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x	a_x^{iai}	a_x^{ia}
25	0,813	5,888
35	0,688	2,985
45	0,331	0,837
55	0,056	0,086

Quoique ces valeurs soient d'ordre normal, leur intervention dans les applications pratiques n'est nullement essentielle comme nous l'avons démontré ci-dessus.

L'exemple numérique met en lumière la conclusion à laquelle nous sommes arrivés dans le chapitre précédent que pour les buts d'une application pratique, les formules approchées sont tout à fait satisfaisantes et qu'il n'est pas nécessaire de procéder aux calculs laborieux permettant d'établir les valeurs précises.

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS WITH LARGE COEFFICIENTS IN THE DIAGONAL

By JOSEF BÍLÝ

In the course of the solution of a certain practical problem occurring in economic life and described below, there arises a system of linear equations which generally has large (in absolute value) coefficients in the diagonal. The aim of this paper is to describe methods of solution of such equations and by the use of matrix calculus to derive the conditions of application of the described methods of solution and their use in the solution of a certain practical problem. The numerical solution of a system of n linearly independent linear equations in n unknowns involves considerable difficulty when the number of unkowns becomes large, since it is necessary to carry out a large number of multiplications each including a certain error and add the results, thus increasing the error. For the solution of such systems of linear equations it is customary to use either the method of successive elimination or the evaluation of determinants in the numerator and denominator of the formula for the solution of systems of linear equations. (An exhaustive review of the methods in question is to be found in Psychometrica, Vol. 6., 1941, Paul S. Dwyer in the article „The solution of simultaneous equations“.) In the method of successive elimination (see Láska-Hruška, Theorie a praxe numerického počítání, Praha 1934, p. 303), we decrease the number of unknowns and equations by one, so that we obtain successively $n-1$ equations in $n-1$ unknowns, etc. until there remains only one equation in one unknown. For equations whose coefficients are symmetrical on either side of the main diagonal, the method of successive elimination is presented in a special form („Doolittle technique", see article by P. S. Dwyer in „The Annals of Mathematical Statistics", Vol. XII, 1941, p. 449, „The Doolittle Technique"; cf. Gauss, „Supplementum theoriae combinationis observationum", C. F. Gauss Werke, Vol. IV, 1873).

In the determinant solution of systems of linear equations the evaluation of the determinants is laborious when the number of equations is large. In the above-mentioned text-book, the method of successively reducing the degree of the determinants is recommended for their evaluation (p. 305). It is also necessary to mention the methods of calculating determinants and matrices derived by Aitken (Studies in practical mathematics, Proceedings of the Royal Society of Edinburgh, Vol. 57, 1937) and Hotelling („Some new methods in matrix calculation", The Annals of Mathematical Statistics, Vol. XIV, 1943). Finally it should be mentioned that the latest developments in calculating machines greatly facilitate the solution of systems of linear equations with a large number of unknowns.

In certain practical problems (e. g. in statics), systems of linear equations arise with large (in absolute value) coefficients in the diagonal compared with the other coefficients. A similar system also arises in connection with a problem from our economic life. The given systems of linear equations can with advantage be solved by the iteration method. In Czech literature, the above-mentioned text-book Láska-Hruška p. 318 discusses the use of the iteration method for the solution of linear equations with large coefficients in the diagonal, and also Karel Čupr, Numerické řešení rovnic, 1st Edit., Prague, 1945, p. 66. Other text-books used here are Runge, Praxis der Gleichungen, Leipzig 1900, pp. 83—91, Runge-König, Vorlesungen über numerisches Rechnen, Berlin 1924, pp. 183—188. The most complete discussion to date concerning the problem in question is the joint article of von Mises and Pollaczek-Geiringer, „Praktische Verfahren der Gleichungsauflösung", published in Zeitschrift für angewandte Mathematik und Mechanik, Vol. 9, 1929, where earlier literature on the subject is given. In Czech literature, there exist three thorough, critical works on the application of the iteration method to the solution of systems of equations by Prof. Dr Václav Hruška („Contribution to the solution of systems of equations by iteration", „Second contribution to the solution of systems of equations by iteration", „Third contribution to the solution of systems of equations by iteration", published in Procedings of the Czech Academy, Class II, Vol. LIII, Nos. 6, 17 and 32).

For the solution of our problem, we shall start out from the consideration given in the work of v. Mises and Pollaczek-Geiringer quoted above.

Let

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i \quad (i = 1, 2, \ldots, n) \tag{1}$$

be a system of n linear equations, linearly independent in n unknowns. Let

$$x_i^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$$

be approximate values of the roots $x_1, x_2, \ldots, x_n$.

