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# On complete linear Weingarten hypersurfaces in locally symmetric Riemannian manifolds

Cícero P. Aquino, Henrique F. de Lima*, Fábio R. dos Santos, Marco Antonio L. Velásquez

*Abstract.* Our aim is to apply suitable generalized maximum principles in order to obtain characterization results concerning complete linear Weingarten hypersurfaces immersed in a locally symmetric Riemannian manifold, whose sectional curvature is supposed to obey standard constraints. In this setting, we establish sufficient conditions to guarantee that such a hypersurface must be either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple.

*Keywords:* locally symmetric Riemannian manifolds; Einstein manifolds; complete linear Weingarten hypersurfaces; totally umbilical hypersurfaces; isoparametric hypersurfaces

*Classification:* Primary 53C42; Secondary 53A10, 53C20, 53C50

## 1. Introduction and statements of the results

The problem to characterize hypersurfaces immersed with either constant mean curvature or constant scalar curvature in a Riemannian space form $\mathbb{Q}_c^{n+1}$ of constant sectional curvature $c$ constitutes a classical theme in the theory of isometric immersions. For instance, Otsuki [14] studied the minimal hypersurfaces in the standard unit Euclidean sphere $\mathbb{S}^{n+1}$ $(n \geq 3)$ with two distinct principal curvatures and proved that if the multiplicities of the two principal curvatures are both greater than 1, then they are the Clifford minimal hypersurfaces. More recently, Wu [18] extended Otsuki's technique in order to prove that, locally, any hypersurface in $\mathbb{S}^{n+1}$ of constant mean curvature and two distinct principal curvatures is an open part of a complete hypersurface of the same curvature properties.

For the case of the scalar curvature, Brasil Jr., Colares and Palmas [7] used the generalized maximum principle of Omori-Yau to characterize complete hypersurfaces with constant normalized scalar curvature in $\mathbb{S}^{n+1}$. In [3], by applying a weak Omori-Yau maximum principle due to Pigola, Rigoli, Setti [15], Alías and

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* Corresponding author.

García-Martínez studied the behavior of the scalar curvature $R$ of a complete hypersurface immersed with constant mean curvature in $\mathbb{Q}_c^{n+1}$, deriving a sharp estimate for the infimum of $R$. Afterwards, Alías, García-Martínez and Rigoli [4] obtained another suitable weak maximum principle for complete hypersurfaces with constant scalar curvature in $\mathbb{Q}_c^{n+1}$, and gave some applications of it in order to estimate the norm of the traceless part of its second fundamental form. In particular, they extended the main theorem of [7] for the context of $\mathbb{Q}_c^{n+1}$.

Meanwhile, Li, Suh and Wei [12] considered the so-called *linear Weingarten* hypersurfaces immersed in $\mathbb{S}^{n+1}$, that is, hypersurfaces of $\mathbb{S}^{n+1}$ whose mean curvature $H$ and normalized scalar curvature $R$ satisfy $R = aH + b$, for some $a, b \in \mathbb{R}$. In this setting, they showed that if $M^n$ is a compact linear Weingarten hypersurface with nonnegative sectional curvature immersed in $\mathbb{S}^{n+1}$ such that $R = aH + b$ with $(n-1)a^2 + 4n(b-1) \geq 0$, then $M^n$ is either totally umbilical or isometric to a Clifford torus. More recently, the first, second and fourth authors [6] obtained another characterization result concerning complete linear Weingarten hypersurfaces immersed in $\mathbb{Q}_c^{n+1}$. Under the assumption that the mean curvature attains its maximum and supposing an appropriate restriction on the norm of the traceless part $\Phi$ of the second fundamental form, they showed that such a hypersurface must be either totally umbilical or isometric to an isoparametric hypersurface of $\mathbb{Q}_c^{n+1}$.

Motivated by the works described above, in this article we deal with complete linear Weingarten hypersurfaces $M^n$ immersed in a wide class of Riemannian manifolds, namely *locally symmetric Riemannian manifolds*. We recall that a Riemannian manifold is said to be locally symmetric when all the covariant derivative components $\bar{R}_{ABCD;E}$ of its curvature tensor vanish identically.

