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# HOCHSCHILD (CO)HOMOLOGY OF YONEDA ALGEBRAS OF RECONSTRUCTION ALGEBRAS OF TYPE $\mathbf{A}_1$

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*Abstract.* The reconstruction algebra is a generalization of the preprojective algebra, and plays important roles in algebraic geometry and commutative algebra. We consider the homological property of this class of algebras by calculating the Hochschild homology and Hochschild cohomology. Let $\Lambda_t$ be the Yoneda algebra of a reconstruction algebra of type $\mathbf{A}_1$ over a field $\Bbbk$. In this paper, a minimal projective bimodule resolution of $\Lambda_t$ is constructed, and the $\Bbbk$-dimensions of all Hochschild homology and cohomology groups of $\Lambda_t$ are calculated explicitly.

*Keywords*: Hochschild cohomology; reconstruction algebra; Yoneda algebra

*MSC 2010*: 16E40, 16G10

## 1. Introduction

It is well known that the preprojective algebra corresponding to the McKay quiver for $G \subseteq \mathrm{SL}(2,\mathbb{C})$ encodes the process of resolving the Gorenstein Kleinian singularity $\mathbb{C}[x,y]^G$, where $\mathbb{C}$ denotes the complex number field (see [4]). For some special finite subgroups $G \subseteq \mathrm{GL}(2,\mathbb{C})$, Iyama and Wemyss introduced the reconstruction algebra and generalized Brieskorn's results (see [17], [24], [23], [22], [21]). Thus we can view reconstruction algebras as a natural geometric generalization of preprojective algebras of extended Dynkin diagrams.

The reconstruction algebra plays important roles in algebraic geometry and commutative algebra. In the papers [24], [23], [22], [21], it is shown that the moduli space of finite dimensional representations of a reconstruction algebra in types $\mathbf{A}$ and $\mathbf{D}$ corresponding to $G \subseteq \mathrm{GL}(2,\mathbb{C})$ contains enough information to construct the

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minimal resolution of $\mathbb{C}^2/G$. Moreover, reconstruction algebras of type $\mathbf{A}$ and $\mathbf{D}$ are isomorphic to the endomorphism ring of the special Cohen-Macaulay modules in the sense of [25]. But we know little about the homology property of reconstruction algebras. In this paper, we consider the homology property of reconstruction algebras of type $\mathbf{A}_1$ by considering the Hochschild homology and cohomology of their Yoneda algebras.

The Hochschild cohomologies of a finite-dimensional algebra are subtle invariants of associative algebras and have played a fundamental role in representation theory. Let $\Lambda$ be a finite-dimensional algebra (associative with unity) over a field $\Bbbk$. Denote by $\Lambda^e := \Lambda \otimes_{\Bbbk} \Lambda^{\mathrm{op}}$ the enveloping algebra of $\Lambda$. Then the $i$-th Hochschild homology and cohomology of $\Lambda$ are identified with the $\Bbbk$-spaces (see [7]):

$$HH_i(\Lambda) = \mathrm{Tor}_i^{\Lambda^e}(\Lambda, \Lambda) \quad \text{and} \quad HH^i(\Lambda) = \mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda).$$

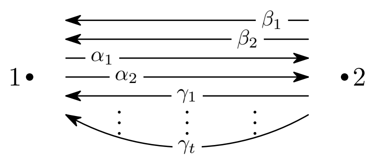
Hochschild homology is closely related to the oriented cycle and the global dimension of algebras [1], [13], [16]; Hochschild cohomology is closely related to simple connectedness, separability and deformation theory [8], [9], [14], [19].

In this paper, we first construct a minimal projective resolution of $\Lambda_t$ in Section 2. Using this minimal projective resolution, in Section 3 we calculate all the $\Bbbk$-dimensions of Hochschild homology groups and cyclic homology groups (in case $\operatorname{char} \Bbbk = 0$) of $\Lambda_t$. In the last section, we calculate all the $\Bbbk$-dimensions of Hochschild cohomology groups of $\Lambda_t$.

## 2. Minimal projective bimodule resolution

Throughout this paper, we fix a field $\Bbbk$ and set $\otimes := \otimes_{\Bbbk}$. In this section, we give the Yoneda algebras $\Lambda_t$ of the reconstruction algebras of type $\mathbf{A}_1$ by quivers with relations, and construct a minimal projective bimodule resolution of $\Lambda_t$.

For a given integer $t \geqslant 0$, recall that the reconstruction algebra $A_t$ of type $\mathbf{A}_1$ (see [23]) is given by the quiver $Q$:

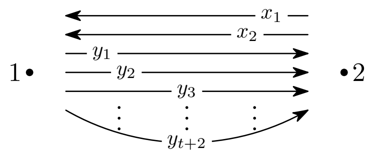



with relations

$$\begin{aligned} R := \{&\alpha_1\beta_2 - \alpha_2\beta_1, \beta_2\alpha_1 - \beta_1\alpha_2, \alpha_1\gamma_1 - \alpha_2\beta_2, \gamma_1\alpha_1 - \beta_2\alpha_2\} \\ &\cup \{\alpha_1\gamma_{i+1} - \alpha_2\gamma_i, \gamma_{i+1}\alpha_1 - \gamma_i\alpha_2\,;\ 1 \leqslant i \leqslant t-1\}, \end{aligned}$$

where we write a composition of paths from right to left. Clearly, $A_t$ are quadratic algebras. In particular, $A_0$ is just the preprojective algebra of type $\widetilde{\mathbf{A}}_1$. The Hochschild homology and cohomology of the Yoneda algebra of $A_0$ is considered in [15], [20]. In this paper, we consider the $t \geqslant 1$ case.