Let us form

$$x_i^{(2)} = x_i^{(1)} + c_i \sum_{j=1}^{n} a_{ij}x_j^{(1)} - b_i c_i \tag{2}$$

and repeat this procedure, so that the l-th step will be

$$x_i^{(l+1)} = x_i^{(l)} + c_i \sum_{j=1}^{n} a_{ij}x_j^{(l)} - b_i c_i =$$

$$= (1 + c_i a_{ii})\, x_i^{(l)} + c_i \sum_{j=1}^{n}{}' a_{ij}x_j^{(l)} - b_i c_i \tag{3}$$

where Σ' denotes that the summation does not include the term with subscript $j = i$. We wish to find $c_1, c_2, \ldots, c_n$ and the conditions that $x_i^{(l+1)}$ will be better approximations to the roots x_i than are $x_i^{(l)}$ $(i = 1, 2, \ldots, n)$. These conditions are fulfilled if

$$\sum_{i=1}^{n} |x_i^{(l+1)} - x_i| < \sum_{i=1}^{n} |x_i^{(l)} - x_i|. \tag{4}$$

Let us denote the error of the l-th approximation

$$z_i^{(l)} = x_i^{(l)} - x_i \tag{5}$$

for which we have the relation

$$z_i^{(l+1)} = z_i^{(l)} + c_i \sum_{j=1}^{n} a_{ij}z_j^{(l)} =$$

$$= (1 + c_i a_{ii})\, z_i^{(l)} + c_i \sum_{j=1}^{n}{}' a_{ij}z_j^{(l)}. \tag{6}$$

The inequality (4) can be written in the form

$$\sum_{i=1}^{n} |z_i^{(l+1)}| < \sum_{i=1}^{n} |z_i^{(l)}|. \tag{7}$$

In our solution, we now make use of the fact that — as follows from Eq. (6) — $z_1^{(l+1)}, z_2^{(l+1)}, \ldots, z_n^{(l+1)}$ are linear transformations of the values $z_1^{(l)}, z_2^{(l)}, \ldots, z_n^{(l)}$, which are in turn linear transformations of $z_1^{(l-1)}, z_2^{(l-1)}, \ldots, z_n^{(l-1)}$ with the same matrix. It is therefore a question of applying successively the same linear transformation. The idea that the iteration procedure for the solution of systems of linear equations involves the application of linear transformations is used by Hotelling in his paper mentioned above, a fact which the writer discovered only after the conclusion of his own work.

$z_i^{(l+1)}$ can be expressed in terms of $z_1^{(1)}, z_2^{(1)}, \dots, z_n^{(1)}$:

$$z_i^{(l+1)} = \sum_{k=1}^{n} a_{ik}^{(l)} z_k^{(1)} \quad (i = 1, 2, \dots, n) \tag{8}$$

where

$$a_{ik}^{(l)} = \sum_{j=1}^{n} a_{ij}^{(1)} a_{jk}^{(l-1)}$$

$$a_{ij}^{(1)} = c_i a_{ij} \text{ if } j \neq i$$

and

$$a_{ii}^{(1)} = 1 + c_i a_{ii}. \tag{9}$$

If we denote the matrices of the transformation and of the compound transformation by

$$A_c = \left\| \begin{matrix} 1 + c_1 a_{11}, & c_1 a_{12}, & \dots, c_1 a_{1n} \\ c_2 a_{21}, & 1 + c_2 a_{22}, & \dots, c_2 a_{2n} \\ \cdot & \cdot & \dots, \cdot \\ c_n a_{n1}, & c_n a_{n2}, & \dots, 1 + c_n a_{nn} \end{matrix} \right\|, \tag{10}$$

$$A_c^l = \left\| \begin{matrix} a_{11}^{(l)}, & a_{12}^{(l)}, & \dots, a_{1n}^{(l)} \\ a_{21}^{(l)}, & a_{22}^{(l)}, & \dots, a_{2n}^{(l)} \\ \cdot & \cdot & \dots, \cdot \\ a_{n1}^{(l)}, & a_{n2}^{(l)}, & \dots, a_{nn}^{(l)} \end{matrix} \right\|, \tag{11}$$

then

$$A_c^{(l+1)} = A_c \, . \, A_c^{(l)} = A_c^{(l)} \, . \, A_c. \tag{12}$$

The necessary and sufficient condition for the convergence of the iteration procedure (3) for any chosen system of first approximate values of the roots $x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}$ is that

$$\lim_{l \to \infty} a_{ik}^{(l)} = 0 \text{ for } i, \; k = 1, 2, 3, \dots, n \tag{13}$$

since then

$$\lim_{l \to \infty} z_i^{(l)} = 0.$$

We shall now derive several easily applicable conditions which suffice for condition (13) to be fulfilled.

