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n-ANGULATED QUOTIENT CATEGORIES INDUCED BY MUTATION PAIRS

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Abstract. Geiss, Keller and Oppermann (2013) introduced the notion of n-angulated category, which is a "higher dimensional" analogue of triangulated category, and showed that certain $(n-2)$-cluster tilting subcategories of triangulated categories give rise to n-angulated categories. We define mutation pairs in n-angulated categories and prove that given such a mutation pair, the corresponding quotient category carries a natural n-angulated structure. This result generalizes a theorem of Iyama-Yoshino (2008) for triangulated categories.

Keywords: n-angulated category; quotient category; mutation pair

MSC 2010: 18E30

1. Introduction

Triangulated categories were introduced by Grothendieck, Verdier [10] and Puppe [9] independently to axiomatize the properties of derived categories and stable homotopy categories, respectively. Triangulated categories are very important both in geometry and algebra.

Geiss, Keller and Oppermann [4] introduced the notion of n-angulated category, which is a "higher dimensional" analogue of triangulated category, and showed that certain $(n-2)$-cluster tilting subcategories of triangulated categories give rise to n-angulated categories. For $n = 3$, n-angulated categories are nothing but triangulated categories. Nowadays the theory of n-angulated categories has been developed further. Bergh and Thaule discussed the axioms for n-angulated categories [3]. They introduced a higher "octahedral axiom" and showed that it is equivalent to the map-

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ping cone axiom. Other examples of n-angulated categories from local rings were given in [2]. The notion of Grothendieck group of an n-angulated category was introduced to give a higher analogue of Thomason's classification theorem for triangulated subcategories (see [1]). Recently, Jasso introduced n-abelian and n-exact categories, and showed that the quotient category of a Frobenius n-exact category has a natural structure of an $(n+2)$-angulated category ([7], Theorem 5.11), which is a higher analogue of a result of Happel ([5], Theorem 2.6).

The aim of this paper is to discuss a construction of n-angulated categories. We define mutation pairs in n-angulated categories and prove that given such a mutation pair, the corresponding quotient category carries a natural n-angulated structure. For $n = 3$, our main result recovers a result of Iyama-Yoshino ([6], Theorem 4.2).

The paper is organized as follows. In Section 2, we recall the definition of an n-angulated category and give some useful facts. In Section 3, we define mutation pairs in n-angulated categories, then state and prove our main results. In Section 4, we give an example to illustrate our main result.

2. n-angulated categories

In this section we recall some basics on n-angulated categories from [3] and [4].

Let $\mathcal{C}$ be an additive category equipped with an automorphism $\Sigma\colon \mathcal{C}\to\mathcal{C}$, and n an integer greater than or equal to three. An *n-Σ-sequence* in $\mathcal{C}$ is a sequence of morphisms

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1.$$

Its *left rotation* is the n-Σ-sequence

$$X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n\Sigma f_1} \Sigma X_2.$$

We can define *right rotation of an n-Σ-sequence* similarly. A *morphism of n-Σ-sequences* is a sequence of morphisms $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ such that the diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3} & & & & \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\Sigma\varphi_1} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1
\end{array}$$

commutes where each row is an n-Σ-sequence. It is an *isomorphism* if $\varphi_1, \varphi_2, \ldots, \varphi_n$ are all isomorphisms in $\mathcal{C}$.

Definition 2.1 ([4]). An *n-angulated category* is a triple $(\mathcal{C}, \Sigma, \Theta)$, where $\mathcal{C}$ is an additive category, Σ is an automorphism of $\mathcal{C}$, and Θ is a class of n-Σ-sequences (whose elements are called *n-angles*), which satisfies the following axioms:

(N1) (a) The class Θ is closed under direct sums and direct summands.

(b) For each object $X \in \mathcal{C}$ the trivial sequence

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Sigma X$$

belongs to Θ.

(c) For each morphism $f_1\colon X_1 \to X_2$ in $\mathcal{C}$, there exists an n-angle whose first morphism is f_1.

(N2) An n-Σ-sequence belongs to Θ if and only if its left rotation belongs to Θ.

(N3) Each commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1
\end{array}$$

with rows in Θ can be completed to a morphism of n-Σ-sequences.

(N4) In the situation of (N3), the morphisms $\varphi_3, \varphi_4, \ldots, \varphi_n$ can be chosen such that the mapping cone

$$X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} \Sigma X_1 \oplus Y_n \xrightarrow{\begin{pmatrix} -\Sigma f_1 & 0 \\ \Sigma\varphi_1 & g_n \end{pmatrix}} \Sigma X_2 \oplus \Sigma Y_1$$

belongs to Θ.

We recall a higher "octahedral axiom" (N4$'$) for an n-angulated category as follows, which was introduced by Bergh and Thaule [3].

(N4$'$) Given a commutative diagram

$$\begin{array}{ccccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\Vert & & \downarrow{\scriptstyle \varphi_2} & & & & & & & & & & \Vert \\
X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1 \\
 & & \downarrow{\scriptstyle h_2} & & & & & & & & & & \\
 & & Z_3 & & & & & & & & & & \\
 & & \downarrow{\scriptstyle h_3} & & & & & & & & & & \\
 & & \vdots & & & & & & & & & & \\
 & & \downarrow{\scriptstyle h_{n-1}} & & & & & & & & & & \\
 & & Z_n & & & & & & & & & & \\
 & & \downarrow{\scriptstyle h_n} & & & & & & & & & & \\
 & & \Sigma X_2 & & & & & & & & & &
\end{array}$$

whose top rows and second column are n-angles. Then there exist morphisms $\varphi_i\colon X_i \to Y_i$ $(i = 3, 4, \ldots, n)$, $\psi_j\colon Y_j \to Z_j$ $(j = 3, 4, \ldots, n)$ and $\phi_k\colon X_k \to Z_{k-1}$ $(k = 4, 5, \ldots, n)$ with the following two properties:

(a) The sequence $(1_{X_1}, \varphi_2, \varphi_3, \ldots, \varphi_n)$ is a morphism of n-angles.