Now define the algebra $\Lambda_t := \Bbbk Q^t/I_t$, where the quiver $Q^t = (Q_0^t, Q_1^t)$ is given by




and the ideal $I_t$ of the path $\Bbbk Q^t$ is generated by

$$R_t := \{x_1y_1, y_1x_1, y_{t+2}x_2, x_2y_{t+2}\} \cup \{x_1y_{i+1} + x_2y_i, y_{i+1}x_1 + y_ix_2\,;\ 1 \leqslant i \leqslant t+1\}.$$

Denote by $e_1$ and $e_2$ the primitive orthogonal idempotents corresponding to points 1 and 2, respectively. Under the right length-lexicographic order, we can order all the paths in $Q^t$ by setting $e_1 \prec e_2 \prec x_1 \prec x_2 \prec y_1 \prec y_2 \prec \ldots \prec y_{t+2}$. Then it is easy to see that the set $R_t$ is just a (noncommutative) quadratic Gröbner basis of $I_t$ (see [10]). Therefore, $\Lambda_t$, $t \geqslant 0$ are Koszul algebras [12]. Note that the reconstruction algebra $A_t$ is just the Koszul dual of $\Lambda_t$; we obtain that the Yoneda algebra $E(A_t)$ is isomorphic to the algebra $\Lambda_t$ (see [2], Theorem 2.10.1).

Given an algebra $\Lambda_t$ for some fixed $t \geqslant 1$, we let

$$\begin{aligned}\mathcal{B} := {} & \{e_1, e_2, x_1, x_2\} \cup \{y_i\,;\ 1 \leqslant i \leqslant t+2\} \cup \{y_ix_1, x_1y_i\,;\ 2 \leqslant i \leqslant t+2\} \\ & \cup \{x_1y_ix_1\,;\ 3 \leqslant i \leqslant t+2\}.\end{aligned}$$

It is easy to see that $\mathcal{B}$ is an ordered $\Bbbk$-basis of $\Lambda_t$, so that $\dim_{\Bbbk} \Lambda_t = 4t+8$.

Since $\Lambda_t$ is Koszul, we construct a minimal projective bimodule resolution for the algebra $\Lambda_t$ using the approach of [11]. First, set

$$\begin{aligned}F^0 &:= \{f_1^{(0,1)} = e_1,\ f_2^{(0,2)} = e_2\};\\ F^1 &:= \{f_1^{(1,2)} = x_1,\ f_2^{(1,2)} = x_2\} \cup \{f_m^{(1,1)} = y_m\,;\ 1 \leqslant m \leqslant t+2\}.\end{aligned}$$

For $n \geqslant 2$, we define inductively the set $F^n$:
if $n$ is odd, $F^n := \{f_m^{(n,1)}\,;\ n \leqslant m \leqslant \frac{1}{2}(n+1)t+2n\} \cup \{f_m^{(n,2)}\,;\ n \leqslant m \leqslant \frac{1}{2}(n-1)t+2n\}$, where

$$f_m^{(n,1)} = \sum_{1 \leqslant i \leqslant m-n+1} y_i f_{m-i}^{(n-1,1)}, \quad f_m^{(n,2)} = \sum_{1 \leqslant j \leqslant 2} x_j f_{m-j}^{(n-1,2)};$$

if $n$ is even, $F^n := \{f_m^{(n,1)}, f_m^{(n,2)};\ n \leqslant m \leqslant \frac{1}{2}nt + 2n\}$, where

$$f_m^{(n,1)} = \sum_{1\leqslant j\leqslant 2} x_j f_{m-j}^{(n-1,1)}, \quad f_m^{(n,2)} = \sum_{1\leqslant i\leqslant m-n+1} y_i f_{m-i}^{(n-1,2)}.$$

It is not difficult to see that $|F^n| = n(t+2)+2$, and

$$f_m^{(n,1)} = \sum_{1\leqslant i\leqslant m-n+1} f_{m-i}^{(n-1,2)} y_i,\ f_m^{(n,2)} = \sum_{1\leqslant j\leqslant 2} f_{m-j}^{(n-1,1)} x_j,$$

where for $l = 1, 2$, $f_m^{(n,l)} = 0$ if $m < n$, or $n$ is odd, $l = 1$ and $m > \frac{1}{2}(n+1)t + 2n$, or $n$ is odd, $l = 2$ and $m > \frac{1}{2}(n-1)t + 2n$, or $n$ is even and $m > \frac{1}{2}nt + 2n$.

For any path $p \in Q^t$, we denote by $\mathfrak{o}(p)$ and $\mathfrak{t}(p)$ the origin and terminus of $p$, respectively. Recall that a non-zero element $x = \sum_{i=1}^{s} a_i p_i \in \Bbbk Q^t$, where $a_i \in \Bbbk$ and $p_i$ is a path in $Q^t$, is said to be uniform if there exist vertices $u, v \in Q_0^t$ such that $\mathfrak{o}(p_i) = u$ and $\mathfrak{t}(p_i) = v$ for all paths $p_i$. It is easy to see that the elements $f_m^{(n,l)}$, $l = 1, 2$ are uniform. Thus for each $f \in F^n$, we denote by $\mathfrak{o}(f)$ and $\mathfrak{t}(f)$, respectively, the common origin and terminus of all the paths occurring in $f$.