Let us consider first the sum of the absolute values of the elements of the matrices (10) and (11) in columns and in rows. From (9) we have for the elements of the i-th column of the matrix:

$$\sum_{j=1}^{n} a_{ji}^{(l+1)} = \sum_{s=1}^{n} a_{si}^{(l)} (a_{1s}^{(1)} + a_{2s}^{(1)} + a_{3s}^{(1)} + \dots + a_{ns}^{(1)}) \tag{14a}$$

and for the sum of the absolute values

$$\sum_{j=1}^{n}|a_{ji}^{(l+1)}| \leqq \sum_{s=1}^{n}|a_{si}^{(l)}|(|a_{1s}^{(1)}| + |a_{2s}^{(1)}| + |a_{3s}^{(1)}| + \dots + |a_{ns}^{(1)}|). \qquad (14b)$$

Similarly for the sum of the k-th row, we have

$$\sum_{j=1}^{n}a_{kj}^{(l+1)} = \sum_{r=1}^{n}a_{kr}^{(l)}(a_{r1}^{(1)} + a_{r2}^{(1)} + a_{r3}^{(1)} + \dots + a_{rn}^{(1)}). \qquad (15a)$$

and for the sum of the absolute values

$$\sum_{j=1}^{n}|a_{kj}^{(l+1)}| \leqq \sum_{r=1}^{n}|a_{kr}^{(l)}|(|a_{r1}^{(1)}| + |a_{r2}^{(1)}| + |a_{r3}^{(1)}| + \dots + |a_{rn}^{(1)}|) \qquad (15b)$$

We require to find the smallest value μ so that

$$|a_{1s}^{(1)}| + |a_{2s}^{(1)}| + |a_{3s}^{(1)}| + \dots + |a_{ns}^{(1)}| \leqq \mu \text{ for } s = 1, 2, 3, \dots, n. \qquad (16)$$

It then follows from (14b) that

$$\sum_{j=1}^{n}|a_{ji}^{(l+1)}| \leqq \mu\sum_{s=1}^{n}|a_{si}^{(l)}| \qquad (17a)$$

and on successive application for $l = 1, 2, 3, \dots, l$, we obtain

$$\sum_{j=1}^{n}|a_{ji}^{(l+1)}| \leqq \mu^{l+1} \text{ for } i = 1, 2, 3, \dots, n. \qquad (17b)$$

If $\mu < 1$, then $\lim\limits_{l\to\infty} a_{ji}^{(l)} = 0$ for $i, j = 1, 2, 3, \dots, n$ and therefore the iteration procedure (3) converges.

We make use of Eq. (15b) in that we require to find the smallest number ν such that

$$|a_{r1}^{(1)}| + |a_{r2}^{(1)}| + \dots + |a_{rn}^{(1)}| \leqq \nu \text{ for } r = 1, 2, \dots, n \qquad (18)$$

and we write Eq. (15b) for $l = 1, 2, \dots, l$

$$\sum_{j=1}^{n}|a_{kj}^{(2)}| \leqq \nu\sum_{r=1}^{n}|a_{kr}^{(1)}| \leqq \nu^2, \text{ etc.}$$

$$\sum_{j=1}^{n}|a_{kj}^{(l+1)}| \leqq \nu^{l+1}. \qquad (19)$$

If $\nu < 1$, then $\lim\limits_{l\to\infty}|a_{kj}^{(l)}| = 0$ and therefore the iteration procedure (3) converges.

Conditions (16) and $\mu < 1$ mean that the sum of the absolute values of the elements of each column of the matrix (10) must be less than 1; condition (18) and $\nu < 1$ mean that the sum of the absolute values of the elements of each row of the matrix (10) must be less than 1.

We immediately see that c_i must have opposite sign to a_{ii}, since, if this were not the case, the element $a_{ii}^{(1)} = 1 + c_i a_{ii}$ would itself render impossible the fulfilment of each of the conditions of convergence. If $|a_{ii}|$ is large compared with $|a_{ik}|$ ($k = 1, 2, \ldots, n$) or compared with $|a_{ji}|$ ($j = 1, 2, \ldots, n$), it is sufficient to choose $c_i = -\frac{1}{a_{ii}}$, thus eliminating the „dangerous" element for the fulfilment of the conditions of convergence and also ensuring that the numbers $|c_i a_{ik}|$ are small, since $|c_i| = \frac{1}{|a_{ii}|}$ is a small number if $|a_{ii}|$ is large. Then the sufficient conditions for the convergence of the iteration procedure (3) are

$$\sum_{j=1}^{n}{}' \left| \frac{a_{js}}{a_{ss}} \right| \leqq \mu < 1, \tag{20}$$

$$\sum_{j=1}^{n}{}' \left| \frac{a_{rj}}{a_{rr}} \right| \leqq \nu < 1. \tag{21}$$

Independently from the procedure of the above-cited article we thus reach criterion (20), a result known in previous works; in addition, we have obtained a new criterion given by condition (21).

