Jaroslav Janko

Difference equation of the policy reserve

Aktuárské vědy, Vol. 8 (1948), No. 2, 68–75

Persistent URL: http://dml.cz/dmlcz/144722

DIFFERENCE EQUATION OF THE POLICY RESERVE.

By Prof. Dr Jaroslav Janko

Starting from a quite general form of assurance being sometimes called integral assurance, the annual premium will be payable in such a way that a person aged x years when effecting the assurance will pay an annual premium $P_{(x)}$ for the first year, when aged $(x + 1)$ years an annual premium $P_{(x)+1}$ for the second year, and accordingly the premiums $P_{(x)+2}, P_{(x)+3}, \ldots$ for the following years. If life-long premiums are being paid the insurer is expecting an income from the assumed population l_x being assured, the value of which in the beginning of the assurance will equal

$$P_{(x)}l_x + P_{(x)+1}l_{x+1}v + P_{(x)+2}l_{x+2}v^2 + \ldots + P_{(x)+\omega-x}l_\omega v^{\omega-x} \qquad (1)$$

The general form of payments will be as follows. If a person aged x years effecting the assurance is supposed to die within the first year, that is from the age of x to the age of $(x + 1)$ years, an annuity μ_0 will be paid at the end of the first year. A person aged from $x + 1$ to $x + 2$ years dying, an annuity μ_1 will be paid at the end of the second year; accordingly a person aged from $x + r$ to $x + r + 1$ years dying in the $(r + 1)^{th}$ year of the assurance, an annuity μ_r will be paid at the end of that year. As long as being alive a lifelong annuity will be paid, that is an annuity μ_0 will be paid at the beginning of the assurance. When being alive at the beginning of the second year an annuity η_1 will be paid. Accordingly when being alive at the beginning of the $(r + 1)^{th}$ year of the assurance an annuity η_r will be paid. The value of all payments made to an assumed population l_x at the beginning of the assurance will be

$$\begin{aligned}&\mu_0 d_x v + \mu_1 d_{x+1}v^2 + \ldots + \mu_{\omega-x}d_\omega v^{\omega+1-x} + \\ &+ \eta_0 l_x + \eta_1 l_{x+1}v + \ldots + \eta_{\omega-x}l_\omega v^{\omega-x}\end{aligned} \qquad (2)$$

According to the principle of equivalence a very general expression for ascertaining net premiums is obtained by making expression (1) equal to expression (2). Multiplying both them by v^x we obtain

$$\begin{aligned}\mu_0 C_x + \mu_1 C_{x+1} + \ldots + \mu_{\omega-x}C_\omega + \eta_0 D_x + \eta_1 D_{x+1} + \ldots + \eta_{\omega-x}D_\omega = \\ = P_{(x)}D_x + P_{(x)+1}D_{x+1} + \ldots + P_{(x)+\omega-x}D_\omega .\end{aligned} \qquad (3)$$

The recurrent equation of premium reserve of this assurance will be

$${}_rV_x p_{x+r-1}v = {}_{r-1}V_x + P_{(x)+r-1} - \eta_{r-1} - \mu_{r-1}q_{x+r-1}v \qquad (4)$$

allowing to ascertain ${}_rV_x$ if ${}_{r-1}V_x$ is known a. s. o.

$${}_{r+1}V_x p_{x+r}v = {}_rV_x + P_{(x)+r} - \eta_r - \mu_r q_{x+r}v \qquad (5)$$

ensuing from it.

From a mathematical point of view recurrent equations of the premium reserve are difference equations of the first ordre. By summing them up expressions for the retrospective and prospective premium reserve are obtained.

When multiplying equation (5) by the expression $v^r {}_r p_x$, where ${}_r p_x = p_x \cdot p_{x+1} \cdots p_{x+r-1}$, we get

$${}_{r+1}V_x\, v^{r+1}\, {}_{r+1}p_x = {}_rV_x\, v^r {}_r p_x + P_{(x)+r} v^r {}_r p_x - \eta_r v^r {}_r p_x - \mu_r v^{r+1} {}_r p_x q_{x+r}.$$

By summing up these equations from 0 to $r - 1$, putting ${}_0V_x = 0$, we arrive at

$${}_rV_x\, v^r {}_r p_x = \sum_{s=0}^{r-1} P_{(x)+s} v^s {}_s p_x - \sum_{s=0}^{r-1} \eta_s v^s {}_s p_x - \sum_{s=0}^{r-1} \mu_s v^{s+1} {}_s p_x q_{x+s} \tag{6}$$

being an expression for the retrospective premium reserve.