Given a hypersurface $M^n$ immersed in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ we will assume that, for certain constants $c_1$ and $c_2$, the sectional curvature $K$ of $\overline{M}^{n+1}$ satisfies the following standard conditions:

$$K(\eta, v) = \frac{c_1}{n}, \tag{1.1}$$

for vectors $\eta \in T^{\perp}M$ and $v \in TM$; and

$$K(u, v) \geq c_2, \tag{1.2}$$

for vectors $u, v \in TM$. In particular, we note that space forms $\mathbb{Q}_c^{n+1}$ of constant sectional curvature $c$ satisfy conditions (1.1) and (1.2) for $\frac{c_1}{n} = c_2 = c$. On the other hand, we also observe that such kind of curvature constrains already appear in the current literature (see, for instance, [16], [17] and [19]).

Denoting by $\bar{R}_{AB}$ the components of the Ricci tensor of a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying condition (1.1), the scalar curvature $\bar{R}$ of

$\overline{M}^{n+1}$ is given by

$$\bar{R} = \sum_{A=1}^{n+1} \bar{R}_{AA} = \sum_{i,j=1}^{n} \bar{R}_{ijij} + 2\sum_{i=1}^{n} \bar{R}_{(n+1)i(n+1)i} = \sum_{i,j=1}^{n} \bar{R}_{ijij} + 2c_1.$$

Moreover, it is well known that the scalar curvature of a locally symmetric Riemannian manifold is constant. So, $\mathcal{R} = \frac{1}{n(n-1)} \sum_{i,j} \bar{R}_{ijij}$ is a constant naturally attached to a locally symmetric Riemannian manifold satisfying condition (1.1).

Now, we are in the position to state our first result.

**Theorem 1.** *Let $\overline{M}^{n+1}$, $n \geq 3$, be a locally symmetric Riemannian manifold satisfying conditions* (1.1) *and* (1.2). *Let $M^n$ be a complete linear Weingarten hypersurface immersed in $\overline{M}^{n+1}$ such that $R = aH + b$ with $b > \mathcal{R}$. Setting $c = 2c_2 - \frac{c_1}{n}$, suppose that $R \geq \mathcal{R} - c$ when $c \leq 0$, and $R > \mathcal{R} - \frac{2}{n}c$ when $c > 0$. If $H$ attains its maximum on $M^n$ and*

$$\sup_M |\Phi|^2 \leq \frac{n(n-1)\left(R + c - \mathcal{R}\right)^2}{(n-2)\left(nR + 2c - n\mathcal{R}\right)},$$

*then $M^n$ is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

We want to point out that, when the ambient space $\overline{M}^{n+1}$ is a Riemannian space form $\mathbb{Q}_c^{n+1}$, from the last part of the proof of Theorem 1.1 of [6] we see that the isoparametric hypersurfaces of $\mathbb{Q}_c^{n+1}$ with two distinct principal curvatures, one of which is simple, are such that $|\Phi|^2 \equiv \frac{n(n-1)R^2}{(n-2)(nR-(n-2)c)}$. Hence, taking into account that in this case $\mathcal{R} = c$, we conclude that our restriction on $|\Phi|$ is, in fact, a mild hypothesis.

Afterwards, we derive another characterization result concerning the case that the ambient space is an Einstein manifold. In what follows, $\mathcal{L}^1(M)$ stands for the space of Lebesgue integrable functions on $M^n$.

**Theorem 2.** *Let $\overline{M}^{n+1}$, $n \geq 3$, be a locally symmetric Einstein manifold satisfying conditions* (1.1) *and* (1.2). *Let $M^n$ be a complete linear Weingarten hypersurface immersed in $\overline{M}^{n+1}$ such that $R = aH + b$ with $(n-1)a^2 + 4n(b - \mathcal{R}) > 0$. Setting $c = 2c_2 - \frac{c_1}{n}$, suppose that $R \geq \mathcal{R} - c$ when $c \leq 0$, and $R > \mathcal{R} - \frac{2}{n}c$ when $c > 0$. If $H$ is bounded, $|\nabla H| \in \mathcal{L}^1(M)$ and*

$$\sup_M |\Phi|^2 \leq \frac{n(n-1)\left(R + c - \mathcal{R}\right)^2}{(n-2)\left(nR + 2c - n\mathcal{R}\right)},$$

*then $M^n$ is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

This paper is organized in the following way: in Section 2 we recall standard facts related to hypersurfaces immersed in a locally symmetric Riemannian manifold and establish a suitable Simons type formula concerning such hypersurfaces.

