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D. P. Misra

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More Multiplicity in the Rate of Interest; Poly-party and Poly-creditor Transactions.

By *D. P. Misra, M. Sc.* (Lucknow).

(Termination.)

P. S. It is understood, unless repugnant, that the right side of an equation is zero, and the left side is arranged in descending or ascending powers of the unknown according as the degree is finite or infinite. Degree finite. Let the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + a^4x^{n-4} + \ldots + a_{n-1}x + a_n = 0 \quad (1)$$

have a positive real root α, so that by hypothesis the expressions

$$\left.\begin{array}{r} a_1\alpha^{n-1} + a_2\alpha^{n-2} + a_3\alpha^{n-3} + a_4\alpha^{n-4} + \ldots + a_{n-1}\alpha + a_n, \\ a_2\alpha^{n-2} + a_3\alpha^{n-3} + a_4\alpha^{n-4} + \ldots + a_{n-1}\alpha + a_n, \\ a_3\alpha^{n-3} + a_4\alpha^{n-4} + \ldots + a_{n-1}\alpha + a_n, \\ a_4\alpha^{n-4} + \ldots + a_{n-1}\alpha + a_n, \\ + \ldots\ldots\ldots\ldots\ldots, \\ a_{n-1}\alpha + a_n, \\ a_n, \end{array}\right\} \quad (2)$$

are all of the same sign except that some, but not all, of them may be zero.

We shall then prove that α is the only real positive root of (1). Now since α is a root of (1), we have

$$a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + a_3\alpha^{n-3} + a_4\alpha^{n-4} + \ldots + a_{n-1}\alpha + a_n = 0. \quad (3)$$

Also since expressions (2) are all of the same sign except that some, but not all, of them may be zero, from (3) it follows that the expressions

$$\left.\begin{array}{l} a_0\alpha^n, \\ a_0\alpha^n + a_1\alpha^{n-1}, \\ a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2}, \\ a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + a_3\alpha^{n-3}, \\ a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + a_3\alpha^{n-3} + a_4\alpha^{n-4}, \\ a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + a_3\alpha^{n-3} + a_4\alpha^{n-4} + \ldots\ldots\ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + a_3x^{n-3} + a\text{-}\alpha^{n-4} + \ldots + a_{n-1}\alpha_t, \end{array}\right\} \quad (4)$$

all have the same sign opposite to that of the expressions (2), except that if some, but not all, of the expressions (2) are zero, the corresponding ones in (4) are also zero.

Now since α is real and positive, from (4) it follows that the expressions

$$\left.\begin{array}{l}
a_0, \\
a_0\alpha + a_1, \\
a_0\alpha^2 + a_1\alpha + a_2, \\
a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3, \\
a_0\alpha^4 + a_1\alpha^3 + a_2\alpha^2 + a_3\alpha + a_4, \\
a_0\alpha^5 + a_1\alpha^4 + a_2\alpha^3 + a_3\alpha^2 + a_4\alpha + a_5, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\
a_0\alpha^{n-1} + a_1\alpha^{n-2} + a_2\alpha^{n-3} + a_3\alpha^{n-4} + a_4\alpha^{n-5} + \dots + a_{n-1},
\end{array}\right\} \qquad (5)$$

are all of the same sign except that, if some, but not all, of the expressions (4) are zero, the corresponding ones in (5) are also zero, since (5) are obtained by dividing (4) by α^n, α^{n-1}, α^{n-2}, α^{n-3}, α^{n-4}, α^{n-5}, . . ., α respectively, which are all positive.

From (1) removing the root α, we get,

$$\begin{aligned}
a_0x^{n-1} &+ (a_0\alpha + a_1)\, x^{n-2} + (a_0\alpha^2 + a_1\alpha + a_2)\, x^{n-3} + \\
&+ (a_0\alpha^3 + a_1\alpha^2 + a_2\alpha + a_3)\, x^{n-4} + \\
&+ (a_0\alpha^4 + a\alpha^3 + a_2\alpha^2 + a_3\alpha + a_4)\, x^{n-5} + \dots \\
&+ (a_0\alpha^{n-2} + a_1\alpha^{n-3} + \dots + a_{n-2})\, x + \\
&+ (a_0\alpha^{n-1} + a_1\alpha^{n-2} + \dots + a_{n-1}) = 0.
\end{aligned} \qquad (6)$$

Now since expressions (5) are all of the same sign except that some, but not all, of them may be zero, the terms of (6) are all of the same sign except that same but not all, of them may be zero. Hence (6) has no change of sign and consequently it cannot have a real positive root.

