant 1\},$$

and the integral of $g \in L^1(\mathbb{S}_s)$ on $\mathbb{S}_s$ is defined by

$$\int_{\mathbb{S}_s} g(w_1, \ldots, w_k)\,\mathrm{d}\mathbf{W_s} := \int_0^1\int_0^{w_1}\cdots\int_0^{w_{s-1}} g(w_1, \ldots, w_s)\,\mathrm{d}w_s \ldots \mathrm{d}w_2\,\mathrm{d}w_1.$$

Let us temporarily set $d = 2$. Then $f^s[\mathbf{x}_{(l,q)}, \Delta^{(0,s)}\mathbf{x}_{(l,q)}]$ becomes

$$f^s[\mathbf{x}_{(l,q)}, \Delta^{(0,s)}\mathbf{x}_{(l,q)}] = \int_{\mathbb{S}_s} \partial^{(0,s)} f\Big(\frac{l}{k}, \frac{q}{k} + \frac{1}{k}(w_1 + \ldots + w_s)\Big)\,\mathrm{d}\mathbf{W_s}.$$

For a general multi-index (t, s) we have

$$\begin{aligned} &f^{t+s}[\mathbf{x}_{(l,q)}, \Delta^{(t,s)}\mathbf{x}_{(l,q)}] \\ &\quad = \int_{\mathbb{S}_s}\int_{\mathbb{S}_t} \partial^{(t,s)} f\Big(\frac{l}{k} + \frac{1}{k}(z_1 + \ldots + z_t), \frac{q}{k} + \frac{1}{k}(w_1 + \ldots + w_s)\Big)\,\mathrm{d}\mathbf{Z_t}\,\mathrm{d}\mathbf{W_s}. \end{aligned}$$

Let $\Box_\gamma^\delta$ be the rectangle defined by $\mathbf{x}_\gamma$ and $\Delta^\delta \mathbf{x}_\gamma$ as the diagonal points. If $\delta = (t, 0)$ or $(0, s)$, $\Box_\gamma^\delta$ degenerates to a segment. For $v \in L^1(\widehat{K})$ and $\Box_\gamma^\delta$ with $\gamma = (l, q)$, we denote the integral as

$$\int_{\Box_\gamma^{(t,s)}} v := \int_{\mathbb{S}_s} \int_{\mathbb{S}_t} v\Big(\frac{l}{k} + \frac{1}{k}(z_1 + \ldots + z_t), \frac{q}{k} + \frac{1}{k}(w_1 + \ldots + w_s)\Big) \, \mathrm{d}\mathbf{Z_t} \, \mathrm{d}\mathbf{W_s}.$$

If $\Box_\gamma^\delta$ degenerates to a segment, the integral is understood as an integral on the segment. By this notation, the difference quotient $f^{t+s}[\mathbf{x}_\gamma, \Delta^{(t,s)} \mathbf{x}_\gamma]$ is written as

$$f^{t+s}[\mathbf{x}_\gamma, \Delta^{(t,s)} \mathbf{x}_\gamma] = \int_{\Box_\gamma^{(t,s)}} \partial^{(t,s)} f.$$

Therefore, if $u \in \mathcal{T}_p^k(\widehat{K})$, then we have

$$0 = u^{t+s}[\mathbf{x}_\gamma, \Delta^{(t,s)} \mathbf{x}_\gamma] = \int_{\Box_\gamma^{(t,s)}} \partial^{(t,s)} u \quad \forall\ \Box_\gamma^{(t,s)} \subset \widehat{K}. \tag{2.1}$$

For the case $d = 3$, the integral $\int_{\Box_\gamma^\delta} v$ is defined in exactly the same manner.

