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Abstract. We study topological spaces that can be represented as the union of a finite collection of dense metrizable subspaces. The assumption that the subspaces are dense in the union plays a crucial role below. In particular, Example 3.1 shows that a paracompact space X which is the union of two dense metrizable subspaces need not be a p-space. However, if a normal space X is the union of a finite family μ of dense subspaces each of which is metrizable by a complete metric, then X is also metrizable by a complete metric (Theorem 2.6). We also answer a question of M.V. Matveev by proving in the last section that if a Lindelöf space X is the union of a finite family μ of dense metrizable subspaces, then X is separable and metrizable.

Keywords: dense subspace; perfect space; Moore space; Čech-complete; p-space; σ-disjoint base; uniform base; pseudocompact; point-countable base; pseudo-ω_1-compact

Classification: Primary 54A25; Secondary 54B05

1. Introduction

In this paper *space* stands for Tychonoff topological space. A space X is called *perfect* if every closed subset of X is a G_δ-set in X. A base of countable order is a base $\mathcal{B}$ such that every strictly decreasing sequence $\{U_n : n \in \omega\}$ of members of $\mathcal{B}$ with a nonempty intersection is a base at each point of this intersection. In terminology and notation we follow [7].

Quite often topological spaces with an amazing combination of properties are constructed as unions of finite collections of metrizable spaces. In particular, Michael line, Mrówka space Ψ, Niemytzkiy half-plane, Alexandroff-Urysohn double circumference, Alexandroff compactification of an uncountable discrete space, and the countable Fréchet-Urysohn fan are spaces of this kind. However, they do not look very similar, each of them has its own non-trivial combination of properties. To provide a general framework for a systematic study of arbitrary unions of metrizable subspaces, M. Ismail and A. Szymanski introduced the concept of *metrizability number* $m(X)$ of a topological space X. This is the smallest cardinal number κ such that X can be represented as the union of κ many metrizable subspaces [10]. In particular, they studied locally compact spaces with finite metrizability number [12].

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In this paper, we are focused on spaces which can be represented as the union of a finite collection of dense metrizable subspaces. The class of such spaces is denoted by $\mathcal{M}_{duf}$.

The properties of spaces in $\mathcal{M}_{duf}$ turn out to be especially interesting. For example, if $X = Y \cup Z$, where Y and Z are dense metrizable subspaces of X, then X is not only first-countable, but has a point-countable base. In fact, even countable unions of dense metrizable subspaces are easily seen to have a σ-disjoint base. M. Ismail and A. Szymanski observed that a locally compact space which is a union of countably many dense (or open) metrizable subspaces is metrizable, and proved that any locally compact space with finite metrizability number contains an open dense metrizable subspace [12]. Their papers contain a rich collection of examples of compacta with finite metrizability number.

Below we establish that if X is the union of a finite family of dense metrizable subspaces each of which is a G_δ-subset of X, then X has a uniform base (Theorem 2.2). If X is the union of a finite family of dense Čech-complete Moore subspaces, then X is a Čech-complete Moore space (Theorem 2.4). Example 3.1 shows that the above results are sharp: we see that a paracompact space X which is the union of two dense metrizable subspaces need not be a p-space and need not have a base of countable order. We also prove that if a Lindelöf or pseudocompact space X is the union of a finite family μ of dense metrizable subspaces, then X is separable and metrizable (see Theorem 4.3 and its corollaries).

2. Some properties of spaces in $\mathcal{M}_{duf}$

In this section, some sufficient conditions for a space in $\mathcal{M}_{duf}$ to be metrizable or to have a uniform base are discussed. Notice that if a perfectly normal space X is a subspace of a space Y which is the union of a countable family of dense metrizable subspaces, then X is metrizable, since every perfectly normal space with a σ-disjoint base is clearly metrizable, by an obvious application of R. Bing's metrization theorem. In connection with this well-known fact, see [1].