Hence the only positive root of (1) is α and no other. Degree infinite. Let the equation

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots = 0. \qquad (7)$$

of an infinite degree have a real positive root α, so that by hypothesis the expressions

$$\left.\begin{array}{r}
a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \dots, \\
a_2\alpha^2 + a_3\alpha^3 + \dots, \\
a_3\alpha^3 + \dots, \\
+ \dots
\end{array}\right\} \qquad (8)$$

are all of the same sign except that some, but not all, of them may be zero. We shall, then, prove that α is the only real positive root of (7).

Since α is a root of (7) by hypothesis, we have

$$a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \dots = 0. \qquad (9)$$

Also since expressions (8) are all of the same sign except that some, but not all, of them may be zero, from (9) it follows that the expressions

$$\left.\begin{array}{l}
a_0, \\
a_0 + a_1\alpha, \\
a_0 + a_1\alpha + a_2\alpha^2 \\
\dots\dots\dots\dots
\end{array}\right\} \qquad (10)$$

are all of the same sign except that, if some, but not all, of the expressions (8) are zero, the corresponding ones in (10) are also zero.

Now since α is positive, dividing the expressions (10) respectively by $\alpha, \alpha^2, \alpha^3, \ldots,$ we easily see that the expressions

$$\left.\begin{array}{l} a_0\alpha^{-1}, \\ a_0\alpha^{-2} + a_1\alpha^{-1}, \\ a_0\alpha^{-3} + a_1\alpha^{-2} + a_2\alpha^{-1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \tag{11}$$

are all of the same sign as (10) except that, if some, but not all, of the expressions (10) are zero, the corresponding ones in (11) are also zero.

Removing the root α from (7) i. e. dividing (7) by $(\alpha - x)$, we get,

$$a_0\alpha^{-1} + (a_0\alpha^{-2} + a_1\alpha^{-1})\, x + (a_0\alpha^{-3} + a_1\alpha^{-2} + a_2\alpha^{-1})\, x^2 + \ldots = 0. \tag{12}$$

It follows from the signs of (11) that all the terms of (12) are of the same sign except that some, but not all, of them may be zero. So (12) cannot have any change of sign and since Decartes' rule of signs has been proved to be true in the first paper in the case of rational integral algebraic equations of even an infinite degree, (12) cannot have a real positive root, as is otherwise obvious.

Hence the only real positive root of (7) is α and no other.

The converse of Lemma I is not necessarily true. E. G., the equation $x^3 - 3x^2 + 3x - 2 = 0$ has $x = 2$ the only one real positive root, still, -2, $(3 \times 2) - 2$, and $(-3 \times 4) + (3 \times 2) - 2$, are not all of the same sign.

Lemma II. If a rational integral algebraic equation of any degree finite or infinite has a negative real root α and if, when α is substituted for the unknown, the expressions obtained by omitting from the left side successively the first term, the first two terms, and so on, are all of the same sign except that some, but not all, of these expressions may be zero, then α is the only real negative root of the equation.

Degree finite. Let the equation (1) have a real negative root α, so that by hypothesis the expressions (2) are all of the same sign except that some, but not all, of them may be zero. We shall, then, prove that α is the only real negative root of (1).

Now since α is a root of (1), (3) still holds. Also since the expressions (2) are all of the same sign except that some, but not all, of them may be zero, from (3) it follows that the expressions (4) are all of the same sign opposite to that of expressions (2), except that if some, but not all, of the expressions (2) are zero, the correspondies ones in (4) are also zero.

Now since α is negative, provided none of the expressions (2) and therefore of (4) and of (5) are zero, it follows from (4) that the expressions (5) are alternately positive and negative, or alternately negative and positive; for all the expressions (5) are obtained by dividing the expressions (4) respectively by $\alpha^n, \alpha^{n-1}, \alpha^{n-2}, \alpha^{n-3}, \alpha^{n-4}, \alpha^{n-5}, \ldots, \alpha$. which are alternately positive and negative or alternately negative and positive according as n is even or odd. In case some, but not all, of the expressions (5) are zero, the expressions (5) of an odd order which are not zero are all of one and the same sign while those of an even order which are not zero are all of the sign opposite to that of those of odd order which are not zero, where in order to assign the order of each expression those whose value is zero are not supposed to be absent from (2), (4), and (5).

In either case, removing the root α from (1), we get (6), where the terms of an odd degree have one and the same sign and those of an even degree too have one and the same sign opposite to that of those of an odd degree, the terms with zero coefficients having obviously no sign.