Besides, in Section 3 we quote some key lemmas that will be essential in the proofs of our results. Finally, in Section 4 we present the proofs of Theorems 1 and 2.

## 2. A Simons type formula

In this section we will introduce some basic facts and notations that will appear in the paper. In what follows, we will suppose that all considered hypersurfaces are orientable and connected.

Let $M^n$ be an $n$-dimensional hypersurface in a Riemannian manifold $\overline{M}^{n+1}$. We choose a local field of orthonormal frame $\{e_A\}$ in $\overline{M}^{n+1}$ with dual coframe $\{\omega_A\}$ such that, at each point of $M^n$, $e_1, \ldots, e_n$ are tangent to $M^n$ and $e_{n+1}$ is normal to $M^n$. We will use the following convention for the indices:

$$1 \le A, B, C, \ldots \le n+1, \; 1 \le i, j, k, \ldots \le n.$$

In this setting, denoting by $\{\omega_{AB}\}$ the connection forms of $\overline{M}^{n+1}$, we have that the structure equations of $\overline{M}^{n+1}$ are given by:

$$d\omega_A = -\sum_i \omega_{Ai} \wedge \omega_i - \omega_{An+1} \wedge \omega_{n+1}, \quad \omega_{AB} + \omega_{BA} = 0, \tag{2.1}$$

$$d\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D} \bar{R}_{ABCD}\omega_C \wedge \omega_D. \tag{2.2}$$

Here, $\bar{R}_{ABCD}$, $\bar{R}_{CD}$ and $\bar{R}$ denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the Riemannian manifold $\overline{M}^{n+1}$.

In this setting, we have

$$\bar{R}_{CD} = \sum_B \bar{R}_{BCDB}, \quad \bar{R} = \sum_A \bar{R}_{AA}.$$

Moreover, the components $\bar{R}_{ABCD;E}$ of the covariant derivative of the Riemannian curvature tensor of $\overline{M}^{n+1}$ are defined by

$$\begin{aligned} \sum_E \bar{R}_{ABCD;E}\omega_E &= d\bar{R}_{ABCD} - \sum_E (\bar{R}_{EBCD}\omega_{EA} \\ &\quad + \bar{R}_{AECD}\omega_{EB} + \bar{R}_{ABED}\omega_{EC} + \bar{R}_{ABCE}\omega_{ED}). \end{aligned}$$

Next, we restrict all the tensors to $M^n$. First of all, $\omega_{n+1} = 0$ on $M^n$, so $\sum_i \omega_{n+1i} \wedge \omega_i = d\omega_{n+1} = 0$ and by Cartan's Lemma we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \; h_{ij} = h_{ji}. \tag{2.3}$$

This gives the second fundamental form of $M^n$, $h = \sum_{ij} h_{ij}\omega_i\omega_j e_{n+1}$ and its square length $S = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature $H$ of $M^n$ is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

The structure equations of $M^n$ are given by

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.4}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l. \tag{2.5}$$

Using the structure equations we obtain the Gauss equation

$$R_{ijkl} = \bar{R}_{ijkl} + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.6}$$

where $R_{ijkl}$ are the components of the curvature tensor of $M^n$.

The Ricci curvature and the normalized scalar curvature of $M^n$ are given, respectively, by

$$R_{ij} = \sum_k \bar{R}_{ikjk} + nHh_{ij} - \sum_k h_{ik}h_{kj} \tag{2.7}$$

and

$$R = \frac{1}{n(n-1)}\sum_i R_{ii}. \tag{2.8}$$

From (2.7) and (2.8) we obtain

$$n(n-1)R = \sum_{i,j} \bar{R}_{ijij} + n^2H^2 - S. \tag{2.9}$$

The first covariant derivatives $h_{ijk}$ of $h_{ij}$ satisfy

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{kj}\omega_{ki} - \sum_k h_{ik}\omega_{kj}. \tag{2.10}$$