Theorem 2.1. *Suppose that a space X is the union of a finite family μ of dense perfect subspaces of X each of which is a G_δ-subset of X. Then X is perfect.*

Proof: Take any closed subset F of X. Take also any $M \in \mu$, and put $F_M = F \cap M$. Since M is perfect, we can fix a decreasing sequence $\eta_M = \{V_i^M : i \in \omega\}$ of open subsets of M such that $F_M = \bigcap \eta_M$. Since M is a G_δ-subset of X, there exists a decreasing sequence $\xi_M = \{W_i^M : i \in \omega\}$ of open subsets of X such that $M = \bigcap \xi_M$. Now we can choose open subsets U_i^M of X for $i \in \omega$ so that $V_i^M = U_i^M \cap M$, $U_i^M \subset W_i^M$, and the sequence $\{U_i^M : i \in \omega\}$ is decreasing. Put $U_i = \bigcup\{U_i^M : M \in \mu\}$ for $i \in \omega$.

Claim 1: $F = \bigcap\{U_i : i \in \omega\}$.

Indeed, $F_M \subset V_i^M \subset U_i^M$ and $F = \bigcup\{F_M : M \in \mu\}$. Therefore, $F \subset \bigcap\{U_i : i \in \omega\}$.

To verify the converse inclusion, take any $z \in X \setminus F$. Then $z \notin F_M$.

Claim 2: There exists $k \in \omega$ such that $z \notin U_k^M$, for every $M \in \mu$.

Fix any $M \in \mu$. Since the sets U_i^M are decreasing for each M, it is enough to show that there exists $k \in \omega$ such that $z \notin U_k^M$. We distinguish two cases.

Case 1: $z \notin M$. Then $z \notin W_k^M$, for some $k \in \omega$, since $M = \bigcap\{W_i^M : i \in \omega\}$. Therefore, $z \notin U_k^M$, since $U_k^M \subset W_k^M$.

Case 2: $z \in M$. Then $z \in M \setminus F_M$. Hence, there exists $k \in \omega$ such that $z \notin V_k^M$. Then $z \notin U_k^M$, since $V_k^M = U_k^M \cap M$. Thus, Claim 2 is established. This completes the proof of Claim 1. Now we see that F is a G_δ-subset of X. □

Theorem 2.2. *Suppose that X is the union of a finite family μ of dense metrizable subspaces of X each of which is a G_δ-subset of X. Then X has a uniform base.*

PROOF: Every metrizable space is perfect. Therefore, it follows from Theorem 2.1 that X is perfect. Observe that the space X has a σ-disjoint base, since X is the union of a countable family of dense metrizable subspaces. Therefore, X has a uniform base, since *every perfect space with a σ-disjoint base has a uniform base* (see [4] for a simple direct proof of this fact). □

The next result is "parallel" to Theorem 2.2.

Theorem 2.3. *Suppose that X is the union of a finite family μ of dense Moore subspaces of X each of which is a G_δ-subset of X. Then X is a Moore space.*

PROOF: For each $M \in \mu$, we fix a decreasing sequence $\eta_M = \{G_n^M : n \in \omega\}$ of open subsets of X such that $M = \bigcap \eta_M$. We also fix a sequence $\xi_M = \{\gamma_n^M : n \in \omega\}$ of families of open subsets of X such that $\{\lambda_n^M : n \in \omega\}$ is a development of M, where $\lambda_n^M = \{V \cap M : V \in \gamma_n^M\}$ and each γ_{n+1}^M is a refinement of γ_n^M. Clearly, we can also assume that every $V \in \gamma_n^M$ is contained in G_n^M. Put $\kappa_n = \bigcup\{\gamma_n^M : M \in \mu\}$.

Claim 1: The family $\{\kappa_n : n \in \omega\}$ is a development for X.

Let us verify the last statement. Fix $x \in X$, and let K be any infinite subset of ω. Take any family $\eta = \{U_n : n \in K\}$ such that $x \in U_n \in \kappa_n$ for every $n \in K$. We have to show that η is a base for X at x. Since μ is finite, we can assume that there exists $M \in \mu$ such that $U_n \in \gamma_n^M$, for every $n \in K$. We need now the following fact:

Claim 2: $x \in M$.

Assume the contrary. Then there exists $j \in \omega$ such that $x \notin G_j^M$. For any $n > j$ with $n \in K$, we have $x \notin U_n$, since $U_n \subset G_j^M$. This is a contradiction. Claim 2 is established.