(b) The n-Σ-sequence

$$(2.1)\quad X_3 \xrightarrow{\begin{pmatrix} f_3 \\ \varphi_3 \end{pmatrix}} X_4 \oplus Y_3 \xrightarrow{\begin{pmatrix} -f_4 & 0 \\ \varphi_4 & -g_3 \\ \phi_4 & \psi_3 \end{pmatrix}} X_5 \oplus Y_4 \oplus Z_3 \xrightarrow{\begin{pmatrix} -f_5 & 0 & 0 \\ -\varphi_5 & -g_4 & 0 \\ \phi_5 & \psi_4 & h_3 \end{pmatrix}} X_6 \oplus Y_5 \oplus Z_4$$

$$\xrightarrow{\begin{pmatrix} -f_6 & 0 & 0 \\ \varphi_6 & -g_5 & 0 \\ \phi_6 & \psi_5 & h_4 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 & 0 \\ (-1)^{n-1}\varphi_{n-1} & -g_{n-2} & 0 \\ \phi_{n-1} & \psi_{n-2} & h_{n-3} \end{pmatrix}} X_n \oplus Y_{n-1} \oplus Z_{n-2}$$

$$\xrightarrow{\begin{pmatrix} (-1)^n\varphi_n & -g_{n-1} & 0 \\ \phi_n & \psi_{n-1} & h_{n-2} \end{pmatrix}} Y_n \oplus Z_{n-1} \xrightarrow{(\psi_n\ h_{n-1})} Z_n \xrightarrow{\Sigma f_2 \cdot h_n} \Sigma X_3$$

is an n-angle, and $h_n \cdot \psi_n = \Sigma f_1 \cdot g_n$.

Theorem 2.2 ([3], Theorem 4.4). *If Θ is a class of n-Σ-sequences satisfying the axioms* (N1), (N2) *and* (N3), *then Θ satisfies* (N4) *if and only if Θ satisfies* (N4′).

To conclude this section, we give three useful facts on n-angulated categories.

Lemma 2.3 ([4], Remarks 2.2 (c)). *If $(\mathcal{C}, \Sigma, \Theta)$ is an n-angulated category, then the opposite category $(\mathcal{C}^{\mathrm{op}}, \Sigma^{-1}, \Phi)$ is also an n-angulated category, where*

$$\Sigma^{-1}X_n \xleftarrow{(-1)^n\Sigma^{-1}f_n} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots \xleftarrow{f_{n-2}} X_{n-1} \xleftarrow{f_{n-1}} X_n$$

is an n-angle in Φ provided that

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

is an n-angle in Θ.

Lemma 2.4. *Let*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

be an n-angle in an n-angulated category $\mathcal{C}$. Then

(1) *the induced sequence*

$$\cdots \longrightarrow \mathcal{C}(-, X_1) \longrightarrow \mathcal{C}(-, X_2) \longrightarrow \cdots \longrightarrow \mathcal{C}(-, X_n) \longrightarrow \mathcal{C}(-, \Sigma X_1) \longrightarrow \cdots$$

is exact;

(2) *the induced sequence*

$$\cdots \longrightarrow \mathcal{C}(\Sigma X_1, -) \longrightarrow \mathcal{C}(X_n, -) \longrightarrow \cdots \longrightarrow \mathcal{C}(X_2, -) \longrightarrow \mathcal{C}(X_1, -) \longrightarrow \cdots$$

is exact.

Proof. (1) We refer to [4], Proposition 2.5 (a).
(2) Follows from (1) and Lemma 2.3. □

The following lemma is due to [3], Lemma 4.1.

Lemma 2.5. *Let* $(\mathcal{C}, \Sigma, \Theta)$ *be an* n*-angulated category. Then each commutative diagram*

$$\begin{array}{ccccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \| & & \downarrow{\scriptstyle \Sigma\varphi_1} \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & X_n & \xrightarrow{g_n} & \Sigma Y_1 \end{array}$$

with rows in Θ *can be completed to a morphism of* n*-*Σ*-sequences such that*

$$X_1 \xrightarrow{\begin{pmatrix} -f_1 \\ \varphi_1 \end{pmatrix}} X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} f_3 & 0 \\ -\varphi_3 & g_2 \end{pmatrix}} \cdots$$

$$\cdots \xrightarrow{\begin{pmatrix} f_{n-2} & 0 \\ (-1)^n \varphi_{n-2} & g_{n-3} \end{pmatrix}} X_{n-1} \oplus Y_{n-2} \xrightarrow{((-1)^{n+1}\varphi_{n-1}, g_{n-2})} Y_{n-1} \xrightarrow{(-1)^n f_n g_{n-1}} \Sigma X_1$$

is an n*-angle.*

3. Main results

Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a subcategory of $\mathcal{C}$. When we say that $\mathcal{D}$ is a subcategory of $\mathcal{C}$, we always mean that $\mathcal{D}$ is full and closed under isomorphisms, direct sums and direct summands. A morphism $f\colon X \to Y$ in $\mathcal{C}$ is called *$\mathcal{D}$-monic* if $\mathcal{C}(Y,D) \xrightarrow{\mathcal{C}(f,D)} \mathcal{C}(X,D) \longrightarrow 0$ is exact for any object $D \in \mathcal{D}$. A morphism $f\colon X \to D$ in $\mathcal{C}$ is called a *left $\mathcal{D}$-approximation of* X if f is $\mathcal{D}$-monic and $D \in \mathcal{D}$. We can define *$\mathcal{D}$-epic* morphism and *right $\mathcal{D}$-approximation* dually.