Let (6) be denoted by $f(x) = 0$, then in $f(-y) = 0$ all the terms are of the same sign; which shows that $f(-y) = 0$ cannot have a real positive root i. e. $f(x) = 0$ cannot have a real negative root.

Hence (1) cannot have a real negative root other than α.

Degree infinite. Let the equation (7) of an infinite degree have a real negative root α, so that by hypothesis the expressions (8) are all of the same sign except that some, but not all, of them may be zero. We shall, then, prove that α is the only real negative root of (7).

Since α is a root of (7) by hypothesis, (9) still holds. Also since expressions (8) are all of the same sign except that some, but not all of them may be zero, it follows that the expressions (10) are all of the same sign opposite to that of the expressions (8), except that, if some, but not all, of the expressions (8) are zero, the corresponding ones in (10) are also zero.

Now since α is negative provided none of the expressions (8) and therefore of (10) and of (11) are zero, the expressions (11) are alternately positive and negative or alternately negative and positive; for expressions (11) are obtained by dividing expressions (10) respectively by $\alpha, \alpha^2, \alpha^3, \ldots$ which are alternately negative and positive. In case some, but not all, of the expressions (8) and therefore of (10) and of (11) are zero, in (11) all the expressions of an even order which are not zero have one and the same sign and all those of an odd order which are not zero also have one and the same sign opposite to that of those of an even order, where the order is assigned as previously.

In either case removing the root α from (7) we get (12) in which all the terms of an even degree are of one and the same sign and also all the terms of an odd degree have one and the same sign opposite to that of those of an even degree, the term independent of x being of even degree zero.

Let (12) be denoted by $f(x) = 0$, so that $f(-y) = 0$ has no changes of sign, i. e. $f(-y) = 0$ cannot have a real positive root. Therefore $f(x) = 0$ i. e. (12) cannot have a real negative root.

Hence (7) has no real negative root other than α.

The converse of Lemma II is not necessarily true, for although the equation $x^3 + 3x^2 + 3x + 2 = 0$ has $x = -2$ the only real negative root, yet 2, $3(-2) + 2$, and $3(-2)^2 + 3(-2) + 2$ are not all of the same sign.

By the help of these two lemmas we can now state and prove Dr. Steffensen's criterion with greater generality and precision, also without any flaw in proof.

Precise statement and Proof of the Criterion. If i is a real positive rate of interest at which a transaction balances and if the transaction remains monocreditor at i, then there is no real positive rate other than i in the transaction.

Let the equation of value, finite or infinite in degree, be $\sum_{n=0}^{t} S_n v^n = 0$, where t is any positive integer finite or infinite, so that by hypothesis $\sum_{n=r}^{t} S_n v^n$ is of one and the same sign for $r > 0$, when evaluated at rate i satisfying

$$\sum_{n=0}^{t} S_n v^n = 0 \tag{13}$$

except that some, but not all, of the expressions $\sum_{n=r}^{t} S_n v^n$ may be zero for $r > 0$.

Since i is $i > 0$, it follows that $\alpha = \dfrac{1}{1+i}$ is a positive real root of (13), and since $\sum_{n=r}^{t} S_n v^n$ if of one and the same sign for $r > 0$, except that for some, but not all, positive integral values of r it may be zero, for $v = \alpha$. Hence, from Lemma I, $\alpha \equiv \dfrac{1}{1+i}$ is the only real positive root of (13). Hence i is the only real positive rate of interest balancing (13).

Corollary I. If in a monocreditor transaction the total repayment is greater than the total advance, there is one and only one real positive rate of interest balancing the a/c. Also there is no properly negative rate of interest balancing the a/c, inadmissible as it would be even if it were. There may be an improperly negative rate of interest at which the transaction may be more creditor, but this rate is inadmissible under the present case.

Let $f(v)$ denote the left side of (13), then $f(0) > 0$ and $f(1) < 0$; so that (13) has at least one root in (0, 1), but being monocreditor it can have only one real positive root; hence there is one and only one real positive rate of interest.

Corollary II. If in a monocreditor transaction the total repayment is equal to the total advance, zero is the only rate of interest. There is no positive or even properly negative rate of interest, inadmissible as the latter would be even if it were. There may be an improperly negative, and therefore inadmissible under the present case, rate.

This follows from $f(1) = 0$ in conjunction with Lemmas I and II.