Thus, we have that $V_n = U_n \cap M \in \lambda_n^M$, for every $n \in K$. Since K is infinite, and $\{\lambda_n^M : n \in \omega\}$ is a development for M, it follows that $\{V_n : n \in K\}$ is a base for X at the point x. Notice that V_n is dense in U_n, since M is dense in X. Therefore, $\{U_n : n \in K\}$ is a base for X at x. Claim 1 is proved. Hence, X is a Moore space. □

Theorem 2.4. *Suppose that X is the union of a finite family μ of dense Čech-complete Moore subspaces of X. Then X is a Čech-complete Moore space.*

PROOF: It is easy to see that X is Čech-complete. Any $M \in \mu$ is a G_δ-subset of X, since M is dense in X and Čech-complete. Therefore, Theorem 2.3 is applicable. Hence, X is a Moore space. □

Similarly, applying Theorems 2.2 and 2.4, we obtain one of the main results in this paper:

Corollary 2.5. *If a space X is the union of a finite family μ of dense subspaces of X such that each $Y \in \mu$ is metrizable by a complete metric, then X is a Čech-complete space with a uniform base.*

Here is an application of the above statement.

Theorem 2.6. *If a normal space X is the union of a finite family μ of dense subspaces each of which is metrizable by a complete metric, then X is also metrizable by a complete metric.*

PROOF: Indeed, it follows from Corollary 2.5 that X is a Čech-complete perfectly normal space with a σ-disjoint base. Therefore, X is metrizable by a complete metric. □

Notice that the problem whether every normal space with a uniform base is metrizable is still open.

3. The main example and other related examples

In this section we show that the results obtained in the preceding section are quite sharp.

It is known that a perfect space with a point-countable base need not have a uniform base. For example, a hereditarily Lindelöf non-metrizable space S with a point-countable base was constructed in [6] under the Continuum Hypothesis CH. The space S is perfect, but does not have a uniform base, since it is a non-metrizable Lindelöf space. Of course, the space S cannot be represented as the union of a countable family of dense metrizable subspaces, since otherwise it would have a σ-disjoint base and a uniform base.

Heath's space H (see [7, 5.4.B]) is the union of two dense open Čech-complete metrizable subspaces. Hence, by Theorem 2.2 and Corollary 2.5, the space H is Čech-complete. Clearly, H has a uniform base which is also a σ-disjoint base. However, H is not metrizable and not normal. Thus, the space X in Theorem 2.2 and Corollary 2.5 need not be metrizable or paracompact. On the other hand, H is metacompact, since it has a uniform base.

Mrówka's space $M(\omega)$, denoted also by Ψ, is the union of two discrete subspaces. However, one of these subspaces is not dense in $M(\omega)$. Even more, $M(\omega)$ *cannot be embedded in a space which is the union of a countable family of dense metrizable subspaces*. Indeed, $M(\omega)$ is separable but not metrizable. Therefore, it does not have a point-countable base. Hence, $M(\omega)$ is not a subspace of a space in $\mathcal{M}_{duf}$.

The next example is a modification of the Michael line.

Example 3.1. Let us show that there exists a space T with the following properties:

1) T is the union of two dense metrizable subspaces;
2) T is paracompact;
3) T is submetrizable (that is, there exists a one-to-one continuous mapping of T onto a metrizable space);
4) T is not perfect;
5) T is not a p-space.

It follows from 1)–5) that T has some other classical properties:

6) T has a G_δ-diagonal;
7) T has a σ-disjoint base;
8) T is collectionwise normal;
9) T is not metrizable;
10) T is not stratifiable;
11) T is not a Moore space;
12) T is not symmetrizable;
13) T is not a σ-space (that is, T does not have a σ-discrete network);
14) T is not Čech-complete;
15) T is an s-space (see the definition in [3]);
16) T is a space of countable type.

Observe that the Michael line has all the properties listed above except for the first one: it cannot be represented as the union of a finite collection of dense metrizable subspaces (see Proposition 3.2).

A construction. We denote by Z the usual Michael line. Hence, $Z = J \cup Q$, where J is the set of irrational numbers and all points of J are isolated in Z, Q is the set of rational numbers with usual intervals in the role of basic open neighbourhoods in Z of the points in Q. On the set $Z \times Q$ we define a new topology $\mathcal{T}^*$ (which is stronger than the product topology) as follows.

The basic open neighbourhoods of the points of $Z \times \{0\}$ are the same as in the product topology of the Michael line with the usual space Q of rational numbers.