Definition 3.1. Let $\mathcal{C}$ be an n-angulated category, and let $\mathcal{D}$ and $\mathcal{Z}$ are subcategories of $\mathcal{C}$ with $\mathcal{D} \subseteq \mathcal{Z}$. The pair $(\mathcal{Z},\mathcal{Z})$ is called a *$\mathcal{D}$-mutation pair* if it satisfies:

(1) For any object $X \in \mathcal{Z}$, there exists an n-angle

$$X \xrightarrow{d_1} D_1 \xrightarrow{d_2} D_2 \xrightarrow{d_3} \cdots \xrightarrow{d_{n-2}} D_{n-2} \xrightarrow{d_{n-1}} Y \xrightarrow{d_n} \Sigma X$$

where $D_i \in \mathcal{D}$, $Y \in \mathcal{Z}$, d_1 is a left $\mathcal{D}$-approximation and d_{n-1} is a right $\mathcal{D}$-approximation.

(2) For any object $Y \in \mathcal{Z}$, there exists an n-angle

$$X \xrightarrow{d_1} D_1 \xrightarrow{d_2} D_2 \xrightarrow{d_3} \cdots \xrightarrow{d_{n-2}} D_{n-2} \xrightarrow{d_{n-1}} Y \xrightarrow{d_n} \Sigma X$$

where $X \in \mathcal{Z}$, $D_i \in \mathcal{D}$, d_1 is a left $\mathcal{D}$-approximation and d_{n-1} is a right $\mathcal{D}$-approximation.

Note that if $\mathcal{C}$ is a triangulated category, i.e., when $n = 3$, our definition of a mutation pair is weaker than Iyama-Yoshino's definition ([6], Definition 2.5), since we do not required $\mathcal{D}$ to be a rigid subcategory.

For a $\mathcal{D}$-mutation pair $(\mathcal{Z},\mathcal{Z})$ in an n-angulated category $\mathcal{C}$, consider the quotient category $\mathcal{Z}/[\mathcal{D}]$ whose objects are objects of $\mathcal{Z}$ and given two objects X, Y, the set of morphisms $(\mathcal{Z}/[\mathcal{D}])(X,Y)$ is defined as the quotient group $\mathcal{Z}(X,Y)/[\mathcal{D}](X,Y)$, where $[\mathcal{D}](X,Y)$ is the subgroup of morphisms from X to Y factoring through some object in $\mathcal{D}$. For any morphism $f\colon X \to Y$ in $\mathcal{Z}$, we denote by $\underline{f}$ the image of f under the quotient functor $\mathcal{Z} \to \mathcal{Z}/[\mathcal{D}]$.

Lemma 3.2. *Let*

$$\begin{array}{ccccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle \Sigma a_1} \\
X & \xrightarrow{d_1} & D_1 & \xrightarrow{d_2} & D_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-2} & \xrightarrow{d_{n-1}} & Y & \xrightarrow{d_n} & \Sigma X,
\end{array}$$

and

$$
\begin{array}{ccccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a'_2} & & \downarrow{\scriptstyle a'_3} & & & & \downarrow{\scriptstyle a'_{n-1}} & & \downarrow{\scriptstyle a'_n} & & \downarrow{\scriptstyle \Sigma a_1} \\
X & \xrightarrow{d_1} & D_1 & \xrightarrow{d_2} & D_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-2} & \xrightarrow{d_{n-1}} & Y & \xrightarrow{d_n} & \Sigma X
\end{array}
$$

be morphisms of n-angles, where $X, Y, X_i \in \mathcal{Z}$ and $D_j \in \mathcal{D}$. Then $\underline{a_n} = \underline{a'_n}$ in the quotient category $\mathcal{Z}/[\mathcal{D}]$.

Proof. Since $d_n a_n = \Sigma a_1 \cdot f_n = d_n a'_n$, we have $d_n(a_n - a'_n) = 0$. By Lemma 2.4 (1) we obtain that $a_n - a'_n$ factors through d_{n-1}, thus $\underline{a_n} = \underline{a'_n}$. □

Now we construct a functor $T\colon \mathcal{Z}/[\mathcal{D}] \to \mathcal{Z}/[\mathcal{D}]$ as follows. For any object $X \in \mathcal{Z}$, fix an n-angle

$$X \xrightarrow{d_1} D_1 \xrightarrow{d_2} D_2 \xrightarrow{d_3} \cdots \xrightarrow{d_{n-2}} D_{n-2} \xrightarrow{d_{n-1}} TX \xrightarrow{d_n} \Sigma X$$

with $D_i \in \mathcal{D}$, $TX \in \mathcal{Z}$, d_1 is a left $\mathcal{D}$-approximation and d_{n-1} is a right $\mathcal{D}$-approximation. For any morphism $f \in \mathcal{Z}(X, X')$, since d_1 is a left $\mathcal{D}$-approximation, there exist morphisms a_i and g which make the following diagram commutative:

$$
\begin{array}{ccccccccccccc}
X & \xrightarrow{d_1} & D_1 & \xrightarrow{d_2} & D_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-2} & \xrightarrow{d_{n-1}} & TX & \xrightarrow{d_n} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} & & & & \downarrow{\scriptstyle a_{n-2}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow{d'_1} & D'_1 & \xrightarrow{d'_2} & D'_2 & \xrightarrow{d'_3} & \cdots & \xrightarrow{d'_{n-2}} & D'_{n-2} & \xrightarrow{d'_{n-1}} & TX' & \xrightarrow{d'_n} & \Sigma X'.
\end{array}
$$

Now we put $T\underline{f} = \underline{g}$. Note that $\Sigma f \cdot d_n = d'_n \cdot g$, which will be used frequently.