We arrive more simply at the iteration procedure (3) with $c_i = -\frac{1}{a_{ii}}$ than in the work of v. Mises and Pollaczek-Geiringer by writing Eq. (1) in the form

$$x_i = \frac{b_i}{a_{ii}} - \sum_{j=1}^{n}{}' \frac{a_{ij}}{a_{ii}} x. \tag{22}$$

In their text-books, Láska-Hruška, Čupr, Runge and Runge-König all start out from this form of the equation.

The matrix A_c can then be written

$$\left\| \begin{matrix} 0, & -\frac{a_{12}}{a_{11}}, & \ldots, & -\frac{a_{1n}}{a_{11}} \\ -\frac{a_{21}}{a_{22}}, & 0, & \ldots, & -\frac{a_{2n}}{a_{22}} \\ \vdots & & & \\ -\frac{a_{n1}}{a_{nn}}, & -\frac{a_{n2}}{a_{nn}}, & \ldots, & 0 \end{matrix} \right\|.$$

We obtain a further criterion of convergence for the iteration procedure (3) if we apply two transformations (6) in turn, i. e. we compute

$$z_i^{(1)}, z_i^{(3)}, z_i^{(5)}, \ldots, z_i^{(2n+1)}.$$

Then we have the relation

$$z_i^{(2l+1)} = \sum_{k=1}^{n} a_{ik}^{(2)} z_k^{(2l-1)} = \sum_{k=1}^{n} a_{ik}^{(2l)} z_k^{(1)}. \tag{23}$$

For the matrix of coefficients, the equation

$$A_c^{2l+2} = A_c^2 \cdot A_c^{2l} \tag{24}$$

holds and for their elements

$$a_{ik}^{(2l+2)} = \sum_{j=1}^{n} a_{ij}^{(2)} a_{jk}^{(2l)}, \tag{25}$$

$$a_{ij}^{(2)} = \sum_{r=1}^{n} a_{ir}^{(1)} a_{rj}^{(1)}. \tag{26}$$

From Lagrange's identity follow the inequalities

$$(a_{ij}^{(2)})^2 \leqq [\sum_{r=1}^{n} (a_{ir}^{(1)})^2]\, [\sum_{r=1}^{n} (a_{rj}^{(1)})^2], \tag{27}$$

$$(a_{ik}^{(2l+2)})^2 \leqq [\sum_{s=1}^{n} (a_{is}^{(2)})^2]\, [\sum_{s=1}^{n} (a_{sk}^{(2l)})^2]. \tag{28}$$

Adding in rows, we obtain

$$\sum_{j=1}^{n} (a_{ij}^{(2)})^2 \leqq [\sum_{r=1}^{n} (a_{ir}^{(1)})^2]\, [\sum_{j=1}^{n} \sum_{r=1}^{n} (a_{rj}^{(1)})^2], \tag{29}$$

$$\sum_{k=1}^{n} (a_{ik}^{(2l+2)})^2 \leqq [\sum_{s=1}^{n} (a_{is}^{(2)})^2]\, [\sum_{k=1}^{n} \sum_{s=1}^{n} (a_{sk}^{(2l)})^2] \tag{30}$$

and then adding in columns

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij}^{(2)})^2 \leqq [\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij}^{(1)})^2]^2 \tag{31}$$

$$\sum_{i=1}^{n} \sum_{k=1}^{n} (a_{ik}^{(2l+2)})^2 \leqq [\sum_{i=1}^{n} \sum_{s=1}^{n} (a_{is}^{(2)})^2]\, [\sum_{k=1}^{n} \sum_{s=1}^{n} (a_{sk}^{(2l)})^2] \tag{32}$$

and hence, on application successively for

$$l = 1, 2, \dots l - 1, l$$

$$\sum_{i=1}^{n} \sum_{k=1}^{n} (a_{ik}^{(2l+2)})^2 \leqq [\sum_{i=1}^{n} \sum_{k=1}^{n} (a_{ik}^{(1)})^2]^{2l+2}. \tag{33}$$

If

$$\sum_{i=1}^{n} \sum_{k=1}^{n} (a_{ik}^{(1)})^2 < 1 \tag{34}$$

then $\lim\limits_{l \to \infty} a_{ik}^{(2l)} = 0$ for all i and k, so that the iteration procedure (3) converges.

Condition (34) is connected with the important conception of the norm of a matrix. By the norm of a matrix is understood the square root of the sum of the products of the elements of a matrix with their conjugate complex numbers. If the matrix consists only of real elements, then its norm is the square root of the sum of the squares of all its elements.