For any $z \in Z$ and any $q \neq 0$, we declare the sets $\{z\} \times V$, where V is an arbitrary open subset of Q in the usual topology such that $q \in V$, to be basic open neighbourhoods of (z, q) with respect to the new topology $\mathcal{T}^*$.

The set $Z \times Q$ with the new topology is denoted by T. It is easy to see that T is zero-dimensional and hence, is Tychonoff.

Put $M_1 = T \setminus (Q \times \{0\})$ and $M_2 = T \setminus (J \times \{0\})$. Clearly, M_1 and M_2 are dense in T and $T = M_1 \cup M_2$. The subspace M_1 of T is metrizable, since it can be represented as the union of a disjoint family of open and closed subspaces of T each of which is homeomorphic to the space Q of rational numbers. The subspace

M_2 of T is also metrizable. This follows from the classical Bing-Nagata-Smirnov metrization theorem and the fact that $Q \times \{0\}$ is a G_δ-subset of M_2. Therefore, T has a σ-disjoint base.

The Michael line Z is homeomorphic to the closed subspace $L = Z \times \{0\}$ of T. Hence, the subspace L of T is not perfect, since the Michael line is not perfect. Therefore, T is not perfect as well. Hence, T does not have a uniform base and is not developable. On the other hand, T is obviously paracompact. It has a G_δ-diagonal, since the identity mapping of the space T onto the product space $R \times Q$ of the usual space R of real numbers with the space Q is continuous. Since the space $R \times Q$ is metrizable, and T is paracompact but not metrizable, it follows that T is not a p-space [2]. However, T is an s-space (see the definition and properties of s-spaces in [3]).

To construct subspaces M_1 and M_2, we could use two dense complementary subspaces Q_1 and Q_2 of Q instead of Q (as the range for the second coordinate). Then the subspaces M_1 and M_2 would become disjoint.

The space T described above, and the Michael line itself, show that the main results obtained in the preceding section are sharp. In particular, using the spaces Z and T, we can verify by standard straightforward arguments that the following statements hold:

(a) *If a space X is the union of a countable family of dense Čech-complete metrizable subspaces, then X need not be a Moore space and need not be Čech-complete.*

Compare this fact with Theorem 2.4 and Corollary 2.5.

(b) *A paracompact space X, which is the union of two dense metrizable subspaces, need not be a p-space and need not have a base of countable order.*

Recall that a paracompact space with a base of countable order is metrizable [1].

(c) *There exists a non-perfect space X which is the union of a countable family of dense perfect subspaces of X each of which is a G_δ-subset of X.*

Compare the last statement with Theorem 2.1.

In the last example we have stated that the Michael line Z cannot be represented as the union of finitely many dense metrizable subspaces. This easily follows from the next simple general statement (which is probably known) an easy proof of which is omitted:

Proposition 3.2. *If the set of non-isolated points of a first-countable space X is countable, and X can be represented as the union of a finite collection of dense metrizable subspaces, then X is metrizable.*

In certain respects, the above example is the best possible. Indeed, if a paracompact space X is a subspace of a space Y which is the union of a countable family of dense metrizable subspaces, and X is either perfect or a p-space, then X is metrizable. This follows from a theorem of V.V. Filippov [8], since X has a point-countable base.

3.1 Unions of finite and locally finite collections of metrizable subspaces. We denote by $\mathcal{M}_{fu}$ the class of spaces which can be represented as the union of a finite family of metrizable subspaces (which are not necessarily dense in the union). The class $\mathcal{M}_{fu}$ is much wider than the class of subspaces of members of $\mathcal{M}_{duf}$. Nevertheless, we show that the results obtained in Section 2 can be applied to members of $\mathcal{M}_{fu}$ and provide certain new information on their structure. Note that S. Oka has studied paracompact perfectly normal spaces which are unions of finite collections of metrizable subspaces and extended to them some important theorems of the dimension theory [15]. A brief concentrated survey of results on finite unions of metrizable spaces is given in the last section of [9]. See this major survey article for the interesting results mentioned there and for further references.

The key role in the study below belongs to the next elementary lemma. It may turn out to be a part of the folklore, but in any case I want to give its proof for the sake of completeness.