We now let

$$P_n := \bigoplus_{f\in F^n} \Lambda_t \mathfrak{o}(f) \otimes \mathfrak{t}(f)\Lambda_t.$$

Define $d_1\colon\ P_1 \to P_0$ by

$$d_1(\mathfrak{o}(f) \otimes \mathfrak{t}(f)) = f \otimes \mathfrak{t}(f) - \mathfrak{o}(f) \otimes f,$$

for $f \in F^1$. Whenever $n \geqslant 2$, the differential $d_n\colon\ P_n \to P_{n-1}$ is given by:
if $n$ is odd,

$$d_n(\mathfrak{o}(f_m^{(n,1)}) \otimes \mathfrak{t}(f_m^{(n,1)})) = \sum_{1\leqslant i\leqslant m-n+1} (y_i \otimes \mathfrak{t}(f_{m-i}^{(n-1,1)}) - \mathfrak{o}(f_{m-i}^{(n-1,2)}) \otimes y_i),$$
$$d_n(\mathfrak{o}(f_m^{(n,2)}) \otimes \mathfrak{t}(f_m^{(n,2)})) = \sum_{1\leqslant j\leqslant 2} (x_j \otimes \mathfrak{t}(f_{m-j}^{(n-1,2)}) - \mathfrak{o}(f_{m-j}^{(n-1,1)}) \otimes x_j);$$

if $n$ is even,

$$d_n(\mathfrak{o}(f_m^{(n,1)}) \otimes \mathfrak{t}(f_m^{(n,1)})) = \sum_{1\leqslant j\leqslant 2} x_j \otimes \mathfrak{t}(f_{m-j}^{(n-1,1)}) + \sum_{1\leqslant i\leqslant m-n+1} \mathfrak{t}(f_{m-i}^{(n-1,2)}) \otimes y_i,$$
$$d_n(\mathfrak{o}(f_m^{(n,2)}) \otimes \mathfrak{t}(f_m^{(n,2)})) = \sum_{1\leqslant i\leqslant m-n+1} y_i \otimes \mathfrak{t}(f_{m-i}^{(n-1,2)}) + \sum_{1\leqslant j\leqslant 2} \mathfrak{o}(f_{m-j}^{(n-1,1)}) \otimes x_j.$$

**Proposition 2.1.** *The complex* $\mathbb{P} = (P_n, d_n)$*:*

$$\ldots \to P_{n+1} \xrightarrow{d_{n+1}} P_n \to \ldots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} \Lambda_t \to 0$$

*is a minimal projective bimodule resolution of* $\Lambda_t$*, where* $\pi$ *is the multiplication map.*

P r o o f. Now we consider the minimal projective bimodule resolution of $\Lambda_t$ constructed in [6], Section 9. Let $X = \{x_0, x_1\} \cup \{y_i;\ 1 \leqslant i \leqslant t+2\}$. Since $\Lambda_t$ is a Koszul algebra, we only need to prove that $F^n$ is a $\Bbbk$-basis of the $\Bbbk$-vector space $K_n := \bigcap_{s+r=n-2} X^s R_t X^r$.

Note that $XK_{n-1} \cap K_{n-1}X \subset K_n$ for all $n$, $m$, and for $j = 1, 2$ we have $f_m^{(n,j)} \in K_n$ by induction on $n$. Denote by $I_t^{\perp}$ the ideal of $\Bbbk(Q^t)^{\mathrm{op}}$ generated by

$$R_t^{\perp} =: \{x_1^{\mathrm{op}} y_{i+1}^{\mathrm{op}} - x_2^{\mathrm{op}} y_i^{\mathrm{op}},\ y_{i+1}^{\mathrm{op}} x_1^{\mathrm{op}} - y_i^{\mathrm{op}} x_2^{\mathrm{op}};\ 1 \leqslant i \leqslant t+1\}.$$

Then $A_t \cong \Bbbk(Q^t)^{\mathrm{op}}/I_t^{\perp}$ is isomorphic to the Yoneda algebra $E(\Lambda_t)$ of $\Lambda_t$, since $\Lambda_t$ is Koszul (cf. [2], Theorem 2.10.1). Therefore, the Betti numbers of a minimal projective resolution of $\Lambda_t$ over $\Lambda_t^e$ are $\dim_{\Bbbk} K_n = n(t+2)+2$. Moreover, since $F^n$ is $\Bbbk$-linearly independent, $F^n$ is a $\Bbbk$-basis of $K_n$.

Finally, by [6], Section 9, and [11], we get that the differential $d_n$ is given as above. $\square$

## 3. Hochschild homology and cyclic homology

In this section, we calculate all the $\Bbbk$-dimensions of Hochschild homology groups and cyclic homology groups (in case $\operatorname{char} \Bbbk = 0$) of $\Lambda_t$ by transforming the Hochschild homology complex to the complex of closed paths.

Let $X$ and $Y$ be the sets of uniform elements in $\Bbbk Q^t$. Then one defines

$$X \odot Y = \{(p,q) \in X \times Y\colon\ \mathfrak{t}(p) = \mathfrak{o}(q) \text{ and } \mathfrak{t}(q) = \mathfrak{o}(p)\}$$

and denotes by $\Bbbk(X \odot Y)$ the vector space spanned by the elements in $X \odot Y$. A pair of uniform elements $(p,q)$ in $\Bbbk Q^t \times \Bbbk Q^t$ is called closed if $(p,q) \in \Bbbk Q^t \odot \Bbbk Q^t$.

Considering the sets $\mathcal{B} \odot F^n$, we have

$$\mathcal{B} \odot F^0 = \{(e_1,e_1),(e_2,e_2)\} \cup \{(y_{i+1}x_1, e_2),(x_2y_i, e_1);\ 1 \leqslant i \leqslant t+1\}.$$

Whenever $n \geqslant 1$, if $n$ is odd, then

$$\begin{aligned}\mathcal{B} \odot F^n = &\Big\{(x_j, f_m^{(n,1)});\ j = 1,2,\ n \leqslant m \leqslant \frac{n+1}{2}t + 2n\Big\} \\ &\cup \Big\{(y_i, f_m^{(n,2)});\ 1 \leqslant i \leqslant t+2,\ n \leqslant m \leqslant \frac{n-1}{2}t + 2n\Big\} \\ &\cup \Big\{(x_1 y_i x_1, f_m^{(n,1)});\ 3 \leqslant i \leqslant t+2,\ n \leqslant m \leqslant \frac{n+1}{2}t + 2n\Big\};\end{aligned}$$

if $n$ is even,

$$\begin{aligned}\mathcal{B} \odot F^n = &\Big\{(e_j, f_m^{(n,j)});\ j = 1,2,\ n \leqslant m \leqslant \frac{n}{2}t + 2n\Big\} \\ &\cup \Big\{(y_i x_1, f_m^{(n,2)}), (x_1 y_i, f_m^{(n,1)});\ 2 \leqslant i \leqslant t+2,\ n \leqslant m \leqslant \frac{n}{2}t + 2n\Big\}.\end{aligned}$$