When summing up, however, from r to n we obtain the sum

$${}_nV_x v^n {}_n p_x = {}_rV_x v^r {}_r p_x + \sum_{s=r}^{n-1} P_{(x)+s} v^s {}_s p_x - \sum_{s=r}^{n-1} \eta_s v^s {}_s p_x - \sum_{s=r}^{n-1} \mu_s v^{s+1} {}_s p_x q_{x+s}$$

or

$${}_rV_x v^r {}_r p_x = \sum_{s=r}^{n-1} \eta_s v^s {}_s p_x + \sum_{s=r}^{n-1} \mu_s v^{s+1} {}_s p_x q_{x+s} + {}_nV_x v^n {}_n p_x - \sum_{s=r}^{n-1} P_{(x)+s} v^s {}_s p_x \tag{7}$$

being the prospective expression of the premium reserve.

As ${}_r p_x = \dfrac{l_{x+r}}{l_x}$ may be expressed by $v^r {}_r p_x = \dfrac{D_{x+r}}{D_x}$ and the equations (6) and (7) are obtained in the form

$${}_rV_x = \frac{1}{D_{x+r}} \left[\sum_{s=0}^{r-1} P_{(x)+s} D_{x+s} - \sum_{s=0}^{r-1} \eta_s D_{x+s} - \sum_{s=0}^{r-1} \mu_s C_{x+s} \right] \tag{6 bis}$$

$${}_rV_x = \frac{1}{D_{x+r}} \left[\sum_{s=r}^{n-1} \eta_s D_{x+s} + \sum_{s=r}^{n-1} \mu_s C_{x+s} + {}_nV_x D_{x+n} - \sum_{s=r}^{n-1} P_{(x)+s} D_{x+s} \right] \tag{7 bis}$$

for

$${}_s p_x q_{x+s} = \frac{l_{x+s}}{l_x} \cdot \frac{d_{x+s}}{l_{x+s}}.$$

Returning now to the recurrent equation (5) we get an expression for the premium reserve by solving it as a complete linear difference equation of the first order with variable coefficients.

II. The general method of solving equation (5) is illustrated by starting from a homogenious linear difference equation of the first order with variable coefficients. It may be put in the following form

$$f(x+1) - m(x)\, f(x) = 0 \tag{8}$$

assuming x being only integers greater or at least equal to a. A function $y(x)$ is introduced so that $y(a) = 1$ and for $x > a$

$$y(x) = m(a)\, m(a + 1)\, m(a + 2) \ldots m(x - 1).$$

On dividing both terms of the equation by $y(x + 1)$ and putting

$$t(x) = \frac{f(x)}{y(x)}$$

we get $t(x + 1) - t(x) = 0$ the first difference being equal to zero (i. e. $\Delta t(x) = 0$) ensuing from it $f(x) = c \cdot y(x)$.

Further on with regard to $f(a) = c$, for $x > a$ follows

$$f(x) = f(a)\, m(a)\, m(ax + 1) \ldots m(x - 1) = f(a) \prod_{i=a}^{x-1} m(i).$$

When passing from a homogenous to a complete linear difference equation of the first order with variable coefficients x will also be assumed to be integer and such that $x \geqq a$ holds. This equation is then expressed by

$$f(x + 1) - m(x)\, f(x) = U(x). \tag{9}$$

The function $y(x)$ is introduced again so as to be $y(a) = 1$,

$$y(x) = m(a)\, m(a + 1) \ldots m(x - 1) \text{ for } x > a;$$

$$t(x) = \frac{f(x)}{y(x)}. \tag{10}$$

Dividing the whole equation by $y(x + 1)$ we get

$$\Delta t(x) = \frac{U(x)}{y(x + 1)},$$

from which follows

$$t(x) = t(a) + \sum_{i=a}^{x-1} \frac{U(i)}{y(i + 1)}$$

so that for $f(x)$ following solution is obtained from equation (10):

$$f(x) = y(x) \left[f(a) + \sum_{i=a}^{x-1} \frac{U(i)}{y(i + 1)} \right]. \tag{11}$$