Lemma 3.3. *Suppose that a space X is the union of a finite family μ of subspaces. Then there exists a finite disjoint family η of open subsets of X such that $\bigcup\eta$ is dense in X, and, for every $V \in \eta$, there exists a subfamily ν of μ such that $V \cap M$ is dense in V for every $M \in \nu$, and $V \subset \bigcup\nu$.*

Proof: We will prove this statement by induction on the number of elements in μ. Let $\mu = \{M_i : i = 1, \dots, k\}$ and $Z_j = \bigcup\{M_i : i \in (\{1, \dots, k\}) \setminus \{j\}\}$, for $j = 1, \dots, k$. We denote by V_j the largest open subset of X such that $V_j \subset Z_j$, for $j = 1, \dots, k$, and let $Y = X \setminus \overline{\bigcup\{V_i : i = 1, \dots, k\}}$ and $Y_i = M_i \setminus \overline{V}$, where $V = \bigcup\{V_i : i = 1, \dots, k\}$ and the closure is taken in X. Clearly, Y is open in X, and $Y = \bigcup\{Y_i : i = 1, \dots, k\}$. It follows that $\bigcup\{Y_i : i \in (\{1, \dots, k\}) \setminus \{j\}$ contains no nonempty open subset of Y. Therefore, Y_j is dense in Y, for every $j = 1, \dots, k$.

By the inductive assumption, the lemma is applicable to each V_j. Hence, we can fix a finite family η_j of open subsets of X such as in the lemma. Put $\eta = \bigcup\{\eta_j : j = 1, \dots, k\} \cup \{Y\}$. Clearly, $\bigcup\eta$ is dense in X, and each member of η has the property mentioned in the lemma. The family η need not be disjoint, but it is finite. Obviously, we can modify η so that it becomes disjoint and satisfies the other conditions of the lemma. □

Theorem 3.4. *Suppose that $X = \bigcup\{M_\alpha : \alpha \in A\}$, where $\gamma = \{M_\alpha : \alpha \in A\}$ is a locally finite family of metrizable subspaces of X. Then:*

1) *if X is perfect, then X has an open dense subspace with a uniform base and a σ-disjoint base;*
2) *if each M_α is Čech-complete, then X has an open dense Čech-complete subspace with a uniform base and a σ-disjoint base.*

Proof: *Claim* 1: For every nonempty open subset U of X, there exists a nonempty open subset V of U such that the set $\{\alpha \in A : M_\alpha \cap V \neq \emptyset\}$ is finite, and if $M_\alpha \cap V \neq \emptyset$, then $M_\alpha \cap V$ is dense in V.

Indeed, since γ is locally finite, we may assume that A is finite. Now Claim 1 easily follows from Lemma 3.3.

To give a slightly more elementary proof of Claim 1, assume that $M_\beta \cap U$ is not dense in U for some $\beta \in A$. Then, for some open nonempty subset U_1 of U, $M_\beta \cap U_1 = \emptyset$. Let $A_1 = \{\alpha \in A : M_\alpha \cap U_1 \neq \emptyset\}$. Then $|A_1| < |A|$. Repeating this step with U_1 and A_1 in the role of U and A, at the end of the process we get the set V we want. Claim 1 is established.

Let us denote by $\mathcal{E}$ the family of all nonempty open subsets V of X such that, for some finite subfamily μ of γ, we have that $V \subset \bigcup \mu$ and $M_\alpha \cap V$ is dense in V, for each $M_\alpha \in \mu$. The next statement immediately follows from Claim 1:

Claim 2: The family $\mathcal{E}$ is a π-base for X.

Claim 3: If X is perfect, then every $V \in \mathcal{E}$ is a space with a uniform base and a σ-disjoint base.

Since every subspace of a perfect space is perfect, this statement immediately follows from the definition of $\mathcal{E}$ and the results established in Section 2.

Claim 4: If X is Čech-complete, then every $V \in \mathcal{E}$ is a Čech-complete space with a uniform base and a σ-disjoint base.

Since every open subspace of a Čech-complete space is Čech-complete, this statement follows from the definition of $\mathcal{E}$ and Corollary 2.5.