Mises and Pollaczek-Geiringer derived the criterion (34) in a somewhat different manner in the above-cited article and called it the Schmidt condition since it is similar to the Schmidt condition in the theory of integral equations.

If $c_i = -\frac{1}{a_{ii}}$, the condition for convergence of the iteration procedure is

$$\sum_{i=1}^{n} \sum_{k=1}^{n}{}' \frac{a_{ik}^2}{a_{ii}^2} < 1 \quad k \neq i. \tag{35}$$

As far as the relation between the condition (34) and the conditions (20) and (21) are concerned, it must be stated that in certain cases the convergence criteria (20) and (21) prove the convergence, but criterion (34) does not, and in other cases the opposite is true. Thus for instance, in the example given in the above-mentioned article,

$$\begin{aligned} x &= a - 0{,}75y \\ y &= b - 0{,}75x \end{aligned}$$

the convergence cannot be shown by the criterion (35) but can be derived however from (20) and (21). In the example

$$\begin{aligned} x &= 1 + 0{,}5y + 0{,}5z \\ y &= 2 + 0{,}1x + 0{,}1z \\ z &= 3 + 0{,}1x + 0{,}5y \end{aligned}$$

the opposite is the case.

For numerical computation it is important to know the error of the l-th approximate value of the root x_i obtained by the iteration procedure. The expression for this error is derived by Mises an Pollaczek-Geiringer in their paper.

Let us denote

$$\delta_i^{(l)} = x_i^{(l)} - x_i^{(l+1)} = z_i^{(l)} - z_i^{(l+1)}. \tag{36}$$

Then, from Eq. (3) we have

$$\delta_i^{(l)} = - c_i \sum_{j=1}^{n} a_{ij} x_j^{(l)} + b_i c_i \tag{37}$$

and since

$$b_i = \sum_{j=1}^{n} a_{ij} x_j$$

then

$$c_i \sum_{j=1}^{n} a_{ij} z_j^{(l)} + \delta_i^{(l)} = 0, \quad i = 1, 2, 3, \ldots, n. \tag{38}$$

From this system of equations for $z_1^{(l)}, z_2^{(l)}, \ldots, z_n^{(l)}$, we calculate

$$z_i^{(l)} = \frac{1}{A} \sum_{k=1}^{n} \frac{\delta_k^{(l)}}{c_k} A_{ki} \tag{39}$$

where A is the determinant of the system of equations (1) and A_{ki} is the cofactor of the element a_{ki} in it. If

$$\left| \frac{\delta_k^{(l)}}{c_k} \right| < \varepsilon \text{ for } k = 1, 2, 3, \ldots, n,$$

then

$$\left| z_i^{(l)} \right| < \varepsilon \sum_{k=1}^{n} \left| \frac{A_{ki}}{A} \right|. \tag{40}$$

If condition (16) is fulfilled and $\mu < 1$ i. e. if $\sum_{i=1} |a_{ik}^{(1)}| \leqq \mu < 1$ then from Eq. (6) written in the form $z_i^{(l+1)} = \sum_{k=1}^{n} a_{ik}^{(1)} z_k^{(l)}$ we obtain the inequality

$$\sum_{i=1}^{n} |z_i^{(l+1)}| \leqq \mu \sum_{i=1}^{n} |z_i^{(l)}|. \tag{41}$$

In view of Eq. (36), we further have

$$\sum_{i=1}^{n} |z_i^{(l)} - z_i^{(l+1)}| = |\delta_1^{(l)}| + |\delta_2^{(l)}| + |\delta_3^{(l)}| + \ldots + |\delta_n^{(l)}| = \delta^{(l)}.$$

Using Eq. (41), we obtain

$$\sum_{i=1}^{n} |\delta_i^{(l)}| = \sum_{i=1}^{n} |z_i^{(l)} - z_i^{(l+1)}| \geqq \sum_{i=1}^{n} |z_i^{(l)}| - \sum_{i=1}^{n} |z_i^{(l+1)}| \geqq \sum_{i=1}^{n} |z_i^{(l)}| - \mu \sum_{i=1}^{n} |z_i^{(l)}| =$$

$$= (1 - \mu) \sum_{i=1}^{n} |z_i^{(l)}|.$$

Hence

$$\sum_{i=1}^{n} |z_i^{(l)}| \leqq \frac{\delta^{(l)}}{1 - \mu} \tag{42}$$

and thus

$$z_i^{(l)} < \frac{\delta^{(l)}}{1 - \mu}.$$

The authors call the described iteration procedure „iteration in simultaneous steps“ (in Gesamtschritten), because in Eq. (6) the approximative values of the roots from the same step i. e. $z_1^{(l)}, z_2^{(l)}, z_3^{(l)}, \ldots, z_n^{(l)}$ are substituted at once into all the equations.

