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A NOTE ON INFINITE aS-GROUPS

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Abstract. Let G be a group. If every nontrivial subgroup of G has a proper supplement, then G is called an aS-group. We study some properties of aS-groups. For instance, it is shown that a nilpotent group G is an aS-group if and only if G is a subdirect product of cyclic groups of prime orders. We prove that if G is an aS-group which satisfies the descending chain condition on subgroups, then G is finite. Among other results, we characterize all abelian groups for which every nontrivial quotient group is an aS-group. Finally, it is shown that if G is an aS-group and $|G| \neq pq, p$, where p and q are primes, then G has a triple factorization.

Keywords: infinite aS-group; supplemented subgroup; nilpotent group

MSC 2010: 20E15, 20E34, 20F18

1. Introduction

A subgroup K of a group G is said to be *supplemented* in G if there exists a subgroup L of G such that $G = KL$. In this case, L is called a *supplement* of K in G. If $K \cap L = \{1\}$, then K is *complemented* in G and L is called a *complement* of K in G. If every nontrivial subgroup of G has a proper supplement (complement), then G is called an *aS-group* (*aC-group*). Both aS- and aC-groups have been investigated by Kappe and Kirtland in [8]. Since the study of supplementation (complementation) in a group can reveal important properties about the structure of the group, this field has received a good deal of attention from several authors (for example see [3], [6] and [8]). In this paper, we investigate infinite aS-groups. For instance, we investigate nilpotent aS-groups. We also characterize abelian groups such that every nontrivial quotient group is an aS-group.

In this paper, the set of all maximal subgroups of a group G is denoted by Max(G). The intersection of all maximal subgroups of a group G, called the *Frattini subgroup* of G, is denoted by $\Phi(G)$. If G has no maximal subgroups, then $\Phi(G) = G$. If K

is a subgroup of G, then we write $K \leqslant G$. By G' we mean the derived subgroup of G. A *subdirect product* is a subgroup G of a direct product $\prod_{i=1}^{n} G_i$ such that every induced projection is surjective. We say G has a *triple factorization* if there exist three proper and nontrivial subgroups H, K and L such that $HK = HL = KL = G$. For more information on triple factorizations see [1] and [9].

2. Nilpotent aS-Groups

Kappe and Kirtland in [8] showed that the class of finite aS-groups coincides with the class of finite aC-groups, which were characterized by Hall [6]. We start with the following result. It should be noted that it was shown in [8] that every subgroup and homomorphic image of an aS-group is also an aS-group.

Theorem 1. *Let G be a finite group. Then the following statements are equivalent:*

(i) *G is an aS-group.*
(ii) *G is an aC-group.*
(iii) *Every cyclic subgroup of prime order of G is complemented in G.*
(iv) *G is supersolvable and its Sylow subgroups are all elementary abelian.*
(v) *G is isomorphic with a subgroup of a direct product of groups of squarefree order.*

Proof. The result follows from Corollary 3.7 in [8], Corollary 2 in [3], and Theorems 1 and 2 in [6]. □

Kappe and Kirtland in [8] have shown that there exists a group such that $\Phi(G) = 1$ but G is not an aS-group. In the next theorem, we show that they are equivalent for a nilpotent group.

Theorem 2. *Let G be a nilpotent group. Then the following statements are equivalent:*

(i) *G is an aS-group.*
(ii) $\Phi(G) = 1$.
(iii) *G is the subdirect product of a family of cyclic groups of prime orders.*

Proof.

(i) $\Rightarrow$ (ii) It follows from Proposition 3.4 in [8].

