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ATOMIC DECOMPOSITION OF PREDICTABLE MARTINGALE HARDY SPACE WITH VARIABLE EXPONENTS

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Abstract. This paper is mainly devoted to establishing an atomic decomposition of a predictable martingale Hardy space with variable exponents defined on probability spaces. More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $p(\cdot)\colon \Omega \to (0,\infty)$ be a $\mathcal{F}$-measurable function such that $0 < \inf_{x\in\Omega} p(x) \leqslant \sup_{x\in\Omega} p(x) < \infty$. It is proved that a predictable martingale Hardy space $\mathcal{P}_{p(\cdot)}$ has an atomic decomposition by some key observations and new techniques. As an application, we obtain the boundedness of fractional integrals on the predictable martingale Hardy space with variable exponents when the stochastic basis is regular.

Keywords: variable exponent; atomic decomposition; martingale Hardy space; fractional integral

MSC 2010: 60G46, 60G42

1. Introduction

Let $\mathcal{P} = \mathcal{P}(\Omega)$ denote the collection of all $\mathcal{F}$-measurable functions $p(\cdot)\colon \Omega \to (0,\infty)$ such that $0 < \inf\limits_{x\in\Omega} p(x) \leqslant \sup\limits_{x\in\Omega} p(x) < \infty$; such a function is called a variable exponent. The space $L_{p(\cdot)}(\Omega)$, the Lebesgue space with variable exponent $p(\cdot)$, is defined as the set of all $\mathcal{F}$-measurable functions f such that for some $\lambda > 0$

$$\int_\Omega \Big(\frac{|f(w)|}{\lambda}\Big)^{p(w)} \,\mathrm{d}\mathbb{P} < \infty,$$

with

$$\|f\|_{p(\cdot)} \equiv \inf\biggl\{\lambda > 0\colon \int_\Omega \Big(\frac{|f(w)|}{\lambda}\Big)^{p(w)} \,\mathrm{d}\mathbb{P} \leqslant 1\biggr\}.$$

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Then $(L_{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a quasi-normed space. When $\Omega = \mathbb{R}^n$, the space $(L_{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is reduced to $(L_{p(\cdot)}(\mathbb{R}^n), \|\cdot\|_{p(\cdot)})$. It should be mentioned that $(L_{p(\cdot)}(\mathbb{R}^n), \|\cdot\|_{p(\cdot)})$ was introduced by Orlicz [20] in 1931 and studied by Kováčik and Rákosník [14], Fan and Zhao [8] and others. Heavily basing on the so-called log-Hölder continuity conditions in [6], namely,

$$|p(x) - p(y)| \leqslant \frac{c_1}{\log(\mathrm{e} + 1/|x-y|)}, \quad x, y \in \mathbb{R}^n,$$

harmonic analysis with variable exponents has got an increasing development in the past years; see [2]–[7], [10], [18], [19], [22] and so on. Especially, in [5], [18], [22], the atomic decompositions of Hardy spaces with variable exponents defined on $\mathbb{R}^n$ were established under the so-called log-Hölder continuity conditions.

In this paper, we establish the atomic decomposition of predictable martingale Hardy spaces with variable exponents defined in a probability space. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(\mathcal{F}_n)_{n\geqslant 0}$ be an increasing filtration of σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$ and let $(\mathbb{E}_{\mathcal{F}_n})_{n\geqslant 0}$ denote the corresponding family of conditional expectations. A sequence of measurable functions $f = (f_n)_{n\geqslant 0} \subset L_1(\Omega)$ is called a martingale with respect to $(\mathcal{F}_n)$ if $\mathbb{E}_{\mathcal{F}_n}(f_{n+1}) = f_n$ for every $n \geqslant 0$. For a martingale relative to $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_n)_{n\geqslant 0})$, define the maximal function of f as

$$M_m f = \sup_{n\leqslant m} |f_n|, \quad Mf = \sup_n |f_n|.$$

Definition 1.1 ($p(\cdot)$-atom). Given $p(\cdot) \in \mathcal{P}$, a measurable function a is called a $p(\cdot)$-atom if there exists a stopping time τ such that

(1) $\mathbb{E}_{\mathcal{F}_n}(a) = 0$, for all $n \leqslant \tau$,
(2) $\|Ma\|_\infty \leqslant \|\chi_{\{\tau<\infty\}}\|_{p(\cdot)}^{-1}$.