Besides this iteration method, the authors mention the iteration procedure by single steps (in Einzelschritten), which is carried out in the following way: when calculating the i-th approximate value in the $(l+1)$-th step, we substitute for $x_1, x_2, \ldots, x_{i-1}$ the values obtained in the $(l+1)$-th step and for $x_i, x_{i+1}, \ldots, x_n$ we then substitute the values obtained in the previous l-th step, so that

$$x_i^{(l+1)} = x_i^{(l)} + c_i \sum_{k=1}^{i-1} a_{ik} x_k^{(l+1)} + c_i \sum_{k=i}^{n} a_{ik} x_k^{(l)} - b_i c_i. \tag{43}$$

If we again introduce $a_{ik}^{(1)} = c_i a_{ik}$ for $i \neq k$, $a_{ii}^{(1)} = 1 + c_i a_{ii}$, we can write

$$z_i^{(l+1)} = \sum_{j=1}^{i-1} a_{ij}^{(1)} z_j^{(l+1)} + \sum_{j=i}^{n} a_{ij}^{(1)} z_j^{(l)}. \tag{44}$$

According to the above authors, the iteration procedure by single steps was presented first by Ph. L. Seidel in the Transactions of the Munich Academy in the year 1874; its convergence is shown for the case where $c_i = -\frac{1}{a_{ii}}$ and the coefficients a_{ik} are symmetrical, i. e. $a_{ik} = a_{ki}$ and the quadratic form $\Sigma a_{ik} x_i x_k$ is positive definite. Since these conditions are not fulfilled in the example under investigation, the proof will not be given here. It will suffice to note that if condition (18) is satisfied and if $\nu < 1$, i. e.

$$\sum_{k=1}^{n} a_{ik} \leq \nu < 1$$

the convergence of the iteration procedure by single steps can be shown. The proof will be given for equations in three unknowns.

The iteration procedure by single steps means that we carry out a linear transformation always on one variable while leaving the rest unchanged and then repeat this procedure. If we apply the transformations carried out successively on $x_1^{(l)}$, $x_2^{(l)}$ and $x_3^{(l)}$, i. e. if we form complete cycle of transformations, we obtain the matrix of the applied transformations,

$$\begin{Vmatrix} \alpha_{11}, \alpha_{12}, \alpha_{13} \\ \alpha_{21}, \alpha_{22}, \alpha_{23} \\ \alpha_{31}, \alpha_{32}, \alpha_{33} \end{Vmatrix} = \begin{Vmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ a_{31}^{(1)}, & a_{32}^{(1)}, & a_{33}^{(1)} \end{Vmatrix} \cdot \begin{Vmatrix} 1, & 0, & 0 \\ a_{21}^{(1)}, & a_{22}^{(1)}, & a_{32}^{(1)} \\ 0, & 0, & 1 \end{Vmatrix} \cdot \begin{Vmatrix} a_{11}^{(1)}, & a_{12}^{(1)}, & a_{13}^{(1)} \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{Vmatrix}. \tag{45}$$

Multiplication of the matrices shows that

$$\begin{Vmatrix} \alpha_{11}, \alpha_{12}, \alpha_{13} \\ \alpha_{21}, \alpha_{22}, \alpha_{23} \\ \alpha_{31}, \alpha_{32}, \alpha_{33} \end{Vmatrix} = \begin{Vmatrix} a_{11}^{(1)}, & a_{12}^{(1)}, & a_{13}^{(1)} \\ a_{21}^{(1)} a_{11}^{(1)}, & a_{21}^{(1)} a_{12}^{(1)} + a_{22}^{(1)}, & a_{21}^{(1)} a_{13}^{(1)} + a_{23}^{(1)} \\ a_{31}^{(1)} a_{11}^{(1)} + a_{32}^{(1)} a_{21}^{(1)} a_{11}^{(1)}, & a_{31}^{(1)} a_{12}^{(1)} + a_{32}^{(1)} a_{21}^{(1)} a_{12}^{(1)} + a_{32}^{(1)} a_{22}^{(1)}, & a_{31}^{(1)} a_{13}^{(1)} + a_{32}^{(1)} a_{21}^{(1)} a_{13}^{(1)} + a_{32}^{(1)} a_{23}^{(1)} + a_{33}^{(1)} \end{Vmatrix}. \tag{46}$$