Thus, $|\mathcal{B} \odot F^n| = (t+2)(nt + 2n + 2)$. Applying the functor $\Lambda_t \otimes_{\Lambda_t^e}$ to the minimal projective bimodule resolution $\mathbb{P} = (P_n, d_n)$, we get a Hochschild homology complex of the algebra $\Lambda_t$. Now, we use the vector spaces $\Bbbk(\mathcal{B} \odot F^n)$ to give a presentation of this Hochschild homology complex.

**Lemma 3.1.** $\Lambda_t \otimes_{\Lambda_t^e} \mathbb{P} = \mathbb{N}$, *where the complex* $\mathbb{N} = (N_n, \tau_n)$, $N_n \cong \Bbbk(\mathcal{B} \odot F^n)$ *and the differential* $\tau_n\colon N_n \to N_{n-1}$ *is given by: for any* $(b, f_m^{(n,1)})$, $(b, f_m^{(n,2)})$ *in* $\Bbbk(\mathcal{B} \odot F^n)$, *if $n$ is odd,*

$$\begin{aligned}\tau_n(b, f_m^{(n,1)}) &= \sum_{1 \leqslant i \leqslant m-n+1} ((by_i, f_{m-i}^{(n-1,1)}) - (y_i b, f_{m-i}^{(n-1,2)})), \\ \tau_n(b, f_m^{(n,2)}) &= \sum_{1 \leqslant j \leqslant 2} ((bx_j, f_{m-j}^{(n-1,2)}) - (x_j b, f_{m-j}^{(n-1,1)}));\end{aligned}$$

*if $n$ is even,*

$$\begin{aligned}\tau_n(b, f_m^{(n,1)}) &= \sum_{1 \leqslant j \leqslant 2} (bx_j, f_{m-j}^{(n-1,1)}) + \sum_{1 \leqslant i \leqslant m-n+1} (y_i b, f_{m-i}^{(n-1,2)}), \\ \tau_n(b, f_m^{(n,2)}) &= \sum_{1 \leqslant j \leqslant 2} (x_j b, f_{m-j}^{(n-1,1)}) + \sum_{1 \leqslant i \leqslant m-n+1} (by_i, f_{m-i}^{(n-1,2)}).\end{aligned}$$

Proof. Let $E$ be the maximal semisimple subalgebra of $\Lambda_t$. Then one can check that

$$N_n = \Lambda_t \otimes_{\Lambda_t^e} P_n = \Lambda_t \otimes_{E^e} \bigoplus_{f \in F^n} (\mathfrak{o}(f) \otimes_{\Bbbk} \mathfrak{t}(f)) \cong \bigoplus_{\alpha,\beta \in \{e_1, e_2\}} \alpha \Lambda_t \beta \otimes_{\Bbbk} \beta F^n \alpha.$$

Thus $N_n \cong \Bbbk(\mathcal{B} \odot F^n)$ as $\Bbbk$-vector spaces. Moreover, from the isomorphisms above, we have the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Lambda_t \otimes_{\Lambda_t^e} P_n \arrow[r, "\mathbf{1}\otimes d_n"] \arrow[d, "\cong"'] & \Lambda_t \otimes_{\Lambda_t^e} P_{n-1} \arrow[r] \arrow[d, "\cong"] & \cdots \\
\cdots \arrow[r] & \Bbbk(\mathcal{B} \odot F^n) \arrow[r, "\tau_n"] & \Bbbk(\mathcal{B} \odot F^{n-1}) \arrow[r] & \cdots
\end{tikzcd}$$

So the differentials $\tau_n$ can be induced by $d_n$ in $\mathbb{P}$. $\square$

Thus, by the definition, $HH_n(\Lambda_t) = \operatorname{Ker}\tau_n / \operatorname{Im}\tau_{n+1}$, and so we have

$$\begin{aligned}
\dim_\Bbbk HH_n(\Lambda_t) &= \dim_\Bbbk \operatorname{Ker}\tau_n - \dim_\Bbbk \operatorname{Im}\tau_{n+1} \\
&= \dim_\Bbbk N_n - \dim_\Bbbk \operatorname{Im}\tau_n - \dim_\Bbbk \operatorname{Im}\tau_{n+1}.
\end{aligned}$$

Consequently, to calculate the dimensions of Hochschild homology groups of $\Lambda_t$, we only need to determine $\dim_\Bbbk \operatorname{Im}\tau_n$ for all $n \geqslant 0$, since $\dim_\Bbbk N_n = |\mathcal{B} \odot F^n|$. To determine $\dim_\Bbbk \operatorname{Im}\tau_n$, we first define an order on $\mathcal{B} \odot F^n$ by the right length-lexicographic order on $\mathcal{B}$ and the relation:

$$(b, f_m^{(n,i)}) \prec (b', f_{m'}^{(n,i')}) \quad \text{if } b \prec b', \text{ or } b = b' \text{ but } m < m',$$

for any $(b, f_m^{(n,i)})$, $(b', f_{m'}^{(n,i')}) \in \mathcal{B} \odot F^n$. Next, we will give the matrix of $\tau_n$ under the ordered bases defined above, and show the $\Bbbk$-dimension of $\tau_n$ by this matrix.