We note that the definition above coincides with the classical one if $p(\cdot) \equiv p$; we refer to [23] for the classical atomic decompositions and martingale theory. We now define the atomic spaces with variable exponents.

Definition 1.2. Given $p(\cdot) \in \mathcal{P}$, let us denote by $H_{p(\cdot)}^{\mathrm{at}}$ the space of those martingales f for which there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $p(\cdot)$-atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of nonnegative real numbers such that

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k \quad \text{a.e.,} \tag{1.1}$$

and

$$\Bigg\| \sum_{k\in\mathbb{Z}} \frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}} \Bigg\|_{p(\cdot)} < \infty.$$

Let

$$\mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\}) \equiv \left\| \sum_{k\in\mathbb{Z}} \frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}} \right\|_{p(\cdot)},$$

where τ_k is the stopping time with respect to the atom a^k. We define

$$\|f\|_{H^{\rm at}_{p(\cdot)}} \equiv \inf \mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\}),$$

where the infimum is taken over all decompositions of the form (1.1). In Section 3, we prove that

$$\mathcal{P}_{p(\cdot)} = H^{\rm at}_{p(\cdot)}$$

with equivalent norms. See Section 2 for the notation $\mathcal{P}_{p(\cdot)}$.

As an application, we prove the boundedness of fractional integrals on predictable martingale Hardy spaces with variable exponents defined on probability spaces. Compared with the Euclidean space $\mathbb{R}^n$, the probability space Ω has no natural metric structure. The main difficulty is how to overcome the log-Hölder continuity of $p(x)$ defined on $(\Omega,\mathcal{F},\mathbb{P})$. It was pointed out in [13] and [24] that the following condition may replace the so-called log-Hölder continuity in some sense. That is, there exists an absolute constant $K_{p(\cdot)} \geqslant 1$ depending only on $p(\cdot)$ such that

$$\mathbb{P}(A)^{p_-(A)-p_+(A)} \leqslant K_{p(\cdot)}, \quad A \in \mathcal{F}, \tag{1.2}$$

where

$$p_+(A) = \sup_{w\in A} p(w), \quad p_-(A) = \inf_{w\in A} p(w).$$

We often denote $K_{p(\cdot)}$ simply by K if there is no confusion. Under the condition (1.2), we prove that the fractional integral operator is bounded on the predictable martingale Hardy spaces with variable exponents by using the atomic decomposition established in Section 3, which can be regarded as the probability version of the result in [22].

Throughout this paper, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{C}$ denote the integer set, nonnegative integer set and complex numbers set, respectively. We denote by C an absolute positive constant, which can vary from line to line, and denote a constant depending only on $p(\cdot)$ by $C_{p(\cdot)}$. The symbol $A \lesssim B$ stands for the inequality $A \leqslant CB$ or $A \leqslant C_{p(\cdot)}B$. If we write $A \approx B$, then it stands for $A \lesssim B \lesssim A$.

2. Preliminaries

In this section, we give some preliminaries necessary to the whole paper. Given $p(\cdot) \in \mathcal{P}$, for a measurable set $A \subset \Omega$ we always denote

$$p_+(A) = \sup_{w \in A} p(w), \quad p_-(A) = \inf_{w \in A} p(w)$$

and

$$p_+ = p_+(\Omega), \quad p_- = p_-(\Omega), \quad \underline{p} = \min\{p_-, 1\}.$$

The space $L_{p(\cdot)} = L_{p(\cdot)}(\Omega)$ is the collection of all measurable functions f defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for some $\lambda > 0$,

$$\varrho(f/\lambda) = \int_\Omega \Big(\frac{|f(w)|}{\lambda}\Big)^{p(w)} \,\mathrm{d}\mathbb{P} < \infty.$$

This becomes a quasi-Banach function space when equipped with the quasi-norm

$$\|f\|_{p(\cdot)} \equiv \inf\{\lambda > 0\colon\ \varrho(f/\lambda) \leqslant 1\}.$$

The following facts are well known; see for example [18].