We have thus transformed the iteration procedure by single steps into the iteration procedure in simultaneous steps with matrix (46). This idea also appears in Hotelling's work. We now add the elements of the last matrix in rows. We obtain

$$\alpha_{11} + \alpha_{12} + \alpha_{13} = a_{11}^{(1)} + a_{12}^{(1)} + a_{13}^{(1)}, \tag{47}$$

$$\alpha_{21} + \alpha_{22} + \alpha_{23} = a_{21}^{(1)}(a_{11}^{(1)} + a_{12}^{(1)} + a_{13}^{(1)}) + a_{22}^{(1)} + a_{23}^{(1)}, \tag{48}$$

$$\begin{aligned} \alpha_{31} + \alpha_{32} + \alpha_{33} = a_{31}^{(1)}(a_{11}^{(1)} + a_{12}^{(1)} + a_{13}^{(1)}) + \\ + a_{32}^{(1)}[a_{21}^{(1)}(a_{11}^{(1)} + a_{12}^{(1)} + a_{13}^{(1)}) + a_{22}^{(1)} + a_{23}^{(1)}] + a_{33}^{(1)}. \end{aligned} \tag{49}$$

Let

$$|a_{i1}^{(1)}| + |a_{i2}^{(1)}| + |a_{i3}^{(1)}| \leqq \mu, \; i = 1, 2, 3;$$

then

$$|\alpha_{11}| + |\alpha_{12}| + |\alpha_{13}| \leqq \mu, \tag{50}$$

$$|\alpha_{21}| + |\alpha_{22}| + |\alpha_{23}| \leqq \mu|a_{21}^{(1)}| + |a_{22}^{(1)}| + |a_{23}^{(1)}|, \tag{51}$$

$$|\alpha_{31}| + |\alpha_{32}| + |\alpha_{33}| \leqq \mu|a_{31}^{(1)}| + |a_{32}^{(1)}| \, [\mu|a_{21}^{(1)}| + |a_{22}^{(1)}| + |a_{23}^{(1)}|] + |a_{33}^{(1)}|. \tag{52}$$

Since $\mu < 1$ we know that

$$|\alpha_{21}| + |\alpha_{22}| + |\alpha_{23}| < |a_{21}^{(1)}| + |a_{22}^{(1)}| + |a_{23}^{(1)}|, \tag{53}$$

$$\begin{aligned} |\alpha_{31}| + |\alpha_{32}| + |\alpha_{33}| < |a_{31}^{(1)}| + |a_{32}^{(1)}|(|a_{21}^{(1)}| + |a_{22}^{(1)}| + |a_{23}^{(1)}|) + \\ + |a_{33}^{(1)}| < |a_{31}^{(1)}| + |a_{32}^{(1)}| + |a_{33}^{(1)}|. \end{aligned} \tag{54}$$

We have thus proved the following statement: If the sum of the absolute values of the elements of each row of the matrix (10) for $n = 3$ is smaller than unity, then the sum of the absolute values of the elements of each row of the matrix (46) is smaller than the sum of the absolute values of the elements of the same row respectively of the matrix (10) (or equal to this sum), thus proving the convergence of the iteration procedure (43) for this special case. In the same way it would be possible to carry out the proof for a system of n linear equations in n unknowns.

We now proceed to the application of the derived results to economic life.

Let us suppose we have n economic units for which we are studying a certain economic phenomenon which manifests itself in various degrees. Let us allocate to each degree of the phenomenon a certain number (index). Let there exist between the economic units a dependence of such a type that the degree of the observed phenomenon in any one unit depends on the degree which it displays in all the other economic units. In this article we shall deal with a special example which arose in Czechoslovakia during the transition from capitalist to socialist economy by the nationalisation of enterprises of certain types. In the example in question, the economic units are joint-stock enterprises, the phenomenon which we are studying is the intrinsic value of their shares; the interdependence of the economic units consists in the fact that the companies hold each others shares.

Let there be m joint-stock companies bound together by the fact that each of them has a certain number of the shares of all or of some of the other $m - 1$ companies. We suppose that a joint-stock company can also hold some of its own shares against the regulations. Such a mutual holding of shares can arise either through friendly commercial relations by one company having an interest in obtaining shares and thus participation in the administration of another company with which it is in special close commercial contact; or it may arise from the competition of two rival joint-stock companies one of which buys the shares of the other on the stock-exchange so that it can control its policy at the general meeting. Very often, however, such a mutual holding of shares is the result of many years of accidental commercial transaction. Suppose we wish to find the intrinsic value of shares of companies mutually bound together in the above-described manner. By the intrinsic value of a share we understand the value of the total assets diminished by all debts, divided by the number of shares. If on the credit side of the balance of any company A there are shares of other companies who also hold shares of company A, then in order to determine the value of a share company A, we require to know beforehand the value of the shares of company A. This problem has arisen several times in connection with the nationalisation of certain enterprises in Czechoslovakia and was pointed out by Ing. Bernat in „Hospodář", Vol. II, No. 16, p. 3.