Then, by exterior differentiation of (2.3), we obtain the *Codazzi equation*

$$h_{ijk} - h_{ikj} = -\bar{R}_{(n+1)ijk}. \tag{2.11}$$

Similarly, the second covariant derivatives $h_{ijkl}$ of $h_{ij}$ are given by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_l h_{ljk}\omega_{li} - \sum_l h_{ilk}\omega_{lj} - \sum_l h_{ijl}\omega_{lk}. \tag{2.12}$$

By exterior differentiation of (2.10), we can get the following *Ricci formula*

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{jm}R_{mikl}. \tag{2.13}$$

At this point, we will assume that the ambient space $\overline{M}^{n+1}$ is a locally symmetric Riemannian manifold. Thus, we have that

$$\bar{R}_{(n+1)ijkl} = \bar{R}_{(n+1)i(n+1)k}h_{jl} + \bar{R}_{(n+1)ij(n+1)}h_{kl} - \sum_m \bar{R}_{mijk}h_{ml}, \tag{2.14}$$

where $\bar{R}_{(n+1)ijkl}$ denotes the covariant derivative of $\bar{R}_{(n+1)ijk}$ as a tensor on $M^n$ so that

$$\begin{aligned}\sum_l \bar{R}_{(n+1)ijkl}\omega_l = d\bar{R}_{(n+1)ijk} - \sum_l \bar{R}_{(n+1)ljk}\omega_{li}\\ - \sum_l \bar{R}_{(n+1)ilk}\omega_{lj} - \sum_l \bar{R}_{(n+1)ijl}\omega_{lk}.\end{aligned}$$

The Laplacian $\Delta h_{ij}$ of $h_{ij}$ is defined by $\Delta h_{ij} = \sum_k h_{ijkk}$. Thus, from equations (2.11), (2.13) and (2.14) we deduce that

$$\begin{aligned}\Delta h_{ij} \;=\;& (nH)_{ij} + nH\bar{R}_{(n+1)i(n+1)j} + nH\sum_k h_{ik}h_{kj}\\ & - \sum_k h_{ij}\bar{R}_{(n+1)k(n+1)k}\\ & - Sh_{ij} + \sum_{k,m}(h_{mi}\bar{R}_{mkjk} + h_{mj}\bar{R}_{mkik} + 2h_{km}\bar{R}_{mijk}).\end{aligned} \tag{2.15}$$

Since $\Delta S = 2\left(\sum_{i,j} h_{ij}\Delta h_{ij} + \sum_{i,j,k} h_{ijk}^2\right)$, taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on $M^n$ such that $h_{ij} = \lambda_i\delta_{ij}$, from equation (2.15) we obtain the following Simons type formula

$$\begin{aligned}\frac{1}{2}\Delta S \;=\;& \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{,ii} + nH\sum_i \lambda_i^3 - S^2\\ & + nH\sum_i \lambda_i \bar{R}_{(n+1)i(n+1)i} - S\sum_i \bar{R}_{(n+1)i(n+1)i}\\ & + \frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_j)^2\bar{R}_{ijij}.\end{aligned} \tag{2.16}$$

Now, let $\Psi = \sum_{i,j}\Psi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor on $M^n$ defined by

$$\Psi_{ij} = nH\delta_{ij} - h_{ij}.$$

Following Cheng-Yau [10], we introduce an operator $\square$ associated to $\Psi$ acting on any smooth function $f$ by

$$\square f = \sum_{i,j} \Psi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{2.17}$$

Setting $f = nH$ in (2.17) and taking again a local frame field $\{e_1, \ldots, e_n\}$ on $M^n$ such that $h_{ij} = \lambda_i \delta_{ij}$, from equation (2.9) we obtain the following:

$$\begin{aligned} \square(nH) &= nH\Delta(nH) - \sum_i \lambda_i (nH)_{,ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_{,i}^2 - \sum_i \lambda_i (nH)_{,ii} \\ &= \frac{n(n-1)}{2}\Delta R + \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_i \lambda_i (nH)_{,ii}. \end{aligned}$$

Consequently, taking into account equation (2.16), we get

$$\begin{aligned} \square(nH) &= \frac{n(n-1)}{2}\Delta R + \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + nH\sum_i \lambda_i^3 - S^2 \\ &\quad + nH\sum_i \lambda_i \bar{R}_{(n+1)i(n+1)i} - S\sum_i \bar{R}_{(n+1)i(n+1)i} \\ &\quad + \frac{1}{2}\sum_{i,j} (\lambda_i - \lambda_j)^2 \bar{R}_{ijij}. \end{aligned} \tag{2.18}$$

**Remark 1.** Concerning the previous computation of $\square(nH)$, when the ambient space is a Riemannian space form, we also would like to suggest the readers to see Corollary 3.3 (case $r = 1$) in [8].