Corollary III. If a monocreditor transaction having total repayment less than total advance consists of a finite number of payments, it cannot have a positive real rate of interest but has one and only one properly negative rate.

(13) is replaced by

$$\sum_{t=0}^{n} S_t v^t = 0 \tag{14}$$

where n is a positive finite integer. Now since the transaction is monocreditor at the rate of interest involved, $S_n < 0$.

Denoting the left side of (14) by $f(v)$, we have $f(\alpha) < 0$, and $f(1) > 0$, so that (14) has at least one root > 1. But, being monocreditor, it can have one and only one real positive root. Hence it has one and only one real positive root $\alpha \equiv \frac{1}{1+i} > 1$, giving $i < 0$.

If the number of payments is actually interminable, all that we can say is that transaction in Corollary III cannot ahev a real positive rate of interest; for $f(0) > 0$, and $f(1) > 0$; which means that $f(v)$ has either none or an even number of zeros in (0, 1). But since a monocreditor transaction cannot have more than one real positive rate of interest, it follows that $f(v)$ is left with the *alternative of having no zero* in (0, 1).

Also since the transaction is said to be monocreditor, it means that the hypothesis postulates the existence of some rate of interest at which the transaction balances. This rate may be properly or improperly negative; or there may be one properly negative rate and another improperly negative one but no third.

The only relevant question is: — Does there exist a properly negative rate of interest at which an interminable transaction with total repayment less than total capital may be monocreditor and balanced?

The answer is certainly not easy, since the sign of $f(+\alpha)$ is doubtful and $f(1) > 0$. Of course a positive real rate of interest cannot exist.

It is to be noted that the converse of the criterion is not necessarily true, as the converses of the lemmas on the basis of which the criterion has been proved are not necessarily true.

In this proof we have not used processes of infinitesimal calculus and therefore the question of uniform convergence does not arise at all. It is sufficient that $\sum_{n=0}^{\infty} S_n v^n$ is convergent. Even absolute convergence is not required.

In concluding this appendix of the present paper I take the opportunity of thanking both Dr. Steffensen and Mr. Lidstone whose elegant rejoinders have been more than suggestive for the preparation of this paper. The author is also greatly in debted to Messrs. Courcouf and Carter for their supports to the first paper.

22. II. 1935.
D. P. Misra

Appendix A.

An actual case of a triparty transaction.

It is interesting to note that the ordinary annuity payment on the basis of sinking fund accumulations really forms a triparty transaction, though monocreditor in general. If the remunerative and reproductive rates of interest are not identical, there will be a third rate of interest paid by the borrower and this may fitly be called „borrowing rate of interest".

It will be proved in the following lines that the borrowing rate of interest always lies outside the interval formed by the remunerative and reproductive rates of interest.

Let £L be the loan advanced to be repaid in n years by equal annual instalments, i', i, and i_1, the remunerative, reproductive, and borrowing rates of interest respectively, then i_1, is given by

$$\left[Li' + \frac{L}{s_{\overline{n}|}}\right] a_{\overline{n}|}^{i_1} = L, \text{ or } i' + \frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}^{i_1}}$$

or
$$i' + \frac{i}{(1+i)^n - 1} = \frac{i_1}{1-(1+i_1)^{-n}}$$

or
$$i' + \frac{i}{(1+i)^n - 1} = \frac{r-1}{1-r^{-n}} \tag{1}$$

or
$$i' + \frac{i}{(1+i)^n - 1} = \frac{r^{n+1} - r^n}{r^n - 1},$$

unless $r = 0$, i. e. unless $i_1 = -1$, which is repugnant;

or $$\left[i' + \frac{i}{(1+i)^n - 1}\right] r^n - i' - \frac{i}{(1+i)^n - 1} = r^{n+1} - r^n,$$
unless $r - 1 = 0$ i. e. unless $i_1 = 0$.

or $$r^{n+1} - \left[1 + i' + \frac{i}{(1+i)^n - 1}\right] r^n + i' + \frac{i}{(1+i)^n - 1} = 0. \quad (2)$$

Let the left side of (2) be denoted, by $f(w)$, then $f(\alpha) > 0$ and $f(0) > 0$, and $f(1 + i') = i'\left[1 - \frac{s'_{\overline{n}|}}{s_{\overline{n}|}}\right]$, where $s'_{\overline{n}|}$ and $s_{\overline{n}|}$ are respectively at i' and i; which $<$ or > 0, according as $i' >$ or < 0. There fore when $i' > i$, $f(1 + i') < 0$, and since $f(\alpha) > 0$ and $f(0) > 0$, it follows that $f(r)$ has an odd number of zeros in the intervals $(0, 1 + i')$ and $(1 + i', \alpha)$ each. But $f(r)$ can have two real positive zeros at most as it has only two changes of sign. Hence one zero must lie in $(0, 1 - i)$, and one in $(1 + i', \alpha)$.