(ii) $\Rightarrow$ (iii) First, we claim that G is an abelian group. Since G is nilpotent, we have $G' \subseteq \Phi(G) = \{1\}$ and so G is abelian. Consider the homeomorphism $\psi\colon G \to \prod G/\mathfrak{m}_i$, where $\mathfrak{m}_i \in \mathrm{Max}(G)$. For every $\mathfrak{m}_i \in$ Max(G) the quotient group

$G/\mathfrak{m}_i$ is cyclic of prime order and $\ker(\psi) = \Phi(\mathrm{G}) = \{1\}$. Thus one can see that G is isomorphic to a subgroup of a direct product of cyclic groups of prime orders. The direct product of cyclic groups of prime orders as a $\mathbb{Z}$-module is semisimple. Therefore, G is isomorphic to a subgroup of direct product of cyclic groups, as every submodule of semisimple module is semisimple. Thus G is a subgroup of the direct product of cyclic groups of prime orders. It is not hard to see that G is a subdirect product of cyclic groups of prime orders.

(iii) $\Rightarrow$ (i) Suppose that G is a subdirect product of cyclic groups of prime orders. Thus G is a subgroup of the direct product of cyclic groups of prime orders and G is semisimple as a $\mathbb{Z}$-module. Hence G is an aS-group. $\square$

The classification of $aPNS$-groups (groups in which every nontrivial subgroup has a proper normal supplement) was done by Kappe and Kirtland in [8]. Now, by Theorem 2, we have the following theorem.

Theorem 3. *Let G be a nilpotent group. Then the following statements are equivalent:*

(i) *G is an aS-group.*

(ii) *G is an $aPNS$-group.*

(iii) *G is the subdirect product of a family of cyclic groups of prime orders.*

(iv) *G is abelian with $\bigcap\limits_{p\in\pi} G^p = \{1\}$, where π is the set of all primes.*

(iv) $\Phi(G) = 1$.

3. Characterization of aS-groups

First we show that every aS-group has a maximal subgroup of finite index.

Theorem 4. *Let G be an aS-group. Then every subgroup has a supplement which is a maximal subgroup of finite index.*

Proof. Let $g \in G$. It is enough to show that the subgroup $\langle g\rangle$ has a supplement which is a maximal subgroup of finite index. Since G is an aS-group, there exists a nontrivial subgroup H such that $\langle g\rangle H = G$. Set $\sum = \{K < G\colon\ H \leqslant K,\ g \notin K\}$. By Zorn's Lemma, $\sum$ has a maximal element, say $\mathfrak{m}$. First, we prove that $\mathfrak{m}$ is a maximal subgroup of G. Assume that there exists a subgroup $\mathfrak{n}$ such that $\mathfrak{m} \subsetneqq \mathfrak{n}$. The equality $\langle g\rangle H = G$ implies that $\mathfrak{n} = G$ and hence $\mathfrak{m}$ is a maximal subgroup.

We now show that $\mathfrak{m}$ is of finite index. With no loss of generality, we can assume that $\langle g\rangle$ is a torsion-free group. Now, consider $\langle g^2\rangle$. Thus, there is a maximal subgroup $\mathfrak{m}$ such that $\langle g^2\rangle\mathfrak{m} = G$. Since $\langle g^2\rangle \subseteq \langle g\rangle$, we have $\langle g\rangle\mathfrak{m} = \langle g^2\rangle\mathfrak{m} = G$.

Hence, there are $n \in \mathbb{Z}$ and $m \in \mathfrak{m}$ such that $g = g^{2n}m$ and so $g^{1-2n} \in \mathfrak{m}$. Since $g^{1-2n} \in \mathfrak{m}$, we deduce that $\mathfrak{m}$ is a maximal subgroup of finite index. $\square$

The following result shows that no aS-group which is not $\mathbb{Z}_p$, where p is prime, is simple.

Corollary 5. *Let G be an aS-group and $G \ncong \mathbb{Z}_p$, where p is prime. Then G is not a simple group.*

P r o o f. If G is finite, then by Theorem 1, G is a supersolvable group. So G is not simple. Now, let G be infinite. It follows from Theorem 4 that G has a maximal subgroup of finite index. So G has a normal subgroup and G is not simple, as desired. $\square$

It was shown in [8] that if G is an aS-group which satisfies the descending chain condition on subgroups, then G is an aC-group. Here, we show that G is indeed a finite group.