## 3. Key lemmas

In this section, we will quote some key lemmas which will be essential in order to prove our classifications of linear Weingarten hypersurfaces in locally symmetric Riemannian manifolds. The first one is a classic algebraic lemma due to M. Okumura in [13], and completed with the equality case in [1] by H. Alencar and M. do Carmo.

**Lemma 1.** *Let $\mu_1, \ldots, \mu_n$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta \geq 0$. Then*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3, \tag{3.1}$$

*and equality holds if and only if at least $(n-1)$ of the numbers $\mu_i$ are equal.*

To obtain the second lemma, we can argue as in the proof of either Lemma 2.1 of [12] or Lemma 3.2 of [6]. In what follows, as before, $\mathcal{R} = \frac{1}{n(n-1)} \sum_{i,j} \bar{R}_{ijij}$.

**Lemma 2.** *Let $M^n$ be a linear Weingarten hypersurface in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying condition* (1.1) *such that $R = aH + b$ for some $a, b \in \mathbb{R}$. Suppose that*

$$(n-1)^2 a^2 + 4n(n-1)(b - \mathcal{R}) \geq 0. \tag{3.2}$$

*Then we have*

$$\sum_{i,j,k} h_{ijk}^2 \geq n^2 |\nabla H|^2. \tag{3.3}$$

*Moreover, if the inequality* (3.2) *is strict and the equality holds in* (3.3) *on $M^n$, then $H$ is constant on $M^n$.*

Proof: Since we are supposing that $R = aH + b$, from equation (2.9) we get

$$2 \sum_{i,j} h_{ij} h_{ijk} = \big(2n^2 H - n(n-1)a\big) H_{,k}.$$

Thus, we get

$$4 \sum_k \left( \sum_{i,j} h_{ij} h_{ijk} \right)^2 = \big(2n^2 H - n(n-1)a\big)^2 |\nabla H|^2.$$

Consequently, using Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned} 4S \sum_{i,j,k} h_{ijk}^2 &= 4 \left( \sum_{i,j} h_{ij}^2 \right) \left( \sum_{i,j,k} h_{ijk}^2 \right) \\ &\geq 4 \sum_k \left( \sum_{i,j} h_{ij} h_{ijk} \right)^2 \\ &= \big(2n^2 H - n(n-1)a\big)^2 |\nabla H|^2. \end{aligned} \tag{3.4}$$

On the other hand, since $R = aH + b$, from equation (2.9) we easily see that

$$\begin{aligned} \big(2n^2 H - n(n-1)a\big)^2 &= n^2(n-1)^2 a^2 + 4n^3(n-1)b \\ &\quad -4n^3(n-1)\mathcal{R} + 4n^2 S. \end{aligned} \tag{3.5}$$

Consequently, from (3.2), (3.4) and (3.5) we have

$$S \sum_{i,j,k} h_{ijk}^2 \geq n^2 S |\nabla H|^2.$$

Therefore, we obtain either $S = 0$ and $\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2$, or $\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2$. Moreover, if inequality (3.2) is strict, from (3.5) we get that

$$\left(2n^2H - n(n-1)a\right)^2 > 4n^2S.$$

Now, let us assume in addition that the equality holds in (3.3) on $M^n$. In this case, we wish to show that $H$ is constant on $M^n$. Suppose, by contradiction, that it does not occur. Consequently, there exists a point $p \in M^n$ such that $|\nabla H(p)| > 0$. So, one deduces from (3.4) that $4S(p)\sum_{i,j,k} h_{ijk}^2(p) > 4n^2S(p)|\nabla H(p)|^2$ and, since $\sum_{i,j,k} h_{ijk}^2(p) = n^2|\nabla H(p)|^2 > 0$, we arrive at a contradiction. Hence, in this case, we conclude that $H$ must be constant on $M^n$. □

In what follows, we will consider the Cheng-Yau's modified operator

$$L = \square - \frac{n-1}{2}a\Delta. \tag{3.6}$$

Related to such operator, we have the following sufficient criteria of ellipticity which extends Lemma 3.3 of [6].