Now the only zero in $(0, 1 + i')$ is $r = 1$ i. e. $i_1 = 0$, which is inadmissible as (2) has been obtained under the condition $r \neq 1$ i. e. $i_1 \neq 0$.

Hence the only admissible zero of $f(r)$ lies in $(1 + i', \alpha)$; from which $1 + i < 1 + i' < 1 + i_1 < \alpha$, i. e. $i < i' < i_1$.

Thus the borrowing rate of interest is greater than the remunerative one, in case the latter itself exceeds the reproductive one.

If $i' < i$ (quite an improbable assumption in practice), $f(1 + i') > 0$, so that either an even number of zeros of $f(r)$ or none of them lies in $(0, 1 + i')$, and either none or an even number in $(1 + i', \alpha)$. But $f(r)$ can have either no or two zeros, so that both the zeros must lie either in $(0, 1 + i')$ or in $(1 + i', \alpha)$, or in neither, but never in both.

Now one of the zeros $r = 1$ obviously lies in $(0, 1 + i')$, inadmissible as it is as a solution of (1). Therefore the other zero must also lie in $(0, 1 + i')$. Hence $0 < 1 + i_1 < 1 + i' < 1 + i$, i. e. $-1 < i_1 < < i' < i$.

Thus the borrowing rate of interest is less than the remunerative one when the latter itself is less than the reproductive one.

In this case it is not obvious that i_1 is always positive. To investigate this let us find the sign of $f(1 - e)$. Now

$$f(1 - \varepsilon) = (1 - \varepsilon)^{n+1} - \left[1 + i' + \frac{i}{(1+i)^n - 1}\right](1 - \varepsilon)^n +$$
$$+ i' + \frac{i}{(1+i)^n - 1} = \left[\left(i' + \frac{1}{\frac{i}{s_{\overline{n}|}}}\right) n - 1\right]\varepsilon +$$

terms containing squares and higher powers of ε.

∴, if Lt $\varepsilon \to 0$ and $\varepsilon > 0$, the sign of the whole expression depends on that of the first term which contains the first power of ε.

$$\therefore \mathop{Lt}_{\varepsilon \to 0} f(1 - \varepsilon) > \text{or} < 0 \text{ according as } \left(i' + \frac{1}{\overset{i}{s}_{\overline{n}|}}\right) n - 1 > 0 \text{ or } < 0$$

i. e. according as $i' + \frac{1}{\overset{i}{s}_{\overline{n}|}} > \text{or} < \frac{1}{n}$.

Hence when $i' + \frac{1}{\overset{i}{s}_{\overline{n}|}} < \frac{1}{n}$, $\mathop{\mathrm{Lt}}_{\varepsilon \to 0} f(1 - \varepsilon) < 0$, and since $f(0) > 0$, it follows that $0 < 1 + i_1 < 1 < 1 + i' < 1 + i$, i. e. $-1 < i_1 < < 0 < i' < i$. i. e. i_1 is negative if $i' < i$ provided $i' + \frac{1}{\overset{i}{s}_{\overline{n}|}} < \frac{1}{n}$.

It can be verified that, for moderate values of $i > i'$, $i' + \frac{1}{\overset{i}{s}_{\overline{n}|}} > \frac{1}{n}$ for all real positive values of n, but, if i be taken only large enough, cases can be constructed in which $i' + \frac{1}{\overset{i}{s}_{\overline{n}|}} < \frac{1}{n}$. E. G. Take $n = 25$, so that $\frac{1}{n} = 0{,}04$, take $i' = 0{,}03$ and then go on taking increasing values of i till $1/\overset{i}{S}_{\overline{n}|}$ gives no significant figures in the first two decimal places. But since no one would like to invest money on a basis of reproductive rate being greater than the remunerative one, the matter is only of theoretical interest.

It has been assumed throughout, that the annual instalment for sinking fund is paid or becomes due at the end of the year, but the conclusion remains unaltered even if the premiums are paid in advance and even if we have annuities due or any other complications such as the frequency of the payment of one or both of the interest and premium may be different from once a year, for a year is only an arbitrary unit of time, providet only the remunerative and reproductive rates of interest are different.