Theorem 6. *If G is an aS-group which satisfies the descending chain condition on subgroups, then G is finite.*

P r o o f. It follows from Theorem 4 that G has a (maximal) subgroup H of finite index whose supplement is a maximal subgroup, say $\mathfrak{m}_1$, such that $[G : \mathfrak{m}_1] < \infty$. If H is finite, then there is nothing to prove. Hence we may assume that H is infinite. Since $H\mathfrak{m}_1 = G$ and $[G : \mathfrak{m}_1] < \infty$, we conclude that $H \cap \mathfrak{m}_1$ is not trivial. If $H \cap \mathfrak{m}_1$ is a finite subgroup, then the Poincaré inequality (see [10], 1.7.10) implies that $[G : H \cap \mathfrak{m}_1] < \infty$ and so G is finite. Thus, we can assume that $H \cap \mathfrak{m}_1$ is infinite. By Theorem 4, $H \cap \mathfrak{m}_1$ has a supplement $\mathfrak{m}_2$ of finite index. Again, we should have $|H \cap \mathfrak{m}_1 \cap \mathfrak{m}_2| > 1$. Now, if $H \cap \mathfrak{m}_1 \cap \mathfrak{m}_2$ is finite, then the Poincaré inequality implies that $[G : H \cap \mathfrak{m}_1 \cap \mathfrak{m}_2] < \infty$ and so G is finite. By repeating this procedure, if $H \cap \bigcap\limits_{i=1}^{i=k} \mathfrak{m}_i$ is not a finite subgroup, for every $k \in \mathbb{N}$, we have a non-stopping descending chain of subgroups $\Big\{H \cap \bigcap\limits_{i \in \mathbb{N}} \mathfrak{m}_i\Big\}$. Since G satisfies the descending chain condition on subgroups, we get a contradiction. Hence, there is $k \in \mathbb{N}$ such that $H \cap \bigcap\limits_{i=1}^{i=k} \mathfrak{m}_i$ is finite. By the Poincaré inequality, one can see that $\Big[G : H \cap \bigcap\limits_{i=1}^{i=k} \mathfrak{m}_i\Big] < \infty$ and so G is finite, as desired. $\square$

Let G be an aS-group which satisfies the descending chain condition on subgroups. Then by Theorem 6, G is finite. Now, it follows from Theorem 1 that G is a supersolvable group. It would be interesting if one could prove the converse of this result.

Theorem 7. *If G is an aS-group and G' satisfies the descending chain condition on its subgroups, then G is solvable.*

P r o o f. If G' is a finite subgroup, then by Proposition 3.5 of [8] and Theorem 1, G' is supersolvable. Since every finite supersolvable group is solvable, G' is solvable and so G is solvable. Thus, we can suppose that G' is an infinite subgroup. By Theorem 4, G' has a supplement of finite index, say $\mathfrak{m}_1$. Since $[G : \mathfrak{m}_1] < \infty$ and G' is infinite, the subgroup $G' \cap \mathfrak{m}_1$ is nontrivial. If $|G' \cap \mathfrak{m}_1| < \infty$, then $[G' : G' \cap \mathfrak{m}_1] = [G : \mathfrak{m}_1] < \infty$. This implies that G' is finite. Hence, we can assume that $G' \cap \mathfrak{m}_1$ is an infinite subgroup. There is a supplement of finite index for $G' \cap \mathfrak{m}_1$. By a similar argument to that of Theorem 6, one concludes that G is solvable. $\square$

Next, we find a class of aS-groups such that every element of this class contains a normal maximal subgroup.