III. Applying this result to the solution of the difference equation (5), which will be expressed with altered symbols, the indices will be put as functional arguments to be in agreement with the foregoing deductions. We shall therefore write

$$V(r + 1)\, p(x + r)\, v = V(r) + P(x + r) - \eta(r) - \mu(r)\, q(x + r)\, v$$

x being considered to be constant. The solution is very much facilitated by writing this equation in form (9) so that it follows

$$V(r+1) - \frac{1}{vp(x+r)} V(r) = [P(x+r) - \eta(r) - \mu(r)\, vq(x+r)] \frac{1}{vp(x+r)} \tag{12}$$

According to equation (10) we have

$$y(r) = \frac{1}{vp(x)} \cdot \frac{1}{vp(x+1)} \cdots \frac{1}{vp(x+r-1)}$$

or

$$y(r) = \frac{1}{v\dfrac{l_{x+1}}{l_x}} \cdot \frac{1}{v\dfrac{l_{x+2}}{l_{x+1}}} \cdots \frac{1}{v\dfrac{l_{x+r}}{l_{x+r-1}}} = \frac{1}{v^r \dfrac{l_{x+r}}{l_x}} = \frac{D_x}{D_{x+r}}. \tag{13}$$

From equation (11) we then get the solution

$$V(r) = y(r) \left[V(a) + \sum_{i=a}^{r-1} \frac{1}{y(i+1)} \cdot \frac{P(x+i) - \eta(i) - \mu(i)\, vq(x+i)}{v \cdot p(x+i)} \right]$$

and with regard to the initial conditions $a = 0$, $V(a) = 0$ we obtain

$$V(r) = y(r) \left[\sum_{i=0}^{r-1} \frac{P(x+i) - \eta(i)}{y(i+1)\, vp(x+i)} - \sum_{i=0}^{r-1} \frac{\mu(i)\, q(x+i)}{y(i+1)\, p(x+1)} \right].$$

As however,

$$\frac{y(r)}{y(i+1)\, vp(x+i)} = \frac{1}{vp(x+i)} \cdot \frac{1}{vp(x+i+1)} \cdots \frac{1}{vp(x+r-1)} =$$

$$= \frac{1}{v^{r-i} \dfrac{l_{x+r}}{l_{x+i}}} = \frac{D_{x+i}}{D_{x+r}} \tag{14}$$

the foregoing result may be expressed in the form of

$$V(r) = \sum_{i=0}^{r-1} [P(x+i) - \eta(i)] \frac{1}{v^{r-i} \dfrac{l_{x+r}}{l_{x+i}}} - \sum_{i=0}^{r-1} \mu(i)\, vq(x+i) \cdot \frac{1}{v^{r-i} \dfrac{l_{x+r}}{l_{x+i}}},$$

and this expression is identical with the retrospective form of the premium reserve which was obtained before. We can prove it by multiplying both sides of the equation by $v^r \dfrac{l_{x+r}}{l_x}$ for we get then

$$V(r)\, v^r \frac{l_{x+r}}{l_x} = \sum_{i=0}^{r-1} P(x+i)\, v^i \frac{l_{x+i}}{l_x} - \sum_{i=0}^{r-1} \eta(i)\, v^i \frac{l_{x+i}}{l_x} - \sum_{i=0}^{r-1} \mu(i)\, v^{i+1} q(x+i) \frac{l_{x+i}}{l_x}$$

or

$$V(r)\, v^r{}_r p_x = \sum_{i=0}^{r-1} P(x+i)\, v^i{}_i p_x - \sum_{i=0}^{r-1} \eta(i)\, v^i{}_i p_x - \sum_{i=0}^{r-1} \mu(i) v^{i+1}{}_i p_x q(x+i)$$

where we retained the indices in symbols for ${}_i p_x$ resp. ${}_r p_x$. The solution of the difference equation may be also given in the form of

$$V(r) = \sum_{i=0}^{r-1} [P(x+i) - \eta(i) - \mu(i)\, vq(x+i)] \frac{D_{x+i}}{D_{x+r}} \tag{15}$$

which can be specified for singular particular kinds of assurance.