Proposition 3.3. *The functor $T\colon \mathcal{Z}/[\mathcal{D}] \to \mathcal{Z}/[\mathcal{D}]$ is a well defined equivalence.*

Proof. By Lemma 3.2, it is easy to see that T is a well defined functor. We can construct another functor $T'\colon \mathcal{Z}/[\mathcal{D}] \to \mathcal{Z}/[\mathcal{D}]$ in a dual manner. For any object $X \in \mathcal{Z}$, fix an n-angle

$$T'X \xrightarrow{d_1} D_1 \xrightarrow{d_2} D_2 \xrightarrow{d_3} \cdots \xrightarrow{d_{n-2}} D_{n-2} \xrightarrow{d_{n-1}} X \xrightarrow{d_n} \Sigma T'X$$

with $D_i \in \mathcal{D}$, $T'X \in \mathcal{Z}$, d_1 being a left $\mathcal{D}$-approximation and d_{n-1} a right $\mathcal{D}$-approximation. For any morphism $f \in \mathcal{Z}(X, X')$, since d'_{n-1} is a right $\mathcal{D}$-approximation, there exist morphisms b_i and g which make the following diagram commutative:

$$\begin{array}{ccccccccccccc}
T'X & \xrightarrow{d_1} & D_1 & \xrightarrow{d_2} & D_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-2} & \xrightarrow{d_{n-1}} & X & \xrightarrow{d_n} & \Sigma T'X \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle b_{n-2}} & & \downarrow{\scriptstyle b_{n-3}} & & & & \downarrow{\scriptstyle b_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Sigma g} \\
T'X' & \xrightarrow{d'_1} & D'_1 & \xrightarrow{d'_2} & D'_2 & \xrightarrow{d'_3} & \cdots & \xrightarrow{d'_{n-2}} & D'_{n-2} & \xrightarrow{d'_{n-1}} & X' & \xrightarrow{d'_n} & \Sigma T'X'.
\end{array}$$

We put $T'\underline{f} = \underline{g}$. The dual of Lemma 3.2 implies that T' is a well defined functor. It is easy to check that T' gives a quasi-inverse of T. □

Definition 3.4. Let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

be an n-angle, where $X_i \in \mathcal{Z}$ and f_1 is $\mathcal{D}$-monic. Then there exists a commutative diagram of n-angles

$$\begin{array}{ccccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle a_n} & & \| \\
X_1 & \xrightarrow{d_1} & D_1 & \xrightarrow{d_2} & D_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-2} & \xrightarrow{d_{n-1}} & TX_1 & \xrightarrow{d_n} & \Sigma X_1.
\end{array} \tag{3.1}$$

The n-T-sequence

$$X_1 \xrightarrow{\underline{f_1}} X_2 \xrightarrow{\underline{f_2}} X_3 \xrightarrow{\underline{f_3}} \cdots \xrightarrow{\underline{f_{n-1}}} X_n \xrightarrow{\underline{a_n}} TX_1$$

is called a *standard n-angle* in $\mathcal{Z}/[\mathcal{D}]$. We define Φ to be the class of n-T-sequences which are isomorphic to standard n-angles.

Lemma 3.5. *Assume that we have a commutative diagram*

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1
\end{array}$$

where the rows are n-angles in $\mathcal{C}$, all $X_i, Y_i \in \mathcal{Z}$ and f_1, g_1 are $\mathcal{D}$-monic. Then we have the commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\underline{f_1}} & X_2 & \xrightarrow{\underline{f_2}} & X_3 & \xrightarrow{\underline{f_3}} & \cdots & \xrightarrow{\underline{f_{n-1}}} & X_n & \xrightarrow{\underline{a_n}} & TX_1 \\
\downarrow{\scriptstyle \underline{\varphi_1}} & & \downarrow{\scriptstyle \underline{\varphi_2}} & & \downarrow{\scriptstyle \underline{\varphi_3}} & & & & \downarrow{\scriptstyle \underline{\varphi_n}} & & \downarrow{\scriptstyle T\underline{\varphi_1}} \\
Y_1 & \xrightarrow{\underline{g_1}} & Y_2 & \xrightarrow{\underline{g_2}} & Y_3 & \xrightarrow{\underline{g_3}} & \cdots & \xrightarrow{\underline{g_{n-1}}} & Y_n & \xrightarrow{\underline{b_n}} & TY_1
\end{array}$$

where the rows are standard n-angles in $\mathcal{Z}/[\mathcal{D}]$.

P r o o f. We only need to show that $T\underline{\varphi_1}\cdot\underline{a_n}=\underline{b_n}\cdot\underline{\varphi_n}$. By the constructions of the morphism $T\underline{\varphi_1}$ and the standard n-angles in $\mathcal{Z}/[\mathcal{D}]$, we have two commutative diagrams of n-angles

$$
\begin{array}{ccccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1\\
\| & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle a_n} & & \|\\
X_1 & \xrightarrow{d_1} & D_1 & \xrightarrow{d_2} & D_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-2} & \xrightarrow{d_{n-1}} & TX_1 & \xrightarrow{d_n} & \Sigma X_1\\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle a'_1} & & \downarrow{\scriptstyle a'_2} & & & & \downarrow{\scriptstyle a'_{n-2}} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \Sigma\varphi_1}\\
Y_1 & \xrightarrow{d'_1} & D'_1 & \xrightarrow{d'_2} & D'_2 & \xrightarrow{d'_3} & \cdots & \xrightarrow{d'_{n-2}} & D'_{n-2} & \xrightarrow{d'_{n-1}} & TY_1 & \xrightarrow{d'_n} & \Sigma Y_1,
\end{array}
$$

and

$$
\begin{array}{ccccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1\\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1}\\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1\\
\| & & \downarrow{\scriptstyle b_2} & & \downarrow{\scriptstyle b_3} & & & & \downarrow{\scriptstyle b_{n-1}} & & \downarrow{\scriptstyle b_n} & & \|\\
Y_1 & \xrightarrow{d'_1} & D'_1 & \xrightarrow{d'_2} & D'_2 & \xrightarrow{d'_3} & \cdots & \xrightarrow{d'_{n-2}} & D'_{n-2} & \xrightarrow{d'_{n-1}} & TY_1 & \xrightarrow{d'_n} & \Sigma Y_1
\end{array}
$$

where $T\underline{\varphi_1}=\underline{\psi_1}$. Lemma 3.2 implies that $T\underline{\varphi_1}\cdot\underline{a_n}=\underline{b_n}\cdot\underline{\varphi_n}$. $\square$

Definition 3.6. Let $\mathcal{C}$ be an n-angulated category. A subcategory $\mathcal{Z}$ is called *extension-closed* if for each morphism $f_n\colon X_n\to\Sigma X_1$ with $X_1, X_n\in\mathcal{Z}$, there exists an n-angle

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

with each $X_i\in\mathcal{Z}$.