It follows from Claims 2, 3, and 4 that if the assumption in 1) is satisfied, then X has a π-base $\mathcal{E}$ each member of which has a uniform base and a σ-disjoint base. Similarly, if the assumption in 2) is satisfied, then X has a π-base $\mathcal{E}$ each member of which is Čech-complete and has a uniform base and a σ-disjoint base.

Take any maximal disjoint subfamily ξ of $\mathcal{E}$. Then $H = \bigcup \xi$ is an open subspace of X, H is dense in X, and X has a uniform base and a σ-disjoint base. Besides, the space H is Čech-complete if the assumption in 2) is satisfied. $\square$

In the same way, we can prove the next statement, just replace the reference to Corollary 2.5 by a reference to Theorem 2.4.

Theorem 3.5. *Suppose that $X = \bigcup\{M_\alpha : \alpha \in A\}$, where $\gamma = \{M_\alpha : \alpha \in A\}$ is a locally finite family of Čech-complete Moore subspaces of X. Then X has an open dense Čech-complete Moore subspace.*

4. Lindelöf spaces and pseudocompact spaces which are unions of dense metrizable subspaces

In this section, we consider Lindelöf spaces and pseudocompact spaces that are members of $\mathcal{M}_{duf}$, and some other closely related situations. A question posed 10–15 years ago by M.V. Matveev in a letter to the author is answered below.

It is well known that there exists a non-metrizable Lindelöf space with a σ-disjoint base, just take the version of the Michael line generated by a Bernstein subset of the space of real numbers. Notice that this space is the union of two metrizable subspaces. However, it is less easy to answer the question posed by

M.V. Matveev: *Does there exist a non-metrizable Lindelöf space X such that $X = Y \cup Z$, where Y and Z are dense metrizable subspaces of X?*

We will call a space X *pseudo-ω_1-compact*, if for every uncountable family $\xi = \{U_\alpha : \alpha \in A\}$ of nonempty open subsets of X there exists an accumulation point in X, i.e. $x \in X$ such that every neighbourhood of x intersects U_α for infinitely many $\alpha \in A$. In particular, every Lindelöf space is pseudo-ω_1-compact. Every pseudocompact space is also pseudo-ω_1-compact.

Let us call a space X *strictly perfect*, if for every closed subset F of X there exists a countable family η of open neighbourhoods of F such that $F = \bigcap\{\overline{U} : U \in \eta\}$. Clearly, every perfectly normal space is strictly perfect.

Proposition 4.1. *Suppose that $X = Y \cup Z$, where Y and Z are dense subspaces of X, Z is strictly perfect, and X is pseudo-ω_1-compact. Then Y is pseudo-ω_1-compact.*

Proof: Assume that Y is not pseudo-ω_1-compact. Then, since Y is dense in X, there exists an uncountable family $\xi = \{W_\alpha : \alpha \in \omega_1\}$ of nonempty open subsets of X such that there is no accumulation point for ξ in Y. Therefore, the set F of accumulation points for ξ in X is a subset of Z. Clearly, F is closed in Z and in X. Let us fix a sequence $\eta = \{U_n : n \in \omega\}$ of open neighbourhoods of F in Z such that F is the intersection of the closures in Z of members of η.

The set F is nowhere dense in Z. Indeed, otherwise F would contain a nonempty open subset of X, since Z is dense in X and F is closed in X. Then F would intersect Y, which is not the case. Hence, $(W_\alpha \cap Z) \setminus F \neq \emptyset$, for each $\alpha \in \omega_1$, and we can fix $n(\alpha) \in \omega$ such that the set $V_\alpha = (W_\alpha \cap Z) \setminus \overline{U_{n(\alpha)}}$ is nonempty. Clearly, there exist an uncountable subset A of ω_1 and $k \in \omega$ such that $n(\alpha) = k$, for every $\alpha \in A$. Then $\gamma = \{V_\alpha : \alpha \in A\}$ is an uncountable family of nonempty open subsets of Z. Notice that $V_\alpha \subset W_\alpha$, for every $\alpha \in A$. We also have that $V_\alpha \cap U_k = \emptyset$, for every $\alpha \in A$. Since $F \subset U_k$, it follows that no point of F is an accumulation point for γ. Hence, no point of X is an accumulation point for γ. For each $\alpha \in A$ we can fix an open subset H_α of X such that $H_\alpha \cap Z = V_\alpha$. Since Z is dense in X, the set V_α is dense in H_α for $\alpha \in A$. It follows that the uncountable family $\{H_\alpha : \alpha \in A\}$ of nonempty open subsets of X also does not have accumulation points in X. Therefore, X is not pseudo-ω_1-compact, a contradiction. □