3. Proof of Theorem 1.3

Let $S \subset \widehat{K}$ be a segment. In the proof of Theorem 1.3, the continuity of the trace operator t defined as $t\colon W^{1,p}(\widehat{K}) \ni v \mapsto v|_S \in L^1(S)$ is crucial. If $d = 2$, the continuity of t is standard and is mentioned in many textbooks such as [7]. For the case $d = 3$, the situation becomes a bit more complicated. If the continuous inclusion $W^{k+1,p}(\widehat{K}) \subset C^0(\widehat{K})$ holds, the continuity of t is obvious. Even if this is not the case, we still have the following lemma. For the proof, see [1], Theorem 4.12; [9], Lemma 2.2, and [17], Theorem 2.1.

Lemma 3.1. *Let $d = 3$ and let $S \subset \widehat{K}$ be an arbitrary segment. Then the following trace operators are well-defined and continuous:*

$$t\colon W^{1,p}(\widehat{K}) \to L^p(S), \quad 2 < p < \infty, \qquad t\colon W^{2,p}(\widehat{K}) \to L^p(S), \quad 1 \leqslant p < \infty.$$

For a multi-index δ, $|\delta| \geqslant 1$, p is taken such that

$$\begin{cases} 2 < p \leqslant \infty, & \text{if } k + 1 - |\delta| = 1,\ d = 3, \\ 1 \leqslant p \leqslant \infty, & \text{if } k + 1 - |\delta| \geqslant 2,\ d = 3 \text{ or } d = 2. \end{cases} \tag{3.1}$$

The set $\Xi_p^{\delta,k} \subset W^{k+1-|\delta|,p}(\widehat{K})$ is then defined by

$$\Xi_p^{\delta,k} := \left\{ v \in W^{k+1-|\delta|,p}(\widehat{K}) \;\middle|\; \int_{\square_{lp}^{\delta}} v = 0 \quad \forall\ \square_{lp}^{\delta} \subset \widehat{K} \right\}.$$

By Lemma 3.1 and (3.1), $\Xi_p^{\delta,k}$ is well-defined. Note that $u \in \mathcal{T}_p^k(\widehat{K})$ implies $\partial^\delta u \in \Xi_p^{\delta,k}$ by definition and (2.1).

Lemma 3.2. *We have* $\Xi_p^{\delta,k} \cap \mathcal{P}_{k-|\delta|} = \{0\}$. *That is, if* $q \in \mathcal{P}_{k-|\delta|}$ *belongs to* $\Xi_p^{\delta,k}$, *then* $q = 0$.

Proof. We notice that $\dim \mathcal{P}_{k-|\delta|} = \#\{\square_{lp}^{\delta} \subset \widehat{K}\}$. For example, if $k = 4$, $d = 2$, and $|\delta| = 2$, then $\dim \mathcal{P}_2 = 6$. This corresponds to the fact that, in $\widehat{K}$, there are six squares with size $1/4$ for $\delta = (1,1)$ and there are six horizontal segments of length $1/2$ for $\delta = (2,0)$. All their vertices (corners and end-points) belong to $\Sigma^4(\widehat{K})$ (see Figure 1). The situation is the same for $d = 3$. Now, suppose that $v \in \mathcal{P}_{k-|\delta|}$ satisfies $\int_{\square_{lp}^{\delta}} q = 0$ for all $\square_{lp}^{\delta} \subset \widehat{K}$. This condition is linearly independent and determines $q = 0$ uniquely. $\square$

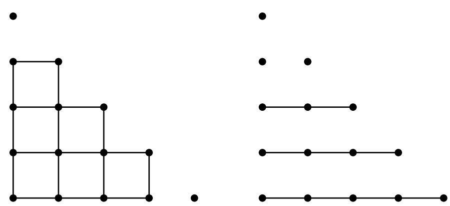

Figure 1. The six squares of size $1/4$ for $\delta = (1,1)$ and the (union of) six segments of length $1/2$ for $\delta = (2,0)$ in $\widehat{K}$.