(1) (Positivity) $\|f\|_{p(\cdot)} \geqslant 0$; $\|f\|_{p(\cdot)} = 0 \Leftrightarrow f \equiv 0$.
(2) (Homogeneity) $\|cf\|_{p(\cdot)} = |c| \cdot \|f\|_{p(\cdot)}$ for $c \in \mathbb{C}$.
(3) (The $\underline{p}$-triangle inequality) $\|f+g\|_{p(\cdot)}^{\underline{p}} \leqslant \|f\|_{p(\cdot)}^{\underline{p}} + \|g\|_{p(\cdot)}^{\underline{p}}$.

We collect some basic lemmas, which will be used in the paper.

Lemma 2.1 (see [3]). *Given $p(\cdot) \in \mathcal{P}$, then for all $f \in L_{p(\cdot)}$ and $\|f\|_{p(\cdot)} \neq 0$ we have*

$$\int_\Omega \left|\frac{f(w)}{\|f\|_{p(\cdot)}}\right|^{p(w)} \mathrm{d}\mathbb{P} = 1.$$

Lemma 2.2 (see [3]). *Given $p(\cdot) \in \mathcal{P}$ and $f \in L_{p(\cdot)}$, then we have*

(1) $\|f\|_{p(\cdot)} < 1$ $(=1,\ >1)$ *if and only if* $\varrho(f) < 1$ $(=1,\ >1)$;
(2) *if* $\|f\|_{p(\cdot)} > 1$, *then* $\varrho(f)^{1/p_+} \leqslant \|f\|_{p(\cdot)} \leqslant \varrho(f)^{1/p_-}$;
(3) *if* $0 < \|f\|_{p(\cdot)} \leqslant 1$, *then* $\varrho(f)^{1/p_-} \leqslant \|f\|_{p(\cdot)} \leqslant \varrho(f)^{1/p_+}$.

Lemma 2.3 (see [3], Hölder's inequality). *Given* $p(\cdot),\ q(\cdot),\ r(\cdot) \in \mathcal{P}$, *such that*

$$\frac{1}{p(w)} = \frac{1}{q(w)} + \frac{1}{r(w)}.$$

Then there exists a constant $C_{p(\cdot)}$ *such that for all* $f \in L_{q(\cdot)}$, $g \in L_{r(\cdot)}$, *and* $fg \in L_{p(\cdot)}$

$$\|fg\|_{p(\cdot)} \leqslant C_{p(\cdot)} \|f\|_{q(\cdot)} \|g\|_{r(\cdot)}.$$

Now we introduce the predictable martingale Hardy space. Let $\mathcal{M}$ be the set of all martingales $f = (f_n)_{n\geqslant 0}$ relative to $\{\mathcal{F}_n\}_{n\geqslant 0}$ such that $f_0 = 0$. For $f \in \mathcal{M}$, denote its martingale difference by $d_n f = f_n - f_{n-1}$ ($n \geqslant 0$, with convention $d_0 f = 0$). If in addition $f_n \in L_{p(\cdot)}$, f is called an $L_{p(\cdot)}$-martingale with respect to $(\mathcal{F}_n)$. In this case we set

$$\|f\|_{p(\cdot)} = \sup_{n\geqslant 0} \|f_n\|_{p(\cdot)}.$$

If $\|f\|_{p(\cdot)} < \infty$, f is called a bounded $L_{p(\cdot)}$-martingale and denoted by $f \in L_{p(\cdot)}$. The stochastic basis $\{\mathcal{F}_n\}_{n\geqslant 0}$ is said to be regular if there exists an absolute constant $R > 0$ such that

$$f_n \leqslant R f_{n-1} \tag{2.1}$$

holds for all nonnegative martingales $f = (f_n)_{n\geqslant 0}$.