Thus we finally conclude, that, if the reproductive and remunerative rates of interest do not coincide, the borrowing rate of interest always lies outside the interval formed by the former two, i. e. it éxceedo both when the remunerative rate is higher than the reproductive one, and is less than both when the remunerative rate is lower than the reproductive one. In the latter case there is fear of its becoming properly negative for a sufficiently high value of the reproductive rate.

So there are three different rates of interest, if two of them are known to differ.

Ex. 1. A loan of £ 1000 is to be repaid by so equal annual instalments by a scheme of accumulative sinking fund, the remunerative and the reproductive rates of interest being respectively 6% p. a. and 4% p. a. Find the rate of interest paid by the borrower and that realised by the lender.

The rate of interest paid by the borrower is given by

$$\left[60 + \frac{1000}{s_{\overline{20}|}^{4\%}}\right] a_{\overline{20}|}^{x} = 1000, \text{ from which } a_{\overline{20}|}^{x} = 10,68483, \text{ showing}$$

$$0,06 < x < 0,07.$$

∴ from first difference interpolation, $x = 0,0689634$ per unit p. a.

The rate of interest y realised by the lender is given by

$$60\, a_{\overline{20}|}^{y} + 1000\,(1 + y)^{-20} = 1000, \text{ or } (1000y - 60)\, a_{\overline{20}|}^{y} = 0,$$

from which, since $a_{\overline{20}|}^{y} \neq 0$ for any real value of y including zero, $1000y - 60 = 0$ i. e. $y = 0,06$, as is other wise obvious.

Ex. 2. A loan of £ 1000 is repaid by 10 equal annual instalments. If the remunerative and reproductive rates of interest are 3% and 6% respectively, find the borrowing rate of interest.

We have $1000\left[0,03 + \frac{1}{s_{\overline{10}|}^{6\%}}\right] a_{\overline{10}|}^{x} = 1000$, or $a_{\overline{10}|}^{x} = 9,44563$, giving $0,01 < x < 0,0125$ i. e. $x = 0,0107357$.

Ex. 3. If in the previous ex. the reproductive and remunerative rates are 6% and 2%, we have, $1000\left[0,02 + \frac{1}{s_{\overline{10}|}^{6\%}}\right] a_{\overline{10}|}^{x} = 1000$, or $a_{\overline{10}|}^{x} = 10,431056$.

Denoting the left side of the last equation by $f(x)$, we have $f(-0,01) = 10,5727135$, and $f(-0,0075) = 10,42523$, giving $-0,01 < x < -0,0075$, from which by first difference interpolation we have $x = -0,007598757$.

It is clear that the borrower pays back £ 958,676 in all by equal annual instalments for 10 years while he actually borrowed £ 1000. In spite of this deficit the lender realised 2% p. a. as he receives £ 20 every year for interest on his invested capital of £ 1000 which is itself received as the accumulated amount of sinking fund at the end of 10 years.

Hence there is nothing magical.

Appendix B.

Question of Investor and Repayer.*)

Mr. W. G. Courcouf who is one of the supporters to the first paper, raises in his letters addressed to the writer some interesting queries about the question of Investor and Repayer. Since the same point may be puzzling many others the writer proposes to deal with this question clearly in this appendix.

It may be asked which should be regarded as the investor and which as repayer in the numerical examples of this paper and the first one. E. g. if in the ex. 2 of the first paper A is called the investor, the repayment exceeds the advance and consequently the negative rates of interest arising from the equation of value become inadmissible. If on the other hand B is called the investor the repayment (A's total payment) falls short of the advance (B's total payment) and consequently all the real rates positive or negative become admissible.

It is clear from these two papers that the writer has called investor the maker of the first payment, for repayment can be obtained only after some investment no matter how small this first payment may be. To suppose otherwise is to put the cart before the horse.

It may be argued that, when B borrows money from A and invests it in some business, B is obviously an investor, though it is A that makes the first payment.

This is no doubt true. But the fact that B is the father of C does not mean that B cannot be the son of A or of anybody else. If B borrows money from A, the latter is the investor between himself and B; while B is the investor between himself and his business and so on.

The rule for the decision of investor and repayer may be put down as follows: —

If the first n payments of any two parties to each other are equal and simultaneous, the investor will be he who makes his $(n + 1)$th payment first or whose $(n + 1)$th payment becomes due first. In case the $(n + 1)$th payments are made or become due simultaneously, the investor is he whose $(n + 1)$th payment is the greater of the two.

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*) This word is used as a technical term here, as to the best of the writer's judgement it is not English.