**Lemma 3.** *Let $M^n$ be a linear Weingarten hypersurface immersed in a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ such that $R = aH + b$ with $b > \mathcal{R}$. Then, $L$ is elliptic.*

Proof: From equation (2.9), since $R = aH + b$ with $b > \mathcal{R} = \frac{1}{n(n-1)}\sum_{i,j}\bar{R}_{ijij}$, we easily see that $H$ cannot vanish on $M^n$ and, by the choice of an appropriate orientation, we may assume that $H > 0$ on $M^n$.

Let us consider the case that $a = 0$. Since $R = b > \mathcal{R}$, from equation (2.9), if we choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on $M^n$ such that $h_{ij} = \lambda_i\delta_{ij}$, we have that $\sum_{i<j}\lambda_i\lambda_j > 0$. Consequently,

$$n^2H^2 = \sum_i \lambda_i^2 + 2\sum_{i<j}\lambda_i\lambda_j > \lambda_i^2$$

for every $i = 1, \ldots, n$ and, hence, we have that $nH - \lambda_i > 0$ for every $i$. Therefore, in this case, we conclude that $L$ is elliptic.

Now, suppose that $a \neq 0$. From equation (2.9) we get that

$$a = -\frac{1}{n(n-1)H}\left(S - n^2H^2 + n(n-1)(b - \mathcal{R})\right).$$

Consequently, for every $i = 1, \ldots, n$, with a straightforward algebraic computation we verify that

$$\begin{aligned} nH - \lambda_i - \frac{n-1}{2}a &= nH - \lambda_i + \frac{1}{2nH}\big(S - n^2H^2 + n(n-1)(b-\mathcal{R})\big) \\ &= \frac{1}{2nH}\left(\sum_{j\neq i}\lambda_j^2 + (\sum_{j\neq i}\lambda_j)^2 + n(n-1)(b-\mathcal{R})\right). \end{aligned}$$

Therefore, since we are assuming that $b > \mathcal{R}$, we also conclude in this case that $L$ is elliptic. □

To close this section, we quote a generalized maximum principle due to Caminha (cf. Proposition 2.1 of [9]; see also the Theorem of Karp [11]). In what follows, $\mathrm{div}X$ denotes the divergence of a smooth vector field $X \in TM$.

**Lemma 4.** *Let $X$ be a smooth vector field on a complete oriented Riemannian manifold $M^n$, such that* $\mathrm{div}X$ *does not change sign on $M^n$. If $|X| \in \mathcal{L}^1(M)$, then* $\mathrm{div}X = 0$.

## 4. Proofs of Theorems 1 and 2

Now, we are in position to prove Theorem 1.

Proof of Theorem 1: From (2.18) and (3.6), if we choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on $M^n$ such that $h_{ij} = \lambda_i\delta_{ij}$, we get

$$\begin{aligned} L(nH) = &\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + nH\sum_i \lambda_i^3 - S^2 \\ &+ \sum_i \bar{R}_{(n+1)i(n+1)i}(nH\lambda_i - S) + \sum_{i,j}(\lambda_i - \lambda_j)^2\bar{R}_{ijij}. \end{aligned} \tag{4.1}$$

Thus, from Lemma 2, we have

$$\begin{aligned} L(nH) \geq H&\sum_i \lambda_i^3 - S^2 + \sum_i \bar{R}_{(n+1)i(n+1)i}(nH\lambda_i - S) \\ &+ \sum_{i,j}(\lambda_i - \lambda_j)^2\bar{R}_{ijij}. \end{aligned} \tag{4.2}$$