Let Γ be the class of nonnegative, non-decreasing and adapted sequences $\lambda = (\lambda_n)_{n\geqslant 0}$ with respect to $\{\mathcal{F}_n\}_{n\geqslant 0}$ and $\lambda_\infty = \lim\limits_{n\to\infty} \lambda_n$. Then we define the predictable variable Hardy martingale spaces $\mathcal{P}_{p(\cdot)}$ as

$$\mathcal{P}_{p(\cdot)} = \{f = (f_n)_{n\geqslant 0} \colon \ \exists\{\lambda_n\} \in \Gamma, \text{ such that } |f_n| \leqslant \lambda_{n-1}, \text{ a.e., } \lambda_\infty \in L_{p(\cdot)}\}$$

equipped with the (quasi)-norm

$$\|f\|_{\mathcal{P}_{p(\cdot)}} = \inf_{\{\lambda_n\}\in\Gamma} \|\lambda_\infty\|_{p(\cdot)}.$$

If we consider the special case $p(\cdot) \equiv p$, then we obtain the classical predictable martingale Hardy spaces $\mathcal{P}_p$.

3. Atomic decompositions

In this section we construct the atomic decomposition of the martingale Hardy space with variable exponent. We refer to [11], [12], [13], [15], [16], [23], [25] for more information on the classical atomic decompositions.

Theorem 3.1. *If $p(\cdot) \in \mathcal{P}$, then the atomic space $H^{\mathrm{at}}_{p(\cdot)}$ is equivalent to predictable spaces $\mathcal{P}_{p(\cdot)}$ by their norms. More precisely, if $f = (f_n)_{n\geqslant 0} \in \mathcal{P}_{p(\cdot)}$ is a martingale, then there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $p(\cdot)$-atoms and a sequence $\mu = (\mu_k)_{k\in\mathbb{Z}}$ of nonnegative real numbers such that for all $n \in \mathbb{N}$*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_{\mathcal{F}_n}(a^k) \tag{3.1}$$

and

$$\|f\|_{H^{\mathrm{at}}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{P}_{p(\cdot)}}.$$

Conversely, if a martingale $f = (f_n)_{n\geqslant 0}$ has a decomposition (3.1), *then $f \in \mathcal{P}_{p(\cdot)}$ and $\|f\|_{\mathcal{P}_{p(\cdot)}} \lesssim \|f\|_{H^{\mathrm{at}}_{p(\cdot)}}$.*

Proof. Assume that $f \in \mathcal{P}_{p(\cdot)}$. Let us consider the stopping times for all $k \in \mathbb{Z}$

$$\tau_k = \inf\{n \in \mathbb{N}\colon\ \lambda_n > 2^k\}, \quad \inf\emptyset = \infty,$$

where $(\lambda_n)_{n\geqslant 0}$ is an adapted, non-decreasing sequence such that $|f_n| \leqslant \lambda_{n-1}$ holds almost everywhere and $\lambda_\infty \in L_{p(\cdot)}$. For each stopping time τ, denote $f_n^\tau = f_{n\wedge\tau}$. It is easy to see that

$$f_n = \sum_{k\in\mathbb{Z}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}).$$

Let

$$\mu_k = 3\cdot 2^k \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)} \quad \text{and} \quad a_n^k = \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

If $\mu_k = 0$ then let $a_n^k = 0$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$. Then $(a_n^k)_{n\geqslant 0}$ is a martingale for each fixed $k \in \mathbb{Z}$. Since $Mf^{\tau_k} \leqslant \lambda_{\tau_k-1} \leqslant 2^k$, we get

$$Ma_n^k \leqslant \frac{Mf^{\tau_{k+1}} + Mf^{\tau_k}}{\mu_k} \leqslant \|\chi_{\{\tau_k<\infty\}}\|^{-1}_{p(\cdot)}.$$

Hence it is easy to check that $(a_n^k)_{n\geqslant 0}$ is a bounded L_2-martingale. Consequently, there exists an element $a^k \in L_2$ such that $\mathbb{E}_{\mathcal{F}_n} a^k = a_n^k$. If $n \leqslant \tau_k$, then $a_n^k = 0$, and $Ma^k \leqslant \|\chi_{\{\tau_k<\infty\}}\|^{-1}_{p(\cdot)}$. Thus we conclude that a^k is really a $p(\cdot)$-atom.