It is easy to see that the definition of an extension-closed subcategory in an n-angulated category is the same as the classical definition in a triangulated category for $n=3$. By definition, each n-angulated subcategory is extension-closed for any $n\geqslant 3$. We can compare the definition of an extension-closed subcategory with the definition of a left extension-closed subcategory given by Jørgensen [8].

Now we can state and prove our main theorem.

Theorem 3.7. *Let $\mathcal{C}$ be an n-angulated category, and let $\mathcal{D}$ and $\mathcal{Z}$ are subcategories of $\mathcal{C}$ with $\mathcal{D} \subseteq \mathcal{Z}$. If $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair and $\mathcal{Z}$ is extension-closed, then the quotient category $\mathcal{Z}/[\mathcal{D}]$ is an n-angulated category with respect to the autoequivalence T and n-angles defined in Definition* 3.4.

Proof. We will check that the class of n-T-sequences Φ, which is defined in Definition 3.4, satisfies the axioms (N1), (N2), (N3) and (N4′). It is easy to see from the definition that (N1) (a) is satisfied.

For any object $X \in \mathcal{Z}$, the identity morphism of X is $\mathcal{D}$-monic. The commutative diagram

$$\begin{array}{ccccccccccccccc}
X & \xrightarrow{1_X} & X & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \Sigma X \\
\| & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \| \\
X & \xrightarrow{d_1} & D_1 & \xrightarrow{d_2} & D_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-2} & \xrightarrow{d_{n-1}} & TX & \xrightarrow{d_n} & \Sigma X
\end{array}$$

shows that

$$X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow TX$$

belongs to Φ. Thus (N1) (b) is satisfied.

For each morphism $f\colon X \to Y$ in $\mathcal{Z}$, let

$$X \xrightarrow{d_1} D_1 \xrightarrow{d_2} D_2 \xrightarrow{d_3} \cdots \xrightarrow{d_{n-2}} D_{n-2} \xrightarrow{d_{n-1}} TX \xrightarrow{d_n} \Sigma X$$

be the n-angle given by the mutation pair. Since the subcategory $\mathcal{Z}$ is extension-closed and $Y, TX \in \mathcal{Z}$, there exists an n-angle

$$Y \xrightarrow{f_1} Y_2 \xrightarrow{f_2} Y_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-2}} Y_{n-1} \xrightarrow{f_{n-1}} TX \xrightarrow{\Sigma f \cdot d_n} \Sigma Y$$

with each $Y_i \in \mathcal{Z}$. By Lemma 2.5, the commutative diagram

$$\begin{array}{ccccccccccccc}
X & \xrightarrow{d_1} & D_1 & \xrightarrow{d_2} & D_2 & \xrightarrow{d_3} & \cdots & \xrightarrow{d_{n-2}} & D_{n-2} & \xrightarrow{d_{n-1}} & TX & \xrightarrow{d_n} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \| & & \downarrow{\scriptstyle \Sigma f} \\
Y & \xrightarrow{f_1} & Y_2 & \xrightarrow{f_2} & Y_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & Y_{n-1} & \xrightarrow{f_{n-1}} & TX & \xrightarrow{\Sigma f \cdot d_n} & \Sigma Y
\end{array}$$

can be completed to a morphism of n-Σ-sequences. Moreover,

$$X \xrightarrow{\begin{pmatrix} -d_1 \\ f \end{pmatrix}} D_1 \oplus Y \xrightarrow{\begin{pmatrix} d_2 & 0 \\ \varphi_2 & f_1 \end{pmatrix}} D_2 \oplus Y_2 \xrightarrow{\begin{pmatrix} d_3 & 0 \\ -\varphi_3 & f_2 \end{pmatrix}} \cdots$$

$$\cdots \xrightarrow{\begin{pmatrix} d_{n-2} & 0 \\ (-1)^n \varphi_{n-2} & f_{n-3} \end{pmatrix}} D_{n-2} \oplus Y_{n-2} \xrightarrow{((-1)^{n+1}\varphi_{n-1}, f_{n-2})} Y_{n-1} \xrightarrow{(-1)^n d_n f_{n-1}} \Sigma X$$

is an n-angle, which induces an n-angle in $\mathcal{Z}/[\mathcal{D}]$

$$X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{f_1}} Y_2 \xrightarrow{\underline{f_2}} \cdots \xrightarrow{\underline{f_{n-3}}} Y_{n-2} \xrightarrow{\underline{f_{n-2}}} Y_{n-1} \xrightarrow{\underline{a_n}} TX$$

since the morphism $\begin{pmatrix} -d_1 \\ f \end{pmatrix}$ is $\mathcal{D}$-monic. Thus (N1) (c) is satisfied.

(N2) We only consider the case of standard n-angles, since the general case can be easily deduced. Let

$$X_1 \xrightarrow{\underline{f_1}} X_2 \xrightarrow{\underline{f_2}} X_3 \xrightarrow{\underline{f_3}} \cdots \xrightarrow{\underline{f_{n-1}}} X_n \xrightarrow{\underline{a_n}} TX_1$$

be a standard n-angle induced by the commutative diagram (3.1). We need to show that its left rotation belongs to Φ.