The constant $A_p^{\delta,k}$ is defined by

$$A_p^{\delta,k} := \sup_{v \in \Xi_p^{\delta,k}} \frac{|v|_{0,p,\widehat{K}}}{|v|_{k+1-|\delta|,p,\widehat{K}}}.$$

The following lemma is an extension of [4], Lemma 2.1.

Lemma 3.3. *Let* p *be given by* (3.1). *Then we have* $A_p^{\delta,k} < \infty$.

Proof. The proof is by contradiction. Suppose that $A_p^{\delta,k} = \infty$. Then there exists a sequence $\{w_k\}_{i=1}^{\infty} \subset \Xi_p^{\delta,k}$ such that $|w_n|_{0,p,\widehat{K}} = 1$ and $\lim_{n\to\infty} |w_n|_{k+1-|\delta|,p,\widehat{K}} = 0$. By [8], Theorem 3.1.1, there exists $\{q_n\} \subset \mathcal{P}_{k-|\delta|}$ such that

$$\|w_n + q_n\|_{k+1-|\delta|,p,\widehat{K}} \leqslant \inf_{q \in \mathcal{P}_{k-|\delta|}} \|w_n + q\|_{k+1-|\delta|,p,\widehat{K}} + \frac{1}{n} \leqslant C|w_n|_{k+1-|\delta|,p,\widehat{K}} + \frac{1}{n}$$

and $\lim_{n\to\infty} \|w_n + q_n\|_{k+1-|\delta|,p,\widehat{K}} = 0$. Because $\{w_n\} \subset W^{k+1-|\delta|,p}(\widehat{K})$ is bounded, $\{q_n\} \subset \mathcal{P}_{k-|\delta|}$ is bounded as well. Hence, there exists a subsequence $\{q_{n_i}\}$ such that q_{n_i} converges to $\bar{q} \in \mathcal{P}_{k-|\delta|}$ and $\lim_{n_i\to\infty} \|w_{n_i} + \bar{q}\|_{k+1-|\delta|,p,\widehat{K}} = 0$. By definition and the continuity of the trace operator, we have $\int_{\square^{\delta}_{lp}} w_{n_i} = 0$ and

$$0 = \lim_{n_i\to\infty} \int_{\square^{\delta}_{lp}} (w_{n_i} + \bar{q}) = \int_{\square^{\delta}_{lp}} \bar{q} \quad \forall\, \square^{\delta}_{lp} \subset \widehat{K}.$$

Therefore, it follows from Lemma 3.1 that $\bar{q} = 0$. This implies that

$$0 = \lim_{n_i\to\infty} \|w_{n_i}\|_{k+1-|\delta|,p,\widehat{K}} \geqslant \lim_{n_i\to\infty} |w_{n_i}|_{0,p,\widehat{K}} = 1,$$

which is a contradiction. $\square$

Proof of Theorem 1.3. The proof is a direct extension of the proof given in [4], Lemma 2.2. Let $d = 2$ initially. Define the linear transformation $F_\alpha\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$(x^*, y^*)^\top = (x, \alpha y)^\top, \quad (x, y)^\top \in \mathbb{R}^2,\ 0 < \alpha \leqslant 1,$$

which squeezes the reference element $\widehat{K}$ perpendicularly to $K_\alpha := F_\alpha(\widehat{K})$. Take an arbitrary $v \in W^{k+1,p}(K_\alpha)$ and define $u \in W^{k+1,p}(\widehat{K})$ by $u(x,y) := v(x, \alpha y)$. To make the formula concise, we introduce the following notation. For a multi-index $\gamma = (a, b) \in \mathbb{N}_0^2$ and a real $t \neq 0$, $(\alpha)^{\gamma t} := \alpha^{bt}$. Let $1 \leqslant p < \infty$ and $1 \leqslant m \leqslant k$. Because $u \in \mathcal{T}^k_p(\widehat{K})$ and $\partial^\delta u \in \Xi^{\delta,k}_p$, we may apply Lemma 3.3 and obtain