Denote $\mathcal{B}_k = \{\tau_k < \infty\} = \{\lambda_\infty > 2^k\}$. Recalling that τ_k is non-decreasing for each $k \in \mathbb{Z}$, we have $\mathcal{B}_k \supset \mathcal{B}_{k+1}$. Then

$$\sum_{k\in\mathbb{Z}} 3\cdot 2^k \chi_{\mathcal{B}_k}(w), \quad w \in \Omega$$

is the sum of the geometric sequence $\{3 \cdot 2^k \chi_{\mathcal{B}_k}(w)\}_{k\in\mathbb{Z}}$. Thus, we can claim that for each $w \in \Omega$ we have

$$\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{B}_k}(w) \approx \sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{B}_k \setminus \mathcal{B}_{k+1}}(w).$$

Indeed, for each fixed $w_0 \in \Omega$ there is $k_0 \in \mathbb{Z}$ such that $w_0 \in \mathcal{B}_{k_0}$ but $w_0 \notin \mathcal{B}_{k_0+1}$. Then

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{B}_k}(w_0) &= \sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{\mathcal{B}_k}(w_0) = \sum_{k=-\infty}^{k_0} 3 \cdot 2^k = 3 \cdot 2^{k_0} \frac{1}{1-2^{-1}} \\ &= 2 \sum_{k=-\infty}^{k_0} 3 \cdot 2^k \chi_{\mathcal{B}_k \setminus \mathcal{B}_{k+1}}(w_0) = 2 \sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{B}_k \setminus \mathcal{B}_{k+1}}(w_0).\end{aligned}$$

This implies

$$\begin{aligned}\mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\}) &= \left\| \sum_{k\in\mathbb{Z}} \frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}} \right\|_{p(\cdot)} \\ &= \left\| \sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)} \lesssim \left\| \sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{B}_k \setminus \mathcal{B}_{k+1}} \right\|_{p(\cdot)} \\ &= \inf\left\{ \lambda > 0\colon \int_\Omega \left(\sum_{k\in\mathbb{Z}} \frac{3 \cdot 2^k \chi_{\mathcal{B}_k \setminus \mathcal{B}_{k+1}}(w)}{\lambda} \right)^{p(w)} \mathrm{d}\mathbb{P} \leqslant 1 \right\} \\ &= \inf\left\{ \lambda > 0\colon \sum_{k\in\mathbb{Z}} \int_{\mathcal{B}_k \setminus \mathcal{B}_{k+1}} \left(\frac{3 \cdot 2^k}{\lambda} \right)^{p(w)} \mathrm{d}\mathbb{P} \leqslant 1 \right\} \\ &\approx \inf\left\{ \lambda > 0\colon \int_\Omega \left(\frac{\lambda_\infty}{\lambda} \right)^{p(w)} \mathrm{d}\mathbb{P} \leqslant 1 \right\}.\end{aligned}$$

Therefore, we obtain

$$\mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\}) \lesssim \|\lambda_\infty\|_{p(\cdot)}.$$

Taking the infimum over all predictable sequences $(\lambda_n)_{n\geqslant 0}$, we conclude that

$$\mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\}) \lesssim \|f\|_{\mathcal{P}_{p(\cdot)}}.$$

To prove the converse part, let

$$\lambda_n = \sum_{k\in\mathbb{Z}} \mu_k \|Ma^k\|_\infty \chi_{\{\tau_k \leqslant n\}}.$$

Then $(\lambda_n)_{n\geqslant 0}$ is a nonnegative, non-decreasing and adapted sequence with the condition $|f_{n+1}| \leqslant \lambda_n$. Since a^k is a $p(\cdot)$-atom for each $k \in \mathbb{Z}$, we get

$$\|f\|_{\mathcal{P}_{p(\cdot)}} \leqslant \|\lambda_\infty\|_{p(\cdot)} \leqslant \bigg\|\sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\bigg\|_{p(\cdot)} = \mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\}),$$

which implies $\|f\|_{\mathcal{P}_{p(\cdot)}} \leqslant \|f\|_{H^{\rm at}_{p(\cdot)}} \lesssim \|f\|_{\mathcal{P}_{p(\cdot)}}$. □

4. Boundedness of fractional integrals on variable Hardy martingale spaces

It is well known that the fractional integrals have occupied a very important role in the classical harmonic analysis. In martingale theory, Chao and Ombe [1] introduced the fractional integrals for dyadic martingales. Sadasue [21] proved the boundedness of fractional integrals on martingale Hardy spaces for $0 < p \leqslant 1$. Recently, Nakai and Sadasue [17] and Hao and Jiao [9] extended the notion of fractional integrals to a more general setting.