Proposition 4.2. *Suppose that $X = \bigcup\{Z_i : i = 1, \dots, n\}$, for some positive $n \in \omega$, where Z_i is a dense strictly perfect subspace of X, for $i = 1, \dots, n$, and X is pseudo-ω_1-compact. Then Z_i is pseudo-ω_1-compact, for each $i = 1, \dots, n$.*

Proof: We will prove this statement by induction on n. For $n = 1$ we have nothing to prove. In the general case of arbitrary $n \in \omega$, we observe that, by Proposition 4.1, the space $\bigcup\{Z_i : i = 1, \dots, n-1\}$ (taken in the role of Y) is pseudo-ω_1-compact. By the inductive assumption, it follows that Z_1 is pseudo-ω_1-compact. Since we can take any Z_i in the role of Z_1, we conclude that each Z_i is pseudo-ω_1-compact. □

Now we are ready to establish one of the main results in this paper:

Theorem 4.3. *Suppose that a pseudo-ω_1-compact space X is the union of a finite family μ of dense metrizable subspaces of X. Then X is separable and metrizable.*

PROOF: This statement immediately follows from Proposition 4.2, since every metrizable space is strictly perfect, and every metrizable pseudo-ω_1-compact space is, obviously, separable. □

Recall that the *extent* $e(X)$ of a space X is countable if every closed discrete subspace of X is countable. Clearly, if the extent of X is countable, then X is pseudo-ω_1-compact. Therefore, we have:

Corollary 4.4. *Suppose that a space X is the union of a finite family μ of dense metrizable subspaces and satisfies at least one of the following conditions:*

1) *the extent of X is countable;*
2) *X is Lindelöf;*
3) *X has a dense open Lindelöf subspace;*
4) *X is pseudocompact.*

Then X is separable and metrizable.

See [5, Theorem 2.15] for yet another result of similar nature on dense unions of nice subspaces.

Example 4.5. Let us assume that the Continuum Hypothesis CH holds. According to [13, Chapter 3, Section 40, Subsection 7], there exists a nonempty subspace L of the space R of real numbers with the following properties:

1) Every nonempty open subset of the space L is uncountable;

2) There exists a countable subset $A \subset L$ such that A is dense in L and $L \setminus U$ is countable, for every open subset U of R such that $A \subset U$.

Fix a subset A of L satisfying condition 2). Let $\mathcal{T}$ be the usual topology on L. Put $\mathcal{B} = \mathcal{T} \cup \{\{x\} : x \in L \setminus A\}$. Clearly, $\mathcal{B}$ is a base for some new topology $\mathcal{T}^*$ on the set L, which is stronger than $\mathcal{T}$. The set L with the new topology $\mathcal{T}^*$ is a topological space L^*.

It follows from conditions 1) and 2) and the definition of the topology $\mathcal{T}^*$ that the next condition holds:

3) $L \setminus A$ is dense in L^*, and $L \setminus U$ is countable, for each $U \in \mathcal{T}^*$ such that $A \subset U$.

Since A is countable, it follows from 3) that L^* is Lindelöf. Condition 1) implies that $L \setminus A$ is an uncountable discrete (in itself) subspace of L^*. Hence, L^* does not have a countable base. Since L^* is Lindelöf, we see that L^* is not metrizable.

Put $X_a = (L \setminus A) \cup \{a\}$, for every $a \in A$. Clearly,

$$L^* = \bigcup \{X_a : a \in A\}.$$

The subspace X_a of L^* is dense in L^*, since $L \setminus A$ is dense in L^*. The subspace X_a is metrizable, since it is first-countable and only one point of X_a is non-isolated.

Since A is countable, we see that the Lindelöf non-metrizable space L^* is the union of a countable family of dense metrizable subspaces. Thus, Theorem 4.3 and Corollary 4.4 do not extend to unions of countable families of dense metrizable subspaces.

Problem 4.6. *Must a space X be Dieudonné complete if it can be represented as the union of two (of finitely many) dense metrizable subspaces?*

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