We solve the problem in the following way: Give the companies in question the numbers 1 to m. Company No. i has a value a_i of assets without the shares of companies bound together, and further it has n_{ij} shares of company No. j ($j = 1, 2, 3, \ldots, m$). We admit also $n_{ii} \neq 0$. The numbers n_{ij} are positive or zero and at least one number n_{ij}, $i \neq j$ is other than zero. The intrinsic value of the shares of company No. j is C_j. If, however, C_j turned out to be negative, then it must be taken as zero since a shareholder is not liable to pay anything for shares which have already been fully paid. Further, let b_i be the total debts of company No. i (i. e. without its own capital). Its net assets then constitute

$$A_i = a_i + \sum_{j=1}^{n} n_{ij} C_j - b_i. \tag{55}$$

The value of one of the N_i shares into which the stock of company No. i is divided is then

$$C_i = \frac{A_i}{N_i} = \frac{a_i - b_i + \sum_{j=1}^{n} n_{ij} C_j}{N_i}. \tag{56}$$

If we introduce $M_i = N_i - n_{ii}$ which represents the number of shares of company No. i in circulation, and further write $a_i - b_i = d_i$ while Σ' has its usual meaning (that the summation does not apply to the term with subscript $j = i$) then

$$C_i = \frac{d_i + \Sigma' n_{ij} C_j}{M_i}. \tag{57}$$

Let us assume that $M_i > 0$, i. e. let us exclude the possibility that the company illegally holds all its own shares, since such a company would not appear among companies holding shares mutually. Some of the numbers d_i can be negative because a_i is the value only of the assets excluding shares of the other companies and can thus be exceeded by the value of debts.

We mentioned above that negative C_i must be replaced by zero. If $C_i < 0$ the company is involved in debt. These companies in debt must be excluded since their intrinsic value is zero. They are either notorious and can be excluded beforehand, or because of the unknown value of their participation, it may be doubtful they are in debt or not. Their position is suspicious if $d_i < 0$.

We exclude companies involved in debt by calculating the values C_i for all companies and then we exclude those companies for which C_i turns out to be negative and carry out the calculation once more.

Equation (57) is in a form suitable for the use of the iteration method provided the conditions of convergence of the iteration procedure (20) or (21) or (34) are satisfied. If these conditions are not fulfilled, it is necessary to solve the system of linear equations

$$-M_i C_i + \sum_{j=1}^{n}{}' n_{ij} C_j = -d_i \quad (i = 1, 2, 3, \ldots, n).$$

Since the extent of participation in other companies is usually relatively small compared with the number of shares of any company in circulation, the conditions of convergence are generally satisfied. The iteration procedure by single steps is particularly suitable if the convergence conditions are fulfilled, since, if we arrange the companies according to decreasing values of $\frac{d_i}{M_i}$, we can at the same time exclude companies involved in debt.

In this case, one advantage of the iteration procedure lies in the fact that it can be easily explained to the layman who has little faith in calculations which he cannot follow.

Example.

Suppose we have three joint-stock companies of which the first has share capital divided into 10 000 shares, second 20 000 shares, third 50 000 shares.

The *first* company holds 1836 shares of the second, 2895 shares of the third company.

The value of its credit balance excluding the above shares is 18 330 000. The value of debts is 12 574 000.

The *second* company holds 547 shares of the first, 5428 shares of the third company.

The value of its credit balance excluding the above shares is 281 788 000. The value of debts is 197 396 000.

The *third* company holds 296 shares of the first, 3238 shares of the second company and 87 of its own shares.

The value of its credit balance exluding the above shares is 385 106 000. The value of debts is 250 295 000.

From these data the following equations arise:

$$C_1 = 575{,}60 + 0{,}18360C_2 + 0{,}28950C_3,$$
$$C_2 = 4219{,}60 + 0{,}02735C_1 + 0{,}27140C_3,$$
$$C_3 = 2700{,}92 + 0{,}00593C_1 + 0{,}06487C_2.$$

The iteration method leads to the following results (calculated to two decimal points and rounded off to the nearest whole number):

	Simultaneous steps			Single steps		
	C_1	C_2	C_3	C_1	C_2	C_3
1st step	576	4220	2701	576	4220	2701
2nd step	2132	4968	2978	2132	5011	3039
3rd step	2350	5086	3036	2375	5109	3046
4th step	2388	5108	3045	2396	5112	3047
5th step	2395	5111	3046			
6th step	2396	5112	3047			

ZPRÁVY

Nová česká učebnicová literatura z oboru statistiky. Ukazatelem zájmu o theorii statistické methody u nás je vydávání českých učebnic tohoto oboru; jsou to zejména Yuleho „Úvod do theorie statistiky“, vydaný v roce 1926, Kohnovy „Základy theorie statistické methody“, vydané v roce 1929 a „Základy statistické indukce“ od profesora Dr Janka, vydané v roce 1937.