In this section, we prove the boundedness of fractional integrals on variable predictable Hardy martingale spaces. We suppose that every σ-algebra $\mathcal{F}_n$ is generated by countable atoms, where $B \in \mathcal{F}_n$ is called an atom provided the following implication holds: if any $A \subset B$ with $A \in \mathcal{F}_n$ satisfies $\mathbb{P}(A) < \mathbb{P}(B)$, then $\mathbb{P}(A) = 0$. Denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. Without loss of generality, we always suppose that the constant in (2.1) satisfies $R \geqslant 2$.

Now we give the definition of fractional integrals; see [21].

Definition 4.1. For $f = (f_n)_{n\geqslant 0} \in \mathcal{M}$, $\alpha > 0$, the fractional integral $I_\alpha f = ((I_\alpha f)_n)_{n\geqslant 0}$ of f is defined by

$$(I_\alpha f)_n = \sum_{k=1}^{n} b_{k-1}^{\alpha} d_k f$$

where b_k is an $\mathcal{F}_k$-measurable function such that for all $B \in A(\mathcal{F}_k)$, for all $\omega \in B$, $b_k(\omega) = \mathbb{P}(B)$.

In order to prove the boundedness of fractional integrals, we need the following lemma; see [24]. For convenience, we give the simple proof.

Lemma 4.2. *Let $p(\cdot), q(\cdot) \in \mathcal{P}$ satisfy* (1.2). *Then for any set $A \in \mathcal{F}$ we have*

$$\|\chi_A\|_{r(\cdot)} \approx \|\chi_A\|_{p(\cdot)} \|\chi_A\|_{q(\cdot)},$$

where

$$\frac{1}{r(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}.$$

Proof. It is not difficult to see that $r(\cdot) \in \mathcal{P}$ and

$$\mathbb{P}(A)^{r_-(A)-r_+(A)} \leqslant K_{r(\cdot)}, \quad A \in \mathcal{F}. \tag{4.1}$$

Next we will show that

$$\mathbb{P}(A)^{1/p_-(A)} \approx \mathbb{P}(A)^{1/p(w)} \approx \mathbb{P}(A)^{1/p_+(A)} \approx \|\chi_A\|_{p(\cdot)}, \quad w \in A. \tag{4.2}$$

Indeed, for every $w \in A$ we have

$$\mathbb{P}(A)^{1/p_-(A)} \leqslant \mathbb{P}(A)^{1/p(w)} \leqslant \mathbb{P}(A)^{1/p_+(A)}.$$

Since $p(\cdot)$ satisfies (1.2), we have

$$\frac{\mathbb{P}(A)^{1/p(w)}}{\mathbb{P}(A)^{1/p_-(A)}} \leqslant \mathbb{P}(A)^{(p_-(A)-p(w))/p_-(A)p(w)} \leqslant K_{p(\cdot)}^{1/p_-^2(\Omega)} =: K,$$

which implies $\mathbb{P}(A)^{1/p(w)} \leqslant K\mathbb{P}(A)^{1/p_-(A)}$. One can check that $\mathbb{P}(A)^{1/p_-(A)} \approx \mathbb{P}(A)^{1/p(w)} \approx \mathbb{P}(A)^{1/p_+(A)}$. Then we arrive at

$$\frac{\chi_A(w)}{\mathbb{P}(A)^{1/p_-(A)}} \approx \frac{\chi_A(w)}{\mathbb{P}(A)^{1/p(w)}}.$$