Consider the following three n-angles in $\mathcal{C}$:

$$\Sigma^{-1}X_n \xrightarrow{(-1)^n\Sigma^{-1}f_n} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-3}} X_{n-2} \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n,$$

$$X_1 \xrightarrow{d_1} D_1 \xrightarrow{d_2} D_2 \xrightarrow{d_3} D_3 \xrightarrow{d_4} \cdots \xrightarrow{d_{n-2}} D_{n-2} \xrightarrow{d_{n-1}} TX_1 \xrightarrow{d_n} \Sigma X_1,$$

$$\Sigma^{-1}X_n \xrightarrow{0} D_1 \xrightarrow{1_{D_1}} D_1 \xrightarrow{0} 0 \xrightarrow{0} \cdots \xrightarrow{0} 0 \xrightarrow{0} X_n \xrightarrow{1_{X_n}} X_n,$$

where the first exists by (N2). Since $d_1 \cdot \Sigma^{-1}f_n = a_2 f_1 \cdot \Sigma^{-1} f_n = 0$, we use (N4′) to get an n-angle in $\mathcal{C}$

$$X_2 \xrightarrow{\begin{pmatrix} f_2 \\ \varphi \end{pmatrix}} X_3 \oplus D_1 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \phi_3 & \psi \end{pmatrix}} X_4 \oplus D_2 \xrightarrow{\begin{pmatrix} -f_4 & 0 \\ \phi_4 & d_3 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_{n-2} & 0 \\ \phi_{n-2} & d_{n-3} \end{pmatrix}}$$

$$X_{n-1} \oplus D_{n-3} \xrightarrow{\begin{pmatrix} (-1)^n\varphi' & 0 \\ \phi_{n-1} & d_{n-2} \end{pmatrix}} X_n \oplus D_{n-2} \xrightarrow{(\psi', d_{n-1})} TX_1 \xrightarrow{\Sigma f_1 \cdot d_n} \Sigma X_2$$

with $\varphi' = f_{n-1}$ and $d_n \cdot \psi' = (-1)^n f_n$. Note that $f_n = d_n a_n$, hence we have $d_n(\psi' - (-1)^n a_n) = 0$. Thus $\psi' - (-1)^n a_n$ factors through d_{n-1}, so that $\underline{\psi'} = (-1)^n \underline{a_n}$. We claim that the morphism $\begin{pmatrix} f_2 \\ \varphi \end{pmatrix}\colon X_2 \to X_3 \oplus D_1$ is $\mathcal{D}$-monic. In fact, for any morphism $f\colon X_2 \to D$ with $D \in \mathcal{D}$, there is a morphism $g\colon D_1 \to D$ such that $ff_1 = gd_1 = g\varphi f_1$ since d_1 is a left $\mathcal{D}$-approximation. Now $(f - g\varphi)f_1 = 0$, hence there exists a morphism $h\colon X_3 \to D$ with $f - g\varphi = hf_2$. So $f = (h\ g)\begin{pmatrix} f_2 \\ \varphi \end{pmatrix}$. Let

$$X_2 \xrightarrow{d'_1} D'_1 \xrightarrow{d'_2} D'_2 \xrightarrow{d'_3} \cdots \xrightarrow{d'_{n-2}} D'_{n-2} \xrightarrow{d'_{n-1}} TX_2 \xrightarrow{d'_n} \Sigma X_2$$

be the n-angle given by the mutation pair. Assume that $T\underline{f_1} = \underline{g_1}$, then we get $d'_n \cdot g_1 = \Sigma f_1 \cdot d_n$ by the definition of $T\underline{f_1}$. Thus we can obtain the following commutative diagram:

$$\begin{array}{ccccccccccc}
X_2 & \xrightarrow{\binom{f_2}{\varphi}} & X_3 \oplus D_1 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \phi_3 & \psi \end{pmatrix}} & \cdots & \xrightarrow{\begin{pmatrix} (-1)^n f_{n-1} & 0 \\ \phi_{n-1} & d_{n-2} \end{pmatrix}} & X_n \oplus D_{n-2} & \xrightarrow{(\psi' \ \ d_{n-1})} & TX_1 & \xrightarrow{\Sigma f_1 \cdot d_n} & \Sigma X_2 \\
\Vert & & \downarrow b_1 & & & & \downarrow b_{n-2} & & \downarrow g_1 & & \Vert \\
X_2 & \xrightarrow{d'_1} & D'_1 & \xrightarrow{d'_2} & \cdots & \xrightarrow{d'_{n-2}} & D'_{n-2} & \xrightarrow{d'_{n-1}} & TX_2 & \xrightarrow{d'_n} & \Sigma X_2.
\end{array}$$

It follows that

$$X_2 \xrightarrow{\underline{f_2}} X_3 \xrightarrow{-\underline{f_3}} X_4 \xrightarrow{\underline{f_4}} \cdots \xrightarrow{-\underline{f_{n-2}}} X_{n-1} \xrightarrow{(-1)^n \underline{f_{n-1}}} X_n \xrightarrow{(-1)^n \underline{a_n}} TX_1 \xrightarrow{T\underline{f_1}} TX_2$$

is an n-angle in $\mathcal{Z}/[\mathcal{D}]$. The commutative diagram

$$\begin{array}{ccccccccccccccc}
X_2 & \xrightarrow{\underline{f_2}} & X_3 & \xrightarrow{-\underline{f_3}} & X_4 & \xrightarrow{\underline{f_4}} & \cdots & \xrightarrow{-\underline{f_{n-2}}} & X_{n-1} & \xrightarrow{(-1)^n \underline{f_{n-1}}} & X_n & \xrightarrow{(-1)^n \underline{a_n}} & TX_1 & \xrightarrow{T\underline{f_1}} & TX_2 \\
\Vert & & \Vert & & \downarrow (-1)^3 & & & & \downarrow (-1)^{n-2} & & \Vert & & \downarrow (-1)^n & & \Vert \\
X_2 & \xrightarrow{\underline{f_2}} & X_3 & \xrightarrow{\underline{f_3}} & X_4 & \xrightarrow{\underline{f_4}} & \cdots & \xrightarrow{\underline{f_{n-2}}} & X_{n-1} & \xrightarrow{\underline{f_{n-1}}} & X_n & \xrightarrow{\underline{a_n}} & TX_1 & \xrightarrow{(-1)^n T\underline{f_1}} & TX_2
\end{array}$$

shows that

$$X_2 \xrightarrow{\underline{f_2}} X_3 \xrightarrow{\underline{f_3}} X_4 \xrightarrow{\underline{f_4}} \cdots \xrightarrow{\underline{f_{n-2}}} X_{n-1} \xrightarrow{\underline{f_{n-1}}} X_n \xrightarrow{\underline{a_n}} TX_1 \xrightarrow{(-1)^n T\underline{f_1}} TX_2$$

belongs to Φ.