We still denote by $\tau_n$ the matrix of the differentials $\tau_n$ under the ordered bases above, and write $\mathbf{I}_n$ for the $n \times n$ identity matrix for any positive integer $n$. For any matrix $T$, we denote by $T^n$ and ${}^nT$ the matrices constructed from $T$ by adding $n$ columns of zeros on the right and left, respectively. Then, from the descriptions of the differentials $\tau_n$ in Lemma 3.1, we obtain:

(1) If $n$ is odd, $\tau_n = \begin{pmatrix} 0 & 0 & 0 \\ A & B & 0 \\ -A & -B & 0 \end{pmatrix}_{(t+2)(nt-t+2n)\times(t+2)(nt+2n+2)}$, where

$$A = \begin{pmatrix} -{}^1X^t & X^{t+1} \\ -{}^2X^{t-1} & X^t \\ \vdots & \vdots \\ -{}^{t+1}X & X^1 \end{pmatrix}, \quad B = \begin{pmatrix} -{}^1X & X^1 & 0 & \dots & 0 & 0 \\ 0 & -{}^1X & X^1 & & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & & -{}^1X & X^1 \end{pmatrix}$$

are $(t+1)(\frac{1}{2}(n-1)t+n) \times (t+2)(n+1)$ matrices, $X = \mathbf{I}_{(n-1)t/2+n}$.

(2) If $n$ is even, $\tau_n = \begin{pmatrix} C & 0 & 0 \\ 0 & D & D \end{pmatrix}_{(t+2)(nt-t+2n)\times(t+2)(nt+2n+2)}$, where

$$C = \begin{pmatrix} Y^1 & Y^1 \\ {}^1Y & {}^1Y \\ Z^{t+1} & Z^{t+1} \\ {}^1Z^t & {}^1Z^t \\ \vdots & \vdots \\ {}^{t+1}Z & {}^{t+1}Z \end{pmatrix}_{(t+2)(nt/2-t+2n)\times(nt+2n+2)},$$

$Y = \mathbf{I}_{(n/2)t+n}$, $Z = \mathbf{I}_{nt/2-t+n}$, and

$$D = \begin{pmatrix} -{}^1Y & Y^1 & 0 & \dots & 0 & 0 \\ 0 & -{}^1Y & Y^1 & & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & & \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & 0 & & -{}^1Y & Y^1 \end{pmatrix}_{t(nt/2+n)\times(t+1)(nt/2+n+1)}.$$

Therefore, we get the following results.

**Lemma 3.2.** *For the map $\tau_n$, $n \geqslant 1$, we have*

$$\operatorname{rank}\tau_n = \begin{cases} \dfrac{n-1}{2}t^2 + \dfrac{3n-1}{2}t + n, & \text{if } n \text{ is odd;} \\ \dfrac{n}{2}t^2 + \dfrac{3n}{2}t + n + 1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. If $n$ is odd, then $\operatorname{rank}\tau_n = \operatorname{rank}(\,A \;\; B\,)$. By elementary transformations, one can check that $\operatorname{rank}\tau_n = \operatorname{rank}(\,A \;\; B\,) = (t+1)((n-1)t/2+n)$.

If $n$ is even, then $\operatorname{rank}\tau_n = \operatorname{rank} C + \operatorname{rank} D$. It is easy to see that $\operatorname{rank} D = t(nt/2+n)$ and $\operatorname{rank} C = tn/2+n+1$, and so $\operatorname{rank}\tau_n = nt^2/2 + 3nt/2 + n + 1$. $\square$

Direct computations show that

$$\dim_{\Bbbk} HH_0(\Lambda_t) = t+3, \qquad \dim_{\Bbbk} HH_1(\Lambda_t) = 2t+4.$$

For the higher degree Hochschild cohomology groups of $\Lambda_t$, we have

**Theorem 3.3.** *Let $\Lambda_t$ be the Yoneda algebra of the reconstruction algebra $A_t$. Then for $n \geqslant 2$,*

$$\dim_{\Bbbk} HH_n(\Lambda_t) = (n+1)t + 2n + 2.$$

P r o o f. Since $\dim_{\Bbbk} HH_n(\Lambda_t) = \dim_{\Bbbk} N_n - \dim_{\Bbbk} \operatorname{Im} \tau_n - \dim_{\Bbbk} \operatorname{Im} \tau_{n+1}$, by direct calculation, we obtain: if $n$ is odd,

$$\begin{aligned}\dim_{\Bbbk} HH_n(\Lambda_t) &= (t+2)(nt+2n+2) - \frac{n-1}{2}t^2 - \frac{3n-1}{2}t - n \\ &\quad - \frac{n+1}{2}t^2 - \frac{3n+3}{2}t - n - 2 \\ &= (n+1)t + 2n + 2;\end{aligned}$$

if $n$ is even,

$$\begin{aligned}\dim_{\Bbbk} HH_n(\Lambda_t) &= (t+2)(nt+2n+2) - \frac{n}{2}t^2 - \frac{3n}{2}t - n - 1 \\ &\quad - \frac{n}{2}t^2 + \frac{3n+2}{2}t - n - 1 \\ &= (n+1)t + 2n + 2.\end{aligned}$$

Hence, we get the theorem. □

Denote by $HC_n(\Lambda_t)$ the $n$-th cyclic homology group of $\Lambda_t$ (cf. [18]).