Theorem 8. *Let G be a torsion aS-group. Then G contains a normal maximal subgroup.*

P r o o f. Let p be the smallest prime integer such that $a^p = 1$, where a runs over all elements of G. Fix the element a and consider the subgroup $\langle a \rangle$. There exists a subgroup $\mathfrak{m}$ such that $\langle a \rangle \mathfrak{m} = G$. We will show that $\mathfrak{m}$ is a normal maximal subgroup of G. First we prove that $[G : \mathfrak{m}] = p$. Clearly, $[G : \mathfrak{m}] \leqslant p$. If $a^i \mathfrak{m} = a^j \mathfrak{m}$, then $a^{i-j} \in \langle a \rangle \cap \mathfrak{m}$, a contradiction. Thus $[G : \mathfrak{m}] = p$ and so $\mathfrak{m}$ is maximal. We claim that if $x \notin \mathfrak{m}$, then $x^i \notin \mathfrak{m}$ for $i = 1, \ldots, p-1$. Assume that $o(x) = l$ and let t be the smallest integer such that $x^t \in \mathfrak{m}$. Let $l = tq + r$ for integers t and q with $0 \leqslant r < t$. If $r \neq 0$, then the equality $1 = x^{tq} x^r$ implies that $x^r \in \mathfrak{m}$ which contradicts the minimality of t. Hence $r = 0$ and $l = tq$. Thus $(x^{l/t})^t = 1$, a contradiction and so the claim is proved.

If $\mathfrak{m}$ is not normal in G, then $y = x^{-1} b x \notin \mathfrak{m}$ for some $x \in G$ and $b \in \mathfrak{m}$. Since $x^i \notin \mathfrak{m}$ and $y^i \notin \mathfrak{m}$ for $i = 1, \ldots, p-1$, we conclude that $\mathfrak{m} x = \mathfrak{m} y^j$ for some j, $1 \leqslant j \leqslant p-1$. Thus $x = m x^{-1} b^j x$ for some $m \in \mathfrak{m}$, and hence $x \in \mathfrak{m}$, which is a contradiction. Therefore, $\mathfrak{m}$ is normal maximal, as desired. $\square$

Factorizable groups have been intensively studied. Both finite and infinite groups have been the subject of many investigations. An important class of factorizable groups are groups which have a triple factorization. For more information on factorizable groups and triple factorization we refer the reader to [1], [2], [4], [7] and [9].

The next theorem states that every aS-group G for which $|G| \neq pq$ or $|G| \neq p$, where p, q are prime numbers, has a triple factorization.

Theorem 9. *Let G be an aS-group such that $|G| \neq pq$ or $|G| \neq p$ where p, q are prime numbers. Then G has a triple factorization.*

P r o o f. First, assume that G is finite. We continue the proof in the following two cases:

Case (i). There exist three distinct prime numbers p, q, r such that $p, q, r \mid |G|$. Since G is an aS-group, there exist subgroups $\mathfrak{m}_1$, $\mathfrak{m}_2$ and $\mathfrak{m}_3$ of indices p, q, r, respectively. It follows from Theorem 1.3.13 in [11] that $G = \mathfrak{m}_1\mathfrak{m}_2 = \mathfrak{m}_2\mathfrak{m}_3 = \mathfrak{m}_3\mathfrak{m}_1$.