Conversely, let

$$X_2 \xrightarrow{\underline{f_2}} X_3 \xrightarrow{\underline{f_3}} X_4 \xrightarrow{\underline{f_4}} \cdots \xrightarrow{\underline{f_{n-1}}} X_n \xrightarrow{\underline{f_n}} TX_1 \xrightarrow{T\underline{f_1}} TX_2$$

be a standard n-angle in $\mathcal{Z}/[\mathcal{D}]$. In a similar way, we can show that its right rotation is also an n-angle.

(N3) We only consider the case of standard n-angles. Suppose that there is a commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{\underline{f_1}} & X_2 & \xrightarrow{\underline{f_2}} & X_3 & \xrightarrow{\underline{f_3}} & \cdots & \xrightarrow{\underline{f_{n-1}}} & X_n & \xrightarrow{\underline{a_n}} & TX_1 \\
\downarrow \underline{\varphi_1} & & \downarrow \underline{\varphi_2} & & & & & & & & \downarrow T\underline{\varphi_1} \\
Y_1 & \xrightarrow{\underline{g_1}} & Y_2 & \xrightarrow{\underline{g_2}} & Y_3 & \xrightarrow{\underline{g_3}} & \cdots & \xrightarrow{\underline{g_{n-1}}} & Y_n & \xrightarrow{\underline{b_n}} & TY_1
\end{array} \tag{3.2}$$

with rows standard n-angles in $\mathcal{Z}/[\mathcal{D}]$. Since $\underline{\varphi_2}\cdot\underline{f_1} = \underline{g_1}\cdot\underline{\varphi_1}$ holds, $\varphi_2 f_1 - g_1\varphi_1$ factors through some object D in $\mathcal{D}$. Assume that $\varphi_2 f_1 - g_1\varphi_1 = gf$, where $f\colon X_1 \to D$ and $g\colon D \to Y_2$. Since f_1 is $\mathcal{D}$-monic, there exists $h\colon X_2 \to D$ such that $f = hf_1$. Note that $(\varphi_2 - gh)f_1 = g_1\varphi_1$, hence by (N3) we have the commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2 - gh} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1
\end{array}$$

with rows n-angles in $\mathcal{C}$. By Lemma 3.5, the diagram (3.2) can be completed to a morphism of n-angles.

(N4′) We only consider the case of standard n-angles. Let

$$X_1 \xrightarrow{\underline{f_1}} X_2 \xrightarrow{\underline{f_2}} X_3 \xrightarrow{\underline{f_3}} \cdots \xrightarrow{\underline{f_{n-1}}} X_n \xrightarrow{\underline{a_n}} TX_1,$$

$$X_1 \xrightarrow{\underline{\varphi_2 f_1}} Y_2 \xrightarrow{\underline{g_2}} Y_3 \xrightarrow{\underline{g_3}} \cdots \xrightarrow{\underline{g_{n-1}}} Y_n \xrightarrow{\underline{b_n}} TX_1,$$

$$X_2 \xrightarrow{\underline{\varphi_2}} Y_2 \xrightarrow{\underline{h_2}} Z_3 \xrightarrow{\underline{h_3}} \cdots \xrightarrow{\underline{h_{n-1}}} Z_n \xrightarrow{\underline{c_n}} TX_2$$

be three standard n-angles in $\mathcal{Z}/[\mathcal{D}]$ which are induced by the following three n-angles in $\mathcal{C}$, respectively:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1,$$

$$X_1 \xrightarrow{\varphi_2 f_1} Y_2 \xrightarrow{g_2} Y_3 \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} Y_n \xrightarrow{g_n} \Sigma X_1,$$

$$X_2 \xrightarrow{\varphi_2} Y_2 \xrightarrow{h_2} Z_3 \xrightarrow{h_3} \cdots \xrightarrow{h_{n-1}} Z_n \xrightarrow{h_n} \Sigma X_2$$

where f_1 and φ_2 are $\mathcal{D}$-monic, thus $\varphi_2 f_1$ is $\mathcal{D}$-monic too. Then we have $f_n = d_n a_n$, $g_n = d_n b_n$ and $h_n = d'_n c_n$ by the definition of standard n-angles in $\mathcal{Z}/[\mathcal{D}]$, where $d_n\colon TX_1 \to \Sigma X_1$ and $d'_n\colon TX_2 \to \Sigma X_2$.

By the axiom (N4′), there exist morphisms $\varphi_i\colon X_i \to Y_i$ $(i = 3, 4, \ldots, n)$, $\psi_j\colon Y_j \to Z_j$ $(j = 3, 4, \ldots, n)$ and $\phi_k\colon X_k \to Z_{k-1}$ $(k = 4, 5, \ldots, n)$ satisfying (N4′) (a) and (N4′) (b). We need to show that in the quotient category $\mathcal{Z}/[\mathcal{D}]$ the corresponding (N4′) (a) and (N4′) (b) are satisfied.