**Corollary 3.4.** *Let $\Lambda_t$ be the Yoneda algebra of the reconstruction algebra $A_t$ and $\operatorname{char} \Bbbk = 0$. Then we have*

$$\dim_{\Bbbk} HC_n(\Lambda_t) = \begin{cases} \dfrac{n+1}{2}(t+2) + 1, & \text{if } n \text{ is odd;} \\[2ex] (t+2)\Big(\dfrac{n}{2} + 1\Big) + 1, & \text{if } n \text{ is even.} \end{cases}$$

P r o o f. By [18], Theorem 4.1.13, we have

$$\begin{aligned}\dim_{\Bbbk} HC_n(\Lambda_t) - \dim_{\Bbbk} HC_n(\Bbbk^2) = &-(\dim_{\Bbbk} HC_{n-1}(\Lambda_t) - \dim_{\Bbbk} HC_{n-1}(\Bbbk^2)) \\ &+(\dim_{\Bbbk} HH_n(\Lambda_t) - \dim_{\Bbbk} HH_n(\Bbbk^2)).\end{aligned}$$

Thus, $\dim_{\Bbbk} HC_n(\Lambda_t) - \dim_{\Bbbk} HC_n(\Bbbk^2) = \sum\limits_{i=0}^{n} (-1)^{n-i}(\dim_{\Bbbk} HH_i(\Lambda_t) - \dim_{\Bbbk} HH_i(\Bbbk^2))$. Moreover, it is well-known that

$$\dim_{\Bbbk} HH_i(\Bbbk^2) = \begin{cases} 2, & \text{if } i = 0; \\ 0, & \text{if } i \geqslant 1, \end{cases} \quad \text{and} \quad \dim_{\Bbbk} HC_i(\Bbbk^2) = \begin{cases} 2, & \text{if } i \text{ is even;} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

Thus, by Theorem 3.3, we obtain the corollary. □

For any finite-dimensional $\Bbbk$-algebra $\Lambda$, we denote by

$$\mathrm{hh.}\dim \Lambda := \inf\{m \in \mathbb{Z};\ \dim_{\Bbbk} HH_n(\Lambda) = 0 \text{ for all } n > m\}$$

and gl.dim $\Lambda$ the Hochschild homology dimension and the global dimension of $\Lambda$, respectively. Then, by the results of Theorem 3.3, we have

**Corollary 3.5.** $\mathrm{gl.}\dim \Lambda_t = \infty = \mathrm{hh.}\dim \Lambda_t$.

Dieter Happel in [14] asked the following question: if the Hochschild cohomology groups $HH^n(\Lambda)$ of a finite dimensional algebra $\Lambda$ over a field $k$ vanish for all sufficiently large $n$, is the global dimension of $\Lambda$ finite? The paper [5] gave the negative answer for the four dimensional algebra $\Bbbk\langle x, y\rangle/(x^2, xy - \mathrm{q}yx, y^2)$.

In [13], Han conjectured that the homology of Happel's question would always hold, namely that a finite-dimensional algebra whose higher Hochschild homology groups vanish must be of finite global dimension. It is known that Han's conjecture holds for commutative algebras, monomial algebras [1], [13]. If the characteristic of the ground field is zero, Han's conjecture also holds for $N$-Koszul algebras, graded local algebras, graded cellular algebras [3]. Our results show that the algebras $\Lambda_t$ also provide the affirmative answer to Han's conjecture.

## 4. Hochschild cohomology and Hilbert series

In this section, we calculate all the $\Bbbk$-dimensions of Hochschild cohomology groups of $\Lambda_t$ by transforming the Hochschild cohomology complex to the complex of parallel paths.

Let $X$ and $Y$ be the sets of uniform elements in $\Bbbk Q^t$. We define

$$X \parallel Y := \{(p, q) \in X \times Y;\ \mathfrak{o}(p) = \mathfrak{o}(q) \text{ and } \mathfrak{t}(p) = \mathfrak{t}(q)\},$$

and denote by $\Bbbk(X \parallel Y)$ the vector space spanned by the elements in $X \parallel Y$, and say a pair of uniform elements $(p, q)$ in $\Bbbk Q^t$ is parallel if $(p, q) \in \Bbbk Q^t \parallel \Bbbk Q^t$.

Considering the sets $\mathcal{B} \parallel F^n$, we have

$$\mathcal{B} \parallel F^0 = \{(e_1, e_1), (e_2, e_2)\} \cup \{(x_2 y_i, e_1), (y_{i+1} x_1, e_2);\ 1 \leqslant i \leqslant t+1\}.$$

Whenever $n \geqslant 1$, if $n$ is odd,

$$\begin{aligned}\mathcal{B} \parallel F^n = &\Big\{(x_j, f_m^{(n,2)});\ j = 1, 2,\ n \leqslant m \leqslant \frac{n-1}{2}t + 2n\Big\}\\ &\cup \Big\{(y_i, f_m^{(n,1)});\ 1 \leqslant i \leqslant t+2,\ n \leqslant m \leqslant \frac{n+1}{2}t + 2n\Big\}\\ &\cup \Big\{(x_1 y_i x_1, f_m^{(n,2)});\ 3 \leqslant i \leqslant t+2,\ n \leqslant m \leqslant \frac{n-1}{2}t + 2n\Big\};\end{aligned}$$

if $n$ is even,

$$\mathcal{B} \parallel F^n = \Big\{(e_j, f_m^{(n,j)});\ j = 1, 2,\ n \leqslant m \leqslant \frac{n}{2}t + 2n\Big\}$$
$$\cup \Big\{(x_1 y_i, f_m^{(n,1)}), (y_i x_1, f_m^{(n,2)});\ 2 \leqslant i \leqslant t + 2,\ n \leqslant m \leqslant \frac{n}{2}t + 2n\Big\}.$$