$$\begin{aligned}
(3.2)\qquad \frac{|v|^p_{m,p,K_\alpha}}{|v|^p_{k+1,p,K_\alpha}} &= \frac{\sum\limits_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}|\partial^\gamma u|^p_{0,p,\widehat{K}}}{\sum\limits_{|\delta|=k+1} \frac{(k+1)!}{\delta!}(\alpha)^{-\delta p}|\partial^\delta u|^p_{0,p,\widehat{K}}} \\
&= \frac{\sum\limits_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}|\partial^\gamma u|^p_{0,p,\widehat{K}}}{\sum\limits_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}\Big(\sum\limits_{|\eta|=k+1-m} \frac{(k+1-m)!}{\eta!(\alpha)^{\eta p}}|\partial^\eta(\partial^\gamma u)|^p_{0,p,\widehat{K}}\Big)} \\
&\leqslant \frac{\sum\limits_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}|\partial^\gamma u|^p_{0,p,\widehat{K}}}{\sum\limits_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}\Big(\sum\limits_{|\eta|=k+1-m} \frac{(k+1-m)!}{\eta!}|\partial^\eta(\partial^\gamma u)|^p_{0,p,\widehat{K}}\Big)} \\
&= \frac{\sum\limits_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}|\partial^\gamma u|^p_{0,p,\widehat{K}}}{\sum\limits_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}|\partial^\gamma u|^p_{k+1-m,p,\widehat{K}}} \\
&\leqslant \frac{\sum\limits_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}|\partial^\gamma u|^p_{0,p,\widehat{K}}}{\sum\limits_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}(A^{\gamma,k}_p)^{-1}|\partial^\gamma u|^p_{0,p,\widehat{K}}} \leqslant C^p_{k,m,p},
\end{aligned}$$

where $C_{k,m,,p} := \max_{|\gamma|=m} A_p^{\gamma,k}$. Here, we use the equality

$$\frac{(k+1)!}{\delta!} = \sum_{\substack{\gamma+\eta=\delta \\ |\gamma|=m, |\eta|=k+1-m}} \frac{m!}{\gamma!} \frac{(k+1-m)!}{\eta!}.$$

Hence, we obtain (1.5) for this case. If $m=0$, we have

$$\begin{aligned} \frac{|v|^p_{0,p,K_\alpha}}{|v|^p_{k+1,p,K_\alpha}} &= \frac{|u|^p_{0,p,\widehat{K}}}{\sum\limits_{|\delta|=k+1} \frac{(k+1)!}{\delta!} (\alpha)^{-\delta p} |\partial^\delta u|^p_{0,p,\widehat{K}}} \\ &\leqslant \frac{|u|^p_{0,p,\widehat{K}}}{\sum\limits_{|\delta|=k+1} \frac{(k+1)!}{\delta!} |\partial^\delta u|^p_{0,p,\widehat{K}}} = \frac{|u|^p_{0,p,\widehat{K}}}{|u|^p_{k+1,p,\widehat{K}}} \leqslant B_k^{0,p}(\widehat{K})^p < \infty. \end{aligned} \tag{3.3}$$