Setting $\Phi_{ij} = h_{ij} - H\delta_{ij}$, we will also consider the following symmetric tensor

$$\Phi = \sum_{i,j}\Phi_{ij}\omega_i \otimes \omega_j.$$

Let $|\Phi|^2=\sum_{i,j}\Phi_{ij}^2$ be the square of the length of $\Phi$. It is easy to check that $\Phi$ is traceless and, recalling that $\mathcal{R}=\frac{1}{n(n-1)}\sum_{i,j}\bar{R}_{ijij}$, from (2.9) we get

$$|\Phi|^2=S-nH^2=n(n-1)H^2+n(n-1)(\mathcal{R}-R). \tag{4.3}$$

Moreover, if we take a local frame field $e_1,\dots,e_n$ at $p\in M^n$, such that

$$h_{ij}=\lambda_i\delta_{ij} \text{ and } \Phi_{ij}=\mu_i\delta_{ij},$$

it is straightforward to check that

$$\sum_i\mu_i=0,\ \sum_i\mu_i^2=|\Phi|^2 \text{ and } \sum_i\mu_i^3=\sum_i\lambda_i^3-3H|\Phi|^2-nH^3.$$

Consequently, by applying Lemma 1 to the real numbers $\mu_1,\dots,\mu_n$, we get

$$\begin{aligned} nH\sum_i\lambda_i^3-S^2 &= -(|\Phi|^2+nH^2)^2+n^2H^4+3nH^2|\Phi|^2+nH\sum_i\mu_i^3 \\ &\geq -|\Phi|^4+nH^2|\Phi|^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3. \end{aligned} \tag{4.4}$$

On the other hand, using curvature conditions (1.1) and (1.2), we get

$$\sum_i\bar{R}_{(n+1)i(n+1)i}(nH\lambda_i-S)=c_1(nH^2-S)=-c_1|\Phi|^2 \tag{4.5}$$

and

$$\begin{aligned} \sum_{i,j}(\lambda_i-\lambda_j)^2\bar{R}_{ijij} &\geq c_2\sum_{i,j}(\lambda_i-\lambda_j)^2 \\ &= 2nc_2(S-nH^2)=2nc_2|\Phi|^2. \end{aligned} \tag{4.6}$$

Hence, setting $c=2c_2-\dfrac{c_1}{n}$, from (4.2), (4.4), (4.5) and (4.6) we obtain that

$$L(nH)\geq|\Phi|^2\left(-|\Phi|^2+nH^2-\frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|+nc\right). \tag{4.7}$$

From (4.3), we obtain

$$n(n-1)H^2=n(n-1)(R-\mathcal{R})+|\Phi|^2. \tag{4.8}$$

Thus, from (4.7) and (4.8) we get

$$L(H)\geq\frac{1}{n(n-1)}|\Phi|^2P_R\left(|\Phi|\right), \tag{4.9}$$

where

$$\begin{aligned} P_R(x) \;&=\; -(n-2)x^2-(n-2)x\sqrt{x^2+n(n-1)(R-\mathcal{R})} \\ &\quad +n(n-1)(R+c-\mathcal{R}). \end{aligned} \tag{4.10}$$

Our restrictions on $R$ guarantee that $P_R(0) > 0$ and the function $P_R(x)$ is strictly decreasing for $x \geq 0$, with $P_R(x^*) = 0$ at

$$x^* = (R+c-\mathcal{R})\sqrt{\frac{n(n-1)}{(n-2)\,(nR+2c-n\mathcal{R})}} > 0.$$

Thus, our hypothesis on $|\Phi|$ guarantees that

$$L(H) \geq \frac{1}{n(n-1)}|\Phi|^2 P_R\,(|\Phi|) \geq 0. \tag{4.11}$$

Consequently, since Lemma 3 guarantees that $L$ is elliptic and as we are supposing that $H$ attains its maximum on $M^n$, from (4.11) we conclude that $H$ is constant on $M^n$. Thus, taking into account equation (4.1), we get

$$\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2 = 0,$$

and it follows that $\lambda_i$ is constant for every $i = 1, \ldots, n$.

If $|\Phi| < x^*$, then from (4.11) we have that $|\Phi| = 0$ and, hence, $M^n$ is totally umbilical. If $|\Phi| = x^*$, since the equality holds in (3.1) of Lemma 1, we conclude that $M^n$ is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple. □

Now, we present the proof of Theorem 2.