Therefore, $|\mathcal{B} \parallel F^n| = (t + 2)(nt + 2n + 2)$. Now we define a complex $\mathbb{M} = (M^n, \eta^n)$ by the sets $\mathcal{B} \parallel F^n$ as follows: First, let $M^n = \Bbbk(\mathcal{B} \parallel F^n)$ for all $n \geqslant 0$. Secondly, define the differential $\eta^1\colon\ M^0 \to M^1$ by

$$\eta^1(e_1, e_1) = -\eta^1(e_2, e_2) = \sum_{1\leqslant j\leqslant 2} (x_j, x_j) - \sum_{1\leqslant i\leqslant t+2} (y_i, y_i),$$
$$\eta^1(y_i x_1, e_2) = -\eta^1(x_1 y_i, e_1) = (x_1 y_{i+1} x_1, x_2) - (x_1 y_i x_1, x_1),$$

and for $n \geqslant 2$, let the differential $\eta^n\colon\ M^{n-1} \to M^n$ be given by: for any $(b, f_m^{(n,1)})$, $(b, f_m^{(n,2)}) \in \Bbbk(\mathcal{B} \parallel F^n)$, if $n$ is odd,

$$\eta^n(b, f_m^{(n-1,1)}) = \sum_{1\leqslant j\leqslant 2} (bx_j, f_{m+j}^{(n,2)}) - \sum_{1\leqslant i\leqslant t+2} (y_i b, f_{m+i}^{(n,1)}),$$
$$\eta^n(b, f_m^{(n-1,2)}) = \sum_{1\leqslant i\leqslant t+2} (by_i, f_{m+i}^{(n,1)}) - \sum_{1\leqslant j\leqslant 2} (x_j b, f_{m+j}^{(n,2)});$$

if $n$ is even,

$$\eta^n(b, f_m^{(n-1,1)}) = \sum_{1\leqslant j\leqslant 2} ((bx_j, f_{m+j}^{(n,2)}) - (x_j b, f_{m+j}^{(n,1)})),$$
$$\eta^n(b, f_m^{(n-1,2)}) = \sum_{1\leqslant i\leqslant t+2} ((by_i, f_{m+i}^{(n,1)}) - (y_i b, f_{m+i}^{(n,2)})).$$

Applying the functor $\mathrm{Hom}_{\Lambda_t^e}(-, \Lambda_t)$ to the minimal projective bimodule resolution $\mathbb{P}$, we get a Hochschild cohomology complex of $\Lambda_t$. In text lemma we show that the complex $\mathbb{M}$ gives a presentation of this Hochschild cohomology complex.

**Lemma 4.1.** $\mathrm{Hom}_{\Lambda_t^e}(\mathbb{P}, \Lambda_t) \cong \mathbb{M}$ *as complexes.*

P r o o f. First, it is easy to see that

$$\mathrm{Hom}_{\Lambda_t^e}(P_n, \Lambda_t) \cong \bigoplus_{f\in F^n} \mathrm{Hom}_{\Lambda_t^e}(\Lambda_t \mathfrak{o}(f) \otimes \mathfrak{t}(f)\Lambda_t, \Lambda_t) \cong \bigoplus_{f\in F^n} \mathfrak{o}(f)\Lambda_t \mathfrak{t}(f) \cong M^n$$

as $\Bbbk$-vector spaces. The corresponding isomorphism $\varphi_n\colon\ M^n \to \mathrm{Hom}_{\Lambda_t^e}(P_n, \Lambda_t)$ is given by $(a, f) \mapsto \xi_{(a,f)}$, where $\xi_{(a,f)}(\mathfrak{o}(g) \otimes \mathfrak{t}(g))$ is $a$ if $f = g$ and is 0 otherwise.

Then we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \Bbbk(\mathcal{B} \parallel F^n) \arrow[r, "\eta^{n+1}"] \arrow[d, "\varphi_n"] & \Bbbk(\mathcal{B} \parallel F^{n+1}) \arrow[r] \arrow[d, "\varphi_{n+1}"] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}_{\Lambda_t^e}(P_n, \Lambda_t) \arrow[r, "d_{n+1}^*"] & \mathrm{Hom}_{\Lambda_t^e}(P_{n+1}, \Lambda_t) \arrow[r] & \cdots
\end{tikzcd}$$

Thus, the isomorphism of complexes is obtained. □

Thus, we can calculate the dimensions of Hochschild cohomology groups of $\Lambda_t$ by the complex $\mathbb{M}$. By the definition, $HH^n(\Lambda_t) = \mathrm{Ker}\,\eta^{n+1}/\,\mathrm{Im}\,\eta^n$, and so

$$\begin{aligned}
\dim_{\Bbbk} HH^n(\Lambda_t) &= \dim_{\Bbbk} \mathrm{Ker}\,\eta^{n+1} - \dim_{\Bbbk} \mathrm{Im}\,\eta^n \\
&= \dim_{\Bbbk} M^n - \dim_{\Bbbk} \mathrm{Im}\,\eta^{n+1} - \dim_{\Bbbk} \mathrm{Im}\,\eta^n.
\end{aligned}$$

Thus, we need to determine $\dim_{\Bbbk} \mathrm{Im}\,\eta^n$ for all $n$. As in the homology case, we will determine the $\Bbbk$-dimension of $\mathrm{Im}\,\eta^n$ by considering the corresponding matrix of $\eta^n$ over an ordered basis of $M^{n-1}$. Let

$$(b, f_m^{(n,i)}) \prec (b', f_{m'}^{(n,i')}) \quad \text{if } b \prec b', \text{ or } b = b' \text{ but } m < m',$$

for any $(b, f_m^{(n,i)})$, $(b', f_{m'}^{(n,i')}) \in \mathcal{B} \parallel F^n$. If we still denote by $\eta^n$ the matrix of $\eta^n$ under the ordered basis $\mathcal{B} \parallel F^n$, then we can check:

(1) if $n$ is odd, $\eta^n = \begin{pmatrix} E & 0 \\ -E & 0 \\ 0 & F \\ 0 & -F \end{pmatrix}_{(t+2)(nt-t+2n)\times(t+2)(nt+2n+2)}$, where

$$E = \begin{pmatrix} U^1 & {}^1U & -U^{t+1} & -{}^1U^t & \ldots & -{}^{t+1}U \end{pmatrix},$$

is an $(((n-1)/2)t+n) \times (((n+1)/2)t^2+3nt+t+4n+4)$ matrix, $U = \mathbf{I}_{((n-1)/2)t+n}$, and

$$F = \begin{pmatrix} -{}^1U & 0 & \ldots & 0 \\ U^1 & -{}^1U & & 0 \\ 0 & U^1 & \ddots & \\ \vdots & & \ddots & -{}^1U \\ 0 & 0 & & U^1 \end{pmatrix}_{(t+1)((n-1)t/2+n)\times t((n-1)t/2+n+1)};$$