Case (ii). $|G| = p^n q^m$, where p and q are prime numbers and $n, m \in \mathbb{N} \cup \{0\}$. Let p be the smallest prime number which divides $|G|$, and $o(g) = p$ for some $g \in G$. Since G is an aS-group, there exists a maximal subgroup $\mathfrak{m}$ such that $\langle g\rangle\mathfrak{m} = G$. Note that $\mathfrak{m}$ is a normal subgroup. If there exists $\mathfrak{m}_i \in \mathrm{Max}(G)$ such that $\mathfrak{m} \cap \mathfrak{m}_i \neq 1$, then we are done, as there exists a subgroup L such that $L(\mathfrak{m} \cap \mathfrak{m}_i) = G$. Since $\mathfrak{m} \cap \mathfrak{m}_i \subseteq \mathfrak{m}_i$, we have $L\mathfrak{m}_i = G$ and so we find the factorization $\mathfrak{m}_i\mathfrak{m} = \mathfrak{m}_i L = \mathfrak{m} L = G$. Thus, we can suppose that $\mathfrak{m} \cap \mathfrak{m}_i = 1$ for each $\mathfrak{m}_i \in \mathrm{Max}(G)$. Let $\mathfrak{m}_1$ be a subgroup of index q. By Theorem 1.3.13 in [11], we have $\mathfrak{m}_1\mathfrak{m} = G$ and so $|G| = pq$.

Now, suppose that G is an infinite group. If $g \in G$ is a torsion free element, then by Theorem 4 there exists a subgroup L of finite index such that $\langle g\rangle L = G$. Therefore, we have $\langle g\rangle \cap L \neq 1$, as g is torsion free and $[G : L] < \infty$. Again, there exists a subgroup K such that $K(\langle g\rangle \cap L) = G$ and we can consider the factorization $\langle g\rangle K = \langle g\rangle L = LK = G$. So we can assume that G is a torsion group. It follows from Corollary 3.4 in [8] that G has a maximal subgroup $\mathfrak{m}$. By Theorem 4, $\mathfrak{m}$ has a supplement K of finite index. Since $\mathfrak{m}$ is infinite, $\mathfrak{m} \cap K \neq 1$. By a similar argument we can show that G has a triple factorization. □

We close this paper with the following result which shows that if G is an abelian group with G/H an aS-group for every nontrivial subgroup H of G, then G itself is an aS-group.

Theorem 10. *For every nontrivial subgroup H of an abelian group G, G/H is an aS-group if and only if G is isomorphic to a direct sum of cyclic groups of prime orders.*

P r o o f. First suppose that for every nontrivial subgroup H of a group G the group G/H is an aS-group. We claim that G is a torsion group. If $G \cong \mathbb{Z}$, we get a contradiction since $\mathbb{Z}_{12}$ is not an aS-group. Thus $G \ncong \mathbb{Z}$. If $\mathbb{Z} \leqslant G$, then by the assumption, $G/12\mathbb{Z}$ is an aS-group. Thus Proposition 3.5 in [8] implies that $\mathbb{Z}/12\mathbb{Z}$ is an aS-group, a contradiction and so the claim is proved. Now, by the primary decomposition (see Theorem 8.4 in [5]), $G \cong \bigoplus A_p$, where A_p is a p-group. Suppose that G is a p-group. Let $K = \langle k\rangle$ be a subgroup of G of order p^i, $i \geqslant 4$ and H a subgroup of K of order p. Since G/H is an aS-group, by Proposition 3.5 in [8] we deduce that K/H is an aS-group, a contradiction. So, the order of each element of G is at most p^3. By the Baer-Prüfer Theorem (see Theorem 17.2 in [5]), G is a direct sum of cyclic groups. If $\mathbb{Z}_{p^3}$ is a direct summand of G, then there exists a subgroup $\langle g\rangle$ such that $G/L \cong \mathbb{Z}_{p^3}$, a contradiction. Hence G is a direct sum of cyclic groups of order at most p^2 and so G is isomorphic to one of the following groups: $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$, $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_p$, or G has at least three summands. If G has at least three summands, then either $G \cong \bigoplus \mathbb{Z}_p$ or there exists a subgroup $\langle g\rangle$ such that either $G/L \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ or $G/L \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$. If $G \cong \bigoplus \mathbb{Z}_p$, then we are done. If not, then we get a contradiction. Now, suppose that $G \cong \bigoplus A_p$, with at least two summands. By a similar argument, one can show that G is a direct sum of cyclic groups of prime orders. The converse is clear. $\square$

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