IV. In the case of pure endowment assurance we have

$$\mu_0 = \mu_1 = \mu_2 = \ldots = 0,\ \eta_n = 1,\ \eta_0 = \eta_1 = \ldots = \eta_{n-1} = \eta_{n+1} = \ldots$$
$$\ldots = \eta_{\omega-x} = 0$$

and thus there will be for $r < n$

$$V(r) = \sum_{i=0}^{r-1} P(x+i) \frac{D_{x+i}}{D_{x+r}}$$

so that with the constant annual premium $P(x+i) = P_{x\overline{n}|}^{\ 1}$ paid for the whole time of the assurance we get

$$V(r) = P_{x\overline{n}|}^{\ 1} \frac{\sum_{i=0}^{r-1} D_{x+i}}{D_{x+r}} = P_{x\overline{n}|}^{\ 1} \frac{N_x - N_{x+r}}{D_{x+r}} = \frac{D_{x+n}}{D_{x+r}} \cdot \frac{N_x - N_{x+r}}{N_x - N_{x+n}}$$

In the case of life assurance we put

$$\eta_0 = \eta_1 = \ldots = \eta_{\omega-x} = 0;\ \ \mu_0 = \mu_1 = \ldots = \mu_{\omega-x} = 1$$

obtaining thus

$$V(r) = \sum_{i=0}^{r-1} P(x+i) \frac{D_{x+i}}{D_{x+r}} - \sum_{i=0}^{r-1} vg(x+i) \frac{D_{x+i}}{D_{x+r}}$$

and the annual premium being constant we have

$$V(r) = P_x \frac{N_x - N_{x+r}}{D_{x+r}} - \sum_{i=0}^{r-1} v \frac{d_{x+i}}{l_{x+i}} \cdot \frac{l_{x+i} v^{x+i}}{D_{x+r}} = P_x \frac{N_x - N_{x+r}}{D_{x+r}} -$$
$$- \frac{{}_{|r}A_x D_x}{D_{x+r}} = \frac{D_x}{D_{x+r}} (P_x\, a_{x\overline{r}|} - {}_{|r}A_x).$$

Starting from integral assurance we arrive at endowment assurance by putting

$$\eta_0 = \eta_1 = \ldots = \eta_{n-1} = \eta_{n+1} = \ldots = \eta_{\omega-x} = 0,\quad \eta_n = 1,$$
$$\mu_0 = \mu_1 = \ldots = \mu_{n-1} = 1.$$

The annual premium being constant we get

$$V(r) = P_{x\overline{n}|}\sum_{i=0}^{r-1}\frac{D_{x+i}}{D_{x+r}} - \sum_{i=0}^{r-1}\frac{C_{x+i}}{D_{x+r}} = P_{x\overline{n}|}\frac{N_x - N_{x+r}}{D_{x+r}} - \frac{M_x - M_{x+r}}{D_{x+r}} =$$

$$= \frac{D_x}{D_{x+r}}\left[\left(\frac{1}{a_{x\overline{n}|}} - d\right)a_{x\overline{r}|} - A^{1}_{x\overline{r}|}\right] = \frac{D_x}{D_{x+r}}\left[\frac{a_{x\overline{r}|}}{a_{x\overline{n}|}} - da_{x\overline{r}|} - 1 + da_{x\overline{r}|} + \right.$$

$$\left. + \frac{D_{x+r}}{D_x}\right] = \frac{D_x}{D_{x+r}}\cdot\frac{a_{x\overline{r}|} - a_{x\overline{n}|}}{a_{x\overline{n}|}} + 1 = 1 - \frac{a_{x+r,\overline{n-r}|}}{a_{x\overline{n}|}}.$$

In case of deponed annuities it is

$$\eta_0 = \eta_1 = \ldots = \eta_{m-1} = 0; \quad \eta_m = \eta_{m+1} = \ldots = \eta_{\omega-x} = 1;$$
$$\mu_0 = \mu_1 = \mu_2 = \ldots = 0.$$