(2) if $n$ is even, $\tau_n = \begin{pmatrix} 0 & G & -G \\ 0 & H & -H \\ 0 & 0 & 0 \end{pmatrix}_{(t+2)(nt-t+2n)\times(t+2)(nt+2n+2)}$, where

$$G = \begin{pmatrix} -{}^1V^t & -{}^2V^{t-1} & \ldots & -{}^{t+1}V \\ V^{t+1} & {}^1V^t & \ldots & {}^tV^1 \end{pmatrix}_{(nt-2t+2n)\times(t+1)((n/2)t+n+1)},$$

$V = \mathbf{I}_{(n/2)t-t+n}$, $W = \mathbf{I}_{nt/2+n}$, and

$$H = \begin{pmatrix} -^1W & 0 & \dots & 0 \\ W^1 & -^1W & & 0 \\ 0 & W^1 & \ddots & \\ \vdots & & \ddots & -^1W \\ 0 & 0 & & W^1 \end{pmatrix}_{(t+2)(nt/2+n)\times(t+1)(nt/2+n+1)}.$$

Therefore, we obtain the following results.

**Lemma 4.2.** *For the map $\eta^n$, $n \geqslant 2$, we have*

$$\operatorname{rank}\eta^n = \begin{cases} \dfrac{n-1}{2}t^2 + \dfrac{3n+1}{2}t + n, & \text{if } n \text{ is odd;} \\ \dfrac{n}{2}t^2 + \dfrac{3n+2}{2}t + n + 1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. If $n$ is odd, then $\operatorname{rank}\eta^n = \operatorname{rank} E + \operatorname{rank} F$. It is easy to see that $\operatorname{rank} E = ((n-1)/2)t + n$ and $\operatorname{rank} F = t((n-1)t/2 + n) + t$, and so $\operatorname{rank}\tau_n = (n-1)t^2/2 + (3n+1)t/2 + n$.

If $n$ is even, then $\operatorname{rank}\eta^n = \operatorname{rank}\binom{G}{H}$. By elementary transformations, one can check that $\operatorname{rank}\eta^n = \operatorname{rank}\binom{G}{H} = (t+1)(nt/2+n) + t + 1$. $\square$

Now, we can get all the $\Bbbk$-dimensions of Hochschild cohomology groups of $\Lambda_t$. First, for the low degree Hochschild cohomology groups of $\Lambda_t$, direct computations show that $\operatorname{rank}\eta^1 = t + 2$ and $\operatorname{rank}\eta^2 = t^2 + 5t + 5$. Therefore, we have

$$\dim_{\Bbbk} HH^0(\Lambda_t) = t + 2, \quad \dim_{\Bbbk} HH^1(\Lambda_t) = t + 3.$$

Secondly, for the higher degree Hochschild cohomology groups of $\Lambda_t$, we have

**Theorem 4.3.** *Let $\Lambda_t$ be the Yoneda algebra of the reconstruction algebra $A_t$. Then for $n \geqslant 2$,*

$$\dim_{\Bbbk} HH^n(\Lambda_t) = (n-1)t + 2n + 2.$$

Proof. Since $\dim_{\Bbbk} HH^n(\Lambda_t) = \dim_{\Bbbk} M^n - \dim_{\Bbbk} \operatorname{Im}\eta^n - \dim_{\Bbbk} \operatorname{Im}\eta^{n+1}$, by direct calculation we obtain: if $n$ is odd,

$$\begin{aligned} \dim_{\Bbbk} HH^n(\Lambda_t) &= (t+2)(nt+2n+2) - \frac{n-1}{2}t^2 - \frac{3n+1}{2}t - n \\ &\quad - \frac{n+1}{2}t^2 - \frac{3n+5}{2}t - n - 2 \\ &= (n-1)t + 2n + 2; \end{aligned}$$

if $n$ is even,

$$\begin{aligned}\dim_{\Bbbk} HH^n(\Lambda_t) &= (t+2)(nt+2n+2) - \frac{n}{2}t^2 - \frac{3n+2}{2}t - n - 1 \\ &\quad - \frac{n}{2}t^2 + \frac{3n+4}{2}t - n - 1 \\ &= (n-1)t + 2n + 2.\end{aligned}$$

This proves the theorem. $\square$

**Remark 4.4.** In the case $t = 0$, we have shown in [15] that $\dim_{\Bbbk} HH^n(\Lambda_0) = \dim_{\Bbbk} HH_n(\Lambda_0)$ for all $n \geqslant 1$. But Theorem 3.3 and Theorem 4.3 show that this property does not hold for $t \geqslant 1$.

Following from Theorem 4.3, we obtain the corollary immediately.

**Corollary 4.5.** *The Hilbert series of the Yoneda algebra of the reconstruction algebra $A_t$ is*

$$\sum_{n=0}^{\infty} \dim_{\Bbbk} HH^n(\Lambda_t)x^n = \frac{tx^2+2}{(1-x)^2} + (t-1)x + t.$$

Denoting by

$$\mathrm{hch.dim}\,\Lambda_t := \inf\{m \in \mathbb{Z};\ \dim_{\Bbbk} HH^n(\Lambda) = 0 \text{ for all } n > m\}$$

the Hochschild cohomology dimension $\Lambda_t$, we have

**Corollary 4.6.** $\mathrm{gl.dim}\,\Lambda_t = \infty = \mathrm{hch.dim}\,\Lambda_t$.

Thus, the algebras $\Lambda_t$ also give the affirmative answer to Happel's question.

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