Setting $p=\infty$ and $1 \leqslant m \leqslant k$, we have

$$\begin{aligned} \frac{|v|_{m,\infty,K_\alpha}}{|v|_{k+1,\infty,K_\alpha}} &= \frac{\max\limits_{|\gamma|=m} \{(\alpha)^{-\gamma} |\partial^\gamma u|_{0,\infty,\widehat{K}}\}}{\max\limits_{|\delta|=k+1} \{(\alpha)^{-\delta} |\partial^\delta u|_{0,\infty,\widehat{K}}\}} \\ &= \frac{\max\limits_{|\gamma|=m} \{(\alpha)^{-\gamma} |\partial^\gamma u|_{0,\infty,\widehat{K}}\}}{\max\limits_{|\gamma|=m} \{(\alpha)^{-\gamma} \max\limits_{|\eta|=k+1-m} \{(\alpha)^{-\eta} |\partial^\eta (\partial^\gamma u)|_{0,\infty,\widehat{K}}\}\}} \\ &\leqslant \frac{\max\limits_{|\gamma|=m} \{(\alpha)^{-\gamma} |\partial^\gamma u|_{0,\infty,\widehat{K}}\}}{\max\limits_{|\gamma|=m} \{(\alpha)^{-\gamma} \max\limits_{|\eta|=k+1-m} \{|\partial^\eta (\partial^\gamma u)|_{0,\infty,\widehat{K}}\}\}} \\ &= \frac{\max\limits_{|\gamma|=m} \{(\alpha)^{-\gamma} |\partial^\gamma u|_{0,\infty,\widehat{K}}\}}{\max\limits_{|\gamma|=m} \{(\alpha)^{-\gamma} |\partial^\gamma u|_{k+1-m,\infty,\widehat{K}}\}} \\ &\leqslant \frac{\max\limits_{|\gamma|=m} \{(\alpha)^{-\gamma} |\partial^\gamma u|_{0,\infty,\widehat{K}}\}}{\max\limits_{|\gamma|=m} \{(\alpha)^{-\gamma} (A_\infty^{\gamma,k})^{-1} |\partial^\gamma u|_{0,\infty,\widehat{K}}\}} \leqslant C_{k,m,\infty}, \end{aligned} \tag{3.4}$$

where $C_{k,m,\infty} := \max_{|\gamma|=m} A_\infty^{\gamma,k}$. Now, the case with $p=\infty$ and $m=0$ is obvious. Therefore, (1.5) is proved.

Next, let $d=3$ and repeat the above proof. Define the linear transformation $F_{\alpha\beta}\colon\ \mathbb{R}^3 \to \mathbb{R}^3$ by

$$(x^*, y^*, z^*)^\top = (x, \alpha y, \beta z)^\top, \quad (x,y,z)^\top \in \mathbb{R}^3,\ 0 < \alpha,\ \beta \leqslant 1,$$

which squeezes the reference tetrahedron $\widehat{K}$ perpendicularly to $K_{\alpha\beta} := F_{\alpha\beta}(\widehat{K})$. Take an arbitrary $v \in W^{k+1,p}(K_{\alpha\beta})$ and define $u \in W^{k+1,p}(\widehat{K})$ by $u(x,y,z) := v(x,\alpha y,\beta z)$. Let p be given by (3.1) with $m = |\delta|$. To make the formula concise, we introduce the following notation. For a multi-index $\gamma = (a,b,c) \in \mathbb{N}_0^3$ and a real $t \neq 0$, $(\alpha,\beta)^{\gamma t} := \alpha^{bt}\beta^{ct}$. Because $u \in \mathcal{T}_p^k(\widehat{K})$ and $\partial^\delta u \in \Xi_p^{\delta,k}$, we may apply Lemma 3.3 as above. Thus, we may repeat (3.2), (3.3), and (3.4) replacing K_α by $K_{\alpha\beta}$, $(\alpha)^{\gamma p}$ by $(\alpha,\beta)^{\gamma p}$, etc. Thus, (1.6) is proved. $\square$

4. Concluding remarks

Theorem 1.3 deals only with right triangles and "right" tetrahedrons. Based on Theorem 1.3, a new error estimation of Lagrange interpolation on triangles is obtained in [13]. It should be emphasized that no geometric condition on triangles is imposed in Theorem 4.1.

Theorem 4.1 (Kobayashi-Tsuchiya [13]). *Let K be an arbitrary triangle. Let $1 \leqslant p \leqslant \infty$, and let k, m be integers such that $k \geqslant 1$ and $0 \leqslant m \leqslant k$. Then, for the kth-order Lagrange interpolation $\mathcal{I}_K^k$ on K, the following estimation holds:*

$$(4.1)\qquad |v - \mathcal{I}_K^k v|_{m,p,K} \leqslant C\Big(\frac{R_K}{h_K}\Big)^m h_K^{k+1-m}|v|_{k+1,p,K} = CR_K^m h_K^{k+1-2m}|v|_{k+1,p,K}$$

for any $v \in W^{k+1,p}(K)$, where the constant C depends only on k, p and is independent of the geometry of K.