That is,

$$\Big(\frac{\chi_A(w)}{\mathbb{P}(A)^{1/p_-(A)}}\Big)^{p(w)} \geqslant \frac{\chi_A(w)}{\mathbb{P}(A)} \geqslant \Big(\frac{\chi_A(w)}{K\mathbb{P}(A)^{1/p_-(A)}}\Big)^{p(w)}.$$

Thus, we have

$$\int_\Omega \Big(\frac{\chi_A(w)}{\mathbb{P}(A)^{1/p_-(A)}}\Big)^{p(w)} \,\mathrm{d}\mathbb{P} \approx \int_\Omega \frac{\chi_A(w)}{\mathbb{P}(A)} \,\mathrm{d}\mathbb{P} = 1.$$

Consequently, we get $\|\chi_A\|_{p(\cdot)} \approx \mathbb{P}(A)^{1/p_-(A)}$ and we get the desired result.

Combining (4.1) and (4.2) we conclude

$$\|\chi_A\|_{r(\cdot)} \approx \mathbb{P}(A)^{1/r(w)} = \mathbb{P}(A)^{1/p(w)+1/q(w)} \approx \|\chi_A\|_{p(\cdot)} \|\chi_A\|_{q(\cdot)}, \quad w \in A.$$

$\square$

Lemma 4.3. *Let $\{\mathcal{F}_n\}_{n\geqslant 0}$ be regular, $f \in \mathcal{M}$ and $\alpha > 0$. Let R be the constant in* (2.1). *If there exists $A \in \mathcal{F}$ such that $Mf \leqslant \chi_A$, then there exists a positive constant $C_\alpha = 2 + (R+1)/(1-(1+1/R)^{\alpha-1})$ independent of f and A such that*

$$M(I_\alpha f) \leqslant C_\alpha \mathbb{P}(A)^\alpha \chi_A.$$

For the proof of Lemma 4.3, *see* [21], *Lemma* 3.5. *In the next lemma, we regard a $p(\cdot)$-atom a as a martingale by $a = (a_n)_{n\geqslant 0} = (E_n(a))_{n\geqslant 0}$, so we can consider the fractional integral $I_\alpha a = ((I_\alpha a)_n)_{n\geqslant 0}$.*

Lemma 4.4. *Let $p(\cdot), q(\cdot) \in \mathcal{P}$ satisfy* (1.2) *and let $\{\mathcal{F}_n\}_{n\geqslant 0}$ be regular. If $p(\cdot) < q(\cdot)$, $\alpha = 1/p(\cdot) - 1/q(\cdot)$ and a is a $p(\cdot)$-atom as in Definition* 1.1, *then we have*

$$\|I_\alpha a\|_{\mathcal{P}_{q(\cdot)}} \lesssim C_\alpha,$$

where C_α is the same constant as in Lemma 4.3.

Proof. Let ν be the stopping time associated with a. Then we have

$$Ma \leqslant \|\chi_{\{\nu<\infty\}}\|_{p(\cdot)}^{-1} \chi_{\{\nu<\infty\}}.$$

This implies

$$M(\|\chi_{\{\nu<\infty\}}\|_{p(\cdot)} a) = \|\chi_{\{\nu<\infty\}}\|_{p(\cdot)} Ma \leqslant \chi_{\{\nu<\infty\}}.$$

By Lemma 4.3, we obtain that

$$M(I_\alpha(\|\chi_{\{\nu<\infty\}}\|_{p(\cdot)} a)) \leqslant C_\alpha \mathbb{P}(\nu<\infty)^\alpha \chi_{\{\nu<\infty\}}.$$

Then, by Lemma 4.2, we have

$$M(I_\alpha a) \leqslant C_\alpha \mathbb{P}(\nu<\infty)^\alpha \|\chi_{\{\nu<\infty\}}\|_{p(\cdot)}^{-1} \chi_{\{\nu<\infty\}} \lesssim C_\alpha \|\chi_{\{\nu<\infty\}}\|_{q(\cdot)}^{-1} \chi_{\{\nu<\infty\}}.$$