By Lemma 3.5, (N4′) (a) holds. For (N4′) (b), we first note that the morphism $\begin{pmatrix} f_3 \\ \varphi_3 \end{pmatrix}\colon X_3 \to X_4 \oplus Y_3$ is $\mathcal{D}$-monic. In fact, for any morphism $f\colon X_3 \to D$ with $D \in \mathcal{D}$, there exists a morphism $g\colon Y_2 \to D$ with $ff_2 = g\varphi_2$ since φ_2 is $\mathcal{D}$-monic. Now $g\varphi_2 f_1 = ff_2 f_1 = 0$, which implies that there exists a morphism $h\colon Y_3 \to D$

with $g = hg_2$. Note that $\varphi_3 f_2 = g_2\varphi_2$, thus $(f - h\varphi_3)f_2 = 0$. Then there exists a morphism $i\colon X_4 \to D$ with $f - h\varphi_3 = if_3$. Hence $f = (i\ h)\binom{f_3}{\varphi_3}$. Therefore the n-angle (2.1) induces an n-angle in $\mathcal{Z}/[\mathcal{D}]$ with the last morphism $\underline{e_n}$ satisfying $d_n'' \cdot e_n = \Sigma f_2 \cdot h_n$, where $d_n''\colon TX_3 \to \Sigma X_3$.

To complete the proof, it suffices to check that $\underline{e_n} = T\underline{f_2} \cdot \underline{c_n}$ and $\underline{c_n \psi_n} = T\underline{f_1} \cdot \underline{b_n}$. Let $T\underline{f_2} = \underline{i_2}$, then by definition we have $\Sigma f_2 \cdot d_n' = d_n'' i_2$. Now $d_n'' i_2 c_n = \Sigma f_2 \cdot d_n' c_n = \Sigma f_2 \cdot h_n = d_n'' e_n$, which implies that $e_n - i_2 c_n$ factors through some object in $\mathcal{D}$, thus $\underline{e_n} = \underline{i_2 c_n} = T\underline{f_2} \cdot \underline{c_n}$. Similarly, let $T\underline{f_1} = \underline{i_1}$, then $\Sigma f_1 \cdot d_n = d_n' i_1$. Note that $d_n' c_n \psi_n = h_n \psi_n = \Sigma f_1 \cdot g_n = \Sigma f_1 \cdot d_n b_n = d_n' i_1 b_n$. Thus $c_n\psi_n - i_1 b_n$ factors through some object in $\mathcal{D}$ and $\underline{c_n}\ \underline{\psi_n} = T\underline{f_1} \cdot \underline{b_n}$. □

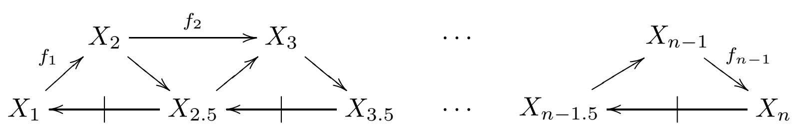
Remark 3.8. In Theorem 3.7, if $n = 3$ and $\mathcal{D}$ is a rigid subcategory of $\mathcal{C}$, then we recover a theorem of Iyama-Yoshino [6], Theorem 4.2.

4. Example

We first recall the standard construction of n-angulated categories given by Geiss-Keller-Oppermann [4], Theorem 1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ an $(n-2)$-cluster tilting subcategory which is closed under Σ^{n-2}, where Σ is the shift functor of $\mathcal{T}$. Then $(\mathcal{C}, \Sigma^{n-2}, \Theta)$ is an n-angulated category, where Θ is the class of all sequences

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma^{n-2} X_1$$

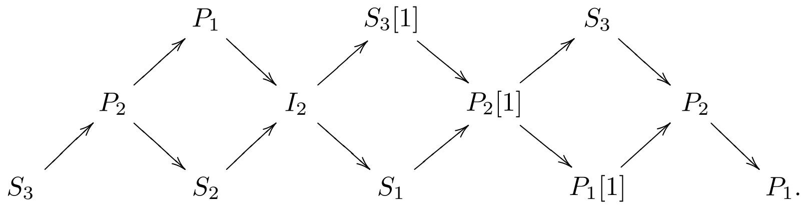
such that there exists a diagram



with $X_i \in \mathcal{C}$ for all $i \in \mathbb{Z}$, such that all the oriented triangles are triangles in $\mathcal{T}$, all the non-oriented triangles commute, and f_n is the composition along the lower edge of the diagram.

Example 4.1. Let $\mathcal{T} = D^b(kQ)/\tau^{-1}[1]$ be the cluster category of type A_3, where Q is the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, $D^b(kQ)$ is the bounded derived category of finite generated modules over kQ, τ is the AR-translation and [1] is the shift functor of $D^b(kQ)$. Then $\mathcal{T}$ is a triangulated category. Its shift functor is also denoted by [1].

The AR-quiver of $\mathcal{T}$ is:



Let $\mathcal{C} = \operatorname{add}(S_3 \oplus P_1 \oplus S_1)$. It is easy to check that $\mathcal{C}$ is a 2-cluster tilting subcategory of $\mathcal{T}$. Moreover, $\mathcal{C}[2] = \mathcal{C}$. Thus $(\mathcal{C}, [2])$ is a 4-angulated category. In fact, $\mathcal{C} \cong \operatorname{proj} A$, where $A = \operatorname{End}_{\mathcal{C}}(S_3 \oplus P_1 \oplus S_1)^{\mathrm{op}}$ is a self-injective cluster tilted algebra. Let $\mathcal{D} = \operatorname{add}(S_3 \oplus S_1)$. Then the 4-angle

$$P_1 \longrightarrow S_1 \longrightarrow S_3 \longrightarrow P_1 \longrightarrow P_1[2]$$

shows that $(\mathcal{C}, \mathcal{C})$ is a $\mathcal{D}$-mutation pair. By Theorem 3.7, the quotient category $\mathcal{C}/[\mathcal{D}]$ is a 4-angulated category.

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