Any tetrahedron can be obtained from a "skinny right" tetrahedron by an affine transformation. To obtain an error estimate, we need to estimate the ratio of the maximum and minimum singular values of the Jacobian matrix of the affine transformation. If we obtained an expression of the ratio in terms of geometric quantities of the tetrahedron, a new error estimation will be obtained. The authors hope that they will report further development of error estimations on tetrahedrons in near future.

References

[1] *R. A. Adams, J. J. F. Fournier*: Sobolev Spaces. Pure and Applied Mathematics 140, Academic Press, New York, 2003.

[2] *T. Apel*: Anisotropic Finite Elements: Local Estimates and Applications. Advances in Numerical Mathematics, Teubner, Stuttgart, 1999.

[3] *K. E. Atkinson*: An Introduction to Numerical Analysis. John Wiley & Sons, New York, 1989.

[4] *I. Babuška, A. K. Aziz*: On the angle condition in the finite element method. SIAM J. Numer. Anal. *13* (1976), 214–226.
[5] *R. E. Barnhill, J. A. Gregory*: Sard kernel theorems on triangular domains with application to finite element error bounds. Numer. Math. *25* (1976), 215–229.
[6] *S. C. Brenner, L. R. Scott*: The Mathematical Theory of Finite Element Methods. Texts in Applied Mathematics 15, Springer, New York, 2008.
[7] *H. Brezis*: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, Springer, New York, 2011.
[8] *P. G. Ciarlet*: The Finite Element Method for Elliptic Problems. Repr., unabridged republ. of the 1978 orig. Classics in Applied Mathematics 40, SIAM, Philadelphia, 2002.
[9] *R. G. Durán*: Error estimates for 3-d narrow finite elements. Math. Comput. *68* (1999), 187–199.
[10] *A. Ern, J.-L. Guermond*: Theory and Practice of Finite Elements. Applied Mathematical Sciences 159, Springer, New York, 2004.
[11] *P. Jamet*: Estimations d'erreur pour des éléments finis droits presque dégénérés. Rev. Franc. Automat. Inform. Rech. Operat. 10, Analyse numer., R-1 *10* (1976), 43–60. (In French.)
[12] *K. Kobayashi, T. Tsuchiya*: A Babuška-Aziz type proof of the circumradius condition. Japan J. Ind. Appl. Math. *31* (2014), 193–210.
[13] *K. Kobayashi, T. Tsuchiya*: A priori error estimates for Lagrange interpolation on triangles. Appl. Math., Praha *60* (2015), 485–499.
[14] *M. Křížek*: On semiregular families of triangulations and linear interpolation. Appl. Math., Praha *36* (1991), 223–232.
[15] *M. Křížek*: On the maximum angle condition for linear tetrahedral elements. SIAM J. Numer. Anal. *29* (1992), 513–520.
[16] *A. Kufner, O. John, S. Fučík*: Function Spaces. Monographs and Textsbooks on Mechanics of Solids and Fluids, Noordhoff International Publishing, Leyden; Publishing House of the Czechoslovak Academy of Sciences, Prague, 1977.
[17] *O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva*: Linear and Quasilinear Equations of Parabolic Type. Translated from Russian original. Translations of Mathematical Monographs 23, AMS, Providence, 1968.
[18] *N. A. Shenk*: Uniform error estimates for certain narrow Lagrange finite elements. Math. Comput. *63* (1994), 105–119.
[19] *T. Yamamoto*: Introduction to Numerical Analysis. Saiensu-sha, 2003. (In Japanese.)

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