Now, let

$$\lambda_n = \|I_\alpha a\|_\infty \chi_{\{\nu\leqslant n\}}.$$

Then $(\lambda_n)_{n\geqslant 0}$ is a nonnegative, non-decreasing and adapted sequence with $(I_\alpha a)_{n+1} \leqslant \lambda_n$. Hence, we have

$$\|I_\alpha a\|_{\mathcal{P}_{q(\cdot)}} \leqslant \|\lambda_\infty\|_{q(\cdot)} \lesssim C_\alpha \frac{1}{\|\chi_{\{\nu<\infty\}}\|_{q(\cdot)}} \|\chi_{\{\nu<\infty\}}\|_{q(\cdot)} = C_\alpha.$$

Therefore $\|I_\alpha a\|_{\mathcal{P}_{q(\cdot)}} \lesssim C_\alpha$, where C_α is the same constant as in Lemma 4.3. $\square$

Lemma 4.5. *Given $p(\cdot)\in\mathcal{P}$. Let $f\in H^{\rm at}_{p(\cdot)}$, i.e., $f=\sum\mu_k a^k$. If $p_+\leqslant 1$, then we have*

$$\sum_{k\in\mathbb{Z}}\mu_k\leqslant\mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\}).$$

P r o o f. Let $\lambda=\sum\limits_{k\in\mathbb{Z}}\mu_k$. By Lemma 2.1, we can obtain that

$$\int_\Omega\biggl(\sum_{k\in\mathbb{Z}}\frac{\mu_k\chi_{\{\tau_k<\infty\}}}{\lambda\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\biggr)^{p(w)}\,\mathrm{d}\mathbb{P}\geqslant\sum_{k\in\mathbb{Z}}\frac{\mu_k}{\lambda}\int_\Omega\biggl(\frac{\chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}}\biggr)^{p(w)}\,\mathrm{d}\mathbb{P}=1.$$

From the definition of $\mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\})$, we get the desired result. $\square$

We now prove the boundedness of fractional integrals on variable Hardy martingale spaces via atomic decomposition.

Theorem 4.6. *Let $p(\cdot),q(\cdot)\in\mathcal{P}$ satisfy* (1.2). *Suppose that $(\Omega,\mathcal{F},P)$ is a complete and non-atomic probability space, and $\{\mathcal{F}_n\}_{n\geqslant 0}$ is a regular stochastic basis. If $p_+\leqslant 1\leqslant q_-$ and $\alpha=1/p(\cdot)-1/q(\cdot)$, then there exists a constant C such that*

$$\|I_\alpha f\|_{\mathcal{P}_{q(\cdot)}}\leqslant C\|f\|_{\mathcal{P}_{p(\cdot)}}$$

for all $f\in\mathcal{P}_{p(\cdot)}$.

P r o o f. Let $f\in\mathcal{P}_{p(\cdot)}$. According to Theorem 3.1, there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $p(\cdot)$-atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of nonnegative real numbers such that for all $n\geqslant 0$,

$$\sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}_{\mathcal{F}_n}a^k=f_n,\quad\text{a.e.}$$

and

$$\mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\})\approx\|f\|_{\mathcal{P}_{p(\cdot)}}.$$

Combining Lemma 4.4 and Lemma 4.5, we can prove Theorem 4.6. Indeed, since $q_-\geqslant 1$, we have

$$\begin{aligned}\|I_\alpha f\|_{\mathcal{P}_{q(\cdot)}}&=\biggl\|\sum_{k\in\mathbb{Z}}\mu_k I_\alpha a^k\biggr\|_{\mathcal{P}_{q(\cdot)}}\leqslant\sum_{k\in\mathbb{Z}}\mu_k\|I_\alpha a^k\|_{\mathcal{P}_{q(\cdot)}}\\&\lesssim C_\alpha\sum_{k\in\mathbb{Z}}\mu_k C_\alpha\mathcal{A}(\{\mu_k\},\{a^k\},\{\tau_k\})\lesssim C_\alpha\|f\|_{\mathcal{P}_{p(\cdot)}}.\end{aligned}$$

$\square$

Remark 4.7. In [13], the atomic decomposition of $H^s_{p(\cdot)}$ is established; but we do not know whether there is a version similar to Lemma 4.3.

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