Proof of Theorem 2: From (2.17) we have that

$$\Box f = \mathrm{trace}(P_1 \circ \nabla^2 f),$$

where $\nabla^2 f$ stands for the self-adjoint linear operator metrically equivalent to the hessian of $f$ and, denoting by $I$ the identity in the algebra of smooth vector fields on $M^n$,

$$P_1 = nHI - h. \tag{4.12}$$

Thus, by using the standard notation $\langle\,,\rangle$ for the induced metric of $M^n$, we get

$$\Box f = \sum_i \langle P_1(\nabla_{e_i}\nabla f), e_i\rangle,$$

where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame on $M^n$. Consequently, we have that

$$\begin{aligned} \operatorname{div}(P_1(\nabla f)) &= \sum_i \langle(\nabla_{e_i}P_1)(\nabla f), e_i\rangle + \sum_i \langle P_1(\nabla_{e_i}\nabla f), e_i\rangle \\ &= \langle \operatorname{div}P_1, \nabla f\rangle + \Box f, \end{aligned} \tag{4.13}$$

where

$$\operatorname{div}P_1 := \operatorname{trace}(\nabla P_1) = \sum_i (\nabla_{e_i}P_1)(e_i).$$

On the other hand, since $\overline{M}^{n+1}$ is an Einstein manifold with $n \geq 3$, there exists a parameter $\lambda \in \mathbb{R}$ such that $\overline{\mathrm{Ric}} = \lambda\langle\,,\rangle$, where $\overline{\mathrm{Ric}}$ denotes the Ricci tensor of $\overline{M}^{n+1}$. Thus, denoting by $\overline{R}$ the curvature tensor of $\overline{M}^{n+1}$, from Lemma 25 of [5] (see also Lemma 3.1 of [2]) we have

$$\langle \operatorname{div}P_1, \nabla f\rangle = \sum_i \langle \overline{R}(\eta, e_i)\nabla f, e_i\rangle = \overline{\mathrm{Ric}}(\eta, \nabla f) = \lambda\langle\eta, \nabla f\rangle = 0,$$

where $\eta$ stands for the unit normal vector field on $M^n$.

Hence, from (4.13), we conclude that

$$\Box f = \operatorname{div}(P_1(\nabla f)). \tag{4.14}$$

From (4.14), we have that

$$L(nH) = \operatorname{div}(P(\nabla H)), \tag{4.15}$$

where

$$P = nP_1 + \frac{n(n-1)}{2}aI. \tag{4.16}$$

Moreover, since $H$ is supposed to be bounded on $M^n$, from equation (2.9) we have that $h$ is also bounded on $M^n$. Consequently, from (4.12) and (4.16) we see that there exists a positive constant $C$ such that $|P| \leq C$. Thus, since we are also assuming that $|\nabla H| \in \mathcal{L}^1(M)$, we obtain that

$$|P(\nabla H)| \leq |P||\nabla H| \leq C|\nabla H| \in \mathcal{L}^1(M). \tag{4.17}$$

Thus, from (4.11), (4.15), (4.17), we can apply Lemma 4 to obtain that $L(nH) = 0$ on $M^n$. Consequently, taking into account that all the inequalities that we have obtained along the proof of Theorem 1 are, in fact, equalities, from equation (4.1) we have that

$$\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2.$$

Hence, since we are assuming $(n-1)a^2 + 4n(b - \mathcal{R}) > 0$, where

$$\mathcal{R} = \frac{1}{n(n-1)} \sum_{i,j} \bar{R}_{ijij},$$

by applying Lemma 2 we get that $H$ is constant on $M^n$.

Therefore, in a similar way as in the last part of the proof of Theorem 1 we conclude that $M^n$ is either totally umbilical or an isoparametric hypersurface with two distinct principal curvatures one of which is simple. □

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Cícero P. Aquino:
Departmento de Matemática, Universidade Federal do Piauí,
64.049-550 Teresina, Piauí, Brazil

*E-mail:* cicero.aquino@ufpi.edu.br

Henrique F. de Lima, Fábio R. dos Santos, Marco Antonio L. Velásquez:
Departamento de Matemática, Universidade Federal de Campina Grande,
58.429-970 Campina Grande, Paraíba, Brazil

*E-mail:* henrique@dme.ufcg.edu.br
fabio@dme.ufcg.edu.br
marco.velasquez@pq.cnpq.br

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