The annual premium $P({}_{m|}a_x) = \dfrac{{}_{m|}a_x}{a_{x\overline{m}|}}$ being constant we get from the general equation (15) and for $r < m$ ${}_rV_x = P({}_{m|}a_x)\sum_{i=0}^{r-1}\dfrac{D_{x+i}}{D_{x+r}}$ $P({}_{m|}a_x)\, a_{x\overline{r}|}\, \dfrac{D_x}{D_{x+r}}$ and for $r > m$ considering that premiums will be paid m-times to the utmost the first sum will be made for $i = 0, 1, 2, \ldots, n-1$, for the other premiums are equal to zero and therefore

$$V(r) = \sum_{i=0}^{m-1} P({}_{m|}a_x)\frac{D_{x+i}}{D_{x+r}} - \sum_{i=m}^{r-1}\frac{D_{x+i}}{D_{x+r}} =$$

$$= \frac{{}_{m|}a_x}{a_{x\overline{m}|}}\cdot a_{x\overline{m}|}\frac{D_x}{D_{x+r}} - {}_{m|m-r}a_x\cdot\frac{D_x}{D_{x+r}} =$$

$$= \frac{D_x}{D_{x+r}}({}_{m|}a_x - {}_{m|m-r}a_x) = \frac{D_x}{D_{x+r}}\cdot {}_{r|}a_x = a_{x+r}.$$

Furthermore it will be shown by two examples how a result can be obtained in a special case by a direct solution of the difference equation being in question.

The recurrent equation of the premium reserve for pure endowment assurance is

$${}_{r+1}V_x p_{x+r} v = {}_rV_x + P^{\ 1}_{x\overline{n}|}$$

which we reduce to the form of the difference equation

$${}_{r+1}V_x - \frac{1}{vp_{x+r}}\cdot {}_rV_x = \frac{P^{\ 1}_{x\overline{n}|}}{vp_{x+r}}$$

and its solution according to (11) will be

$$ {}_rV_x = y(r)\left[{}_0V_x + \sum_{i=0}^{r-1} \frac{P_{\frac{1}{xn|}}}{y(i+1)\, vp_{x+i}}\right] $$

or

$$ {}_rV_x = P_{\frac{1}{xn|}} \sum_{i=0}^{r-1} \frac{y(r)}{y(i+1)\, vp_{x+i}} $$

resulting with regard to (14) in

$$ {}_rV_x = P_{\frac{1}{xn|}} \sum_{i=0}^{r-1} \frac{D_{x+i}}{D_{x+r}} = P_{\frac{1}{xn|}} \frac{N_x - N_{x+r}}{D_{x+r}} = \frac{D_{x+n}}{D_{x+r}} \cdot \frac{N_x - N_{x+r}}{N_x - N_{x+n}}. $$

The recurrent equation for the premium reserve of assurance with a fixed term of payment may be expressed by

$$ {}_{r+1}V_x = ({}_rV_x + {}_xP_{\overline{n|}})\,(1+i) - q_{x+r}\,(v^{n-r-1} - {}_{r+1}V_x) $$

and by arranging it we get

$$ {}_{r+1}V_x - (1 - q_{x+r})\, v - {}_rV_x = P_{\overline{n|}} - q_{x+r} v^{n-r} $$

or

$$ {}_{r+1}V_x - \frac{1}{vp_{x+r}}\, {}_rV_x = \frac{1}{vp_{x+r}}\,({}_xP_{\overline{n|}} - q_{x+r} v^{n-r}) $$

and its solution is quite similar as in the foregoing example

$$ {}_rV_x = y(r) \sum_{i=0}^{r-1} ({}_xP_{\overline{n|}} - q_{x+i} v^{n-i}) \frac{1}{y(i+1) vp(x+i)} $$

or

$$ {}_rV_x = \sum_{i=0}^{r-1} ({}_xP_{\overline{n|}} - q_{x-i} v^{n-i}) \frac{D_{x+i}}{D_{x+r}} = $$

$$ = {}_xP_{\overline{xn|}} \sum_{i=0}^{r-1} \frac{D_{x+i}}{D_{x+r}} - \sum_{i=0}^{r-1} \frac{v^{n-i-1} C_{x+i}}{D_{x+r}} = $$

$$ = \frac{v^n}{a_{\overline{xn|}}} a_{\overline{xr|}} \frac{D_x}{D_{x+r}} - \frac{1}{D_{x+r}} (v^{n-1} C_x + v^{n-2} C_{x+1} + \ldots + v^{n-r-2} C_{x+r-1}) = $$

$$ = v^n \frac{N_n - N_{x+r}}{N_x - N_{x+n}} \cdot \frac{D_x}{D_{x+r}} - \frac{1}{D_{x+r}} \{v^{n+x}[(l_x - l_{x+1}) + (l_{x+1} - l_{x+2}) + $$

$$ + \ldots + (l_{x+r+1} - l_{x+r})]\} = $$

$$ = v^n \frac{D_x}{D_{x+r}} \cdot \frac{N_x - N_{x+r}}{N_x - N_{n+n}} - \frac{1}{D_{x+r}} \cdot v^{n+x}(l_x - l_{x+r}) = $$

$$ = v^n \frac{D_x}{D_{x+r}} \cdot \frac{N_x - N_{x+r}}{N_x - N_{x+n}} - v^n \frac{D_x}{D_{x+r}} + v^{n-r} = $$

$$= - v^n \frac{D_x}{D_{x+r}} \cdot \frac{N_{x+r} - N_{x+n}}{N_x - N_{x+n}} + v^{n-r} =$$

$$= v^{n-r} - \frac{v^n D_x}{N_x - N_{x+n}} \cdot \frac{N_{x+r} - N_{x+n}}{D_{x+r}} = v^{n-r} - {}_kP_{\overline{n|}} a_{x+r, \overline{n-r|}}.$$

We have illustrated the solution of the difference equation of the premium reserve on the hand by the specialisation of the general solution and on the other hand by the direct solution in the special cases. The number of examples could be increased at discretion. Let us consider however only one case of the premium reserve ascertained by the method of sufficient premium. The recurrent formula for this premium reserve is quite similar, for

$${}_{r+1}V_x^d = [{}_rV_x^d + (1 - \gamma) B_{x+r} - \beta - \eta_r] (1 + i) - q_{x+r} (\mu_r - {}_{r+1}V_x^d)$$

or

$${}_{r+1}V_x^d (1 - q_{x+r}) v - {}_rV_x^d = (1 - \gamma) B_{x+r} - \beta - \eta_r - q_{x+r} v \mu_r$$

from which it is quite clear to derive that there is again a complete linear difference equation of the first order with variable coefficients in which contrary to (12) there is only a change of the expression $U(i)$

$${}_{r+1}V_x^d - \frac{1}{vp_{x+r}} {}_rV_x^d = \frac{1}{vp_{x+r}} [(1 - \gamma) B_{x+r} - \beta - \eta_r - q_{x+r} v \mu_r].$$

The initial condition will be in this case ${}_0V_x^d = - \alpha$. According to (11) we get thus the solution

$${}_rV_x^d = y(r) \left[- \alpha + \sum_{i=0}^{r-1} \frac{(1 - \gamma) B_{x+i} - \beta - \eta_i}{y(i + 1) \, vp_{x+i}} - \sum_{i=0}^{r-1} \frac{\mu_i q_{x+i}}{y(i + 1) p_{x+i}} \right],$$

which may be written with regard to (13) and (14)

$${}_rV_x^d = - \alpha \frac{D_x}{D_{x+r}} + \sum_{i=0}^{r-1} [(1 - \gamma) B_{x+i} - \beta - \eta_i] \frac{D_{x+i}}{D_{x+r}} - \sum_{i=0}^{r-1} \mu_i v q_{x+i} \frac{D_{x-i}}{D_{x-r}}.$$

The annual premium B_x being constant and sufficient, being $r < n$ the retrospective reserve for endowment assurance will be

$${}_rV_x^d = - \alpha \frac{D_x}{D_{x+r}} + [(1 - \gamma) B_x - \beta] \sum_{i=0}^{r-1} \frac{D_{x+i}}{D_{x+r}} - \sum_{i=0}^{r-1} \frac{C_{x+i}}{D_{x+r}}$$

or

$${}_rV_x^d = \frac{D_x}{D_{x+r}} \{ - \alpha + [(1 - \gamma) B_x - \beta] \, a_{x\overline{r|}} - A^1_{x\overline{r|}} \}.$$