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On a New Form of Life Assurance.

By *J. F. Steffensen* (Copenhagen).

1. Life Insurance has two different aspects, both of them important and not necessarily antagonistic. The Company wishes to make a reasonable profit, and the person insured wants to provide for his survivors as efficiently as his means allow. In the present paper I am mainly looking at the side of the problem which concerns the policyholder, trying to help him to find a form of policy which, for the money he is able to spend, will yield a more rational protection than the endowment assurance for an insufficient amount which is too often the outcome of the canvassing.

From certain points of view the reversionary annuity is an excellent form of protection, in particular because the premium is so small that it enables the policyholder to secure an annuity large enough for supporting his widow in a substantial and lasting way. But there are obvious drawbacks. If the wife dies first there is no insurance. In the case of divorce — not a negligible possibility at the present day — the policyholder may not be interested in keeping up the reversionary annuity; at any rate he cannot, if desired, substitute another wife. Moreover, the policy has no commercial value, because the policy value is not payable in cash, and policy loans are excluded.[1])

If these restrictions can be removed without materially increasing the premium, as I think they can be, it will be a considerable advantage. I therefore suggest a modification of the reversionary annuity that will have the desired effect. If, in the well-known expression for the present value of a reversionary annuity

$$\bar{a}_{x|y} = \int_0^\infty v^t \, {}_tp_{xy} \, \mu_{x+t} \, \bar{a}_{y+t} \, \mathrm{d}t \tag{1}$$

[1]) In the valuation of Widows' Funds by the reversionary method, account is sometimes taken of the probability of marrying more than once. Even then, there is no insurance if the policyholder leaves no widow, and the policy has no commercial value. Moreover, the introduction of the rates of marriage complicates the calculation considerably, unless methods of averaging are resorted to which may be defended in a group insurance but not in the individual policy I have in view. — The reader is referred to D. A. Porteous' book „Pension and Widows' and Orphans' Funds“ and to the literature quoted there.

we replace ${}_tp_{xy}$ by ${}_tp_x$, we obtain an expression which I shall denote by $\bar{a}_{x||y}$, that is

$$\bar{a}_{x||y} = \int_0^\infty v^t \, {}_tp_x \, \mu_{x+t} \, \bar{a}_{y+t} \, dt \tag{2}$$

which is the present value of the new form of insurance. In order to have a short name for it, I shall tentatively call it a Death Annuity; but this is immaterial, and I shall only be too pleased if somebody can suggest a better name.

A death annuity is evidently nothing but a decreasing life insurance where the sum insured t years after entry is $\bar{a}_{y+t}$. But this sum has been fixed in such a way that the death annuity covers not only everything that is covered by the reversionary annuity, but certain important benefits besides. If, at the issue of the policy, (x) had a wife (y), the death annuity covers one per annum from the moment of the husband's death as long as his widow be alive after that moment, just as in the case of the reversionary annuity. If the wife dies before the husband, or if her claim to the benefit has ceased before the husband's death, the insurance continues automatically as a decreasing life insurance for the amount $\bar{a}_{y+t}$ at the time t. But the husband may at any time substitute another life for (y). If, at the time ϑ, he substitutes a life (z), and there is no alteration in the payment of the premium (a question to which I shall presently return), the annual amount S of the new annuity is determined by the equation

$$S \,.\, \bar{a}_{x+\vartheta||z} = \bar{a}_{x+\vartheta||y+\vartheta}.$$

It is essential for the practical working of this new form of insurance that the substitution of another life for (y) should not require another medical examination of the policyholder, or meet other obstacles. It is therefore advisable, although the risk of selection against the company may not be very significant, to employ a somewhat lighter mortality for the annuitant than for (x). On the other hand, it is necessary for the policyholder to inform the company if he substitutes another life, as otherwise the annual amount S of the new annuity cannot be fixed at the time of the substitution, but only at the death of the policyholder. But this inconvenience is so small that it hardly counts, because the policyholder himself is interested in settling the amount at once.

Another question to consider is the payment of the premium. In the case of a reversionary annuity the premium must cease when either of the two lives involved fails. But in the case of a death annuity the insurance is continued if (y) dies before (x), and it is therefore possible to make the premium payable on (x)'s life alone, which is also desirable in order to make the premium as small as possible, so that a comparison with the premium for a reversionary annuity does not exhibit too great a difference in favour of the latter, from the policy-holder's point of view.

2. Before proceeding to the comparison of premiums, it will be suitable to examine whether the expression (2) for the present value of a death annuity presents any particular facilities for the calculation. In the general case this does not seem to be so, and the ordinary method would therefore be to write

$$\bar{a}_{x|y} = \frac{1}{D_x}\int_0^\infty D_{x+t}\,\mu_{x+t}\,\bar{a}_{y+t}\,\mathrm{d}t \tag{3}$$

and to calculate the integral

$$\overline{N}_{x|y} = \int_0^\infty D_{x+t}\,\mu_{x+t}\,\bar{a}_{y+t}\,\mathrm{d}t \tag{4}$$

by numerical integration, whereafter

$$\bar{a}_{x|y} = \frac{\overline{N}_{x|y}}{D_x}. \tag{5}$$

There is, however, one special case, suited for a preliminary investigation, where the calculation is easy, and that is when $y = x$, and the two lives follow the same Makeham formula

$$\mu_x = A + Bc^x = A + Be^{\gamma x}. \tag{6}$$

In this case we find, by (3),

$$\bar{a}_{x|x} = \frac{1}{D_x}\int_x^\infty \overline{N}_x \mu_x\,\mathrm{d}x; \tag{7}$$

but we have, by (6),

$$\mu_x\,\mathrm{d}x = A\,\mathrm{d}x + \frac{1}{\gamma}\mathrm{d}\mu_x,$$

so that

$$\bar{a}_{x|x} = \frac{A}{D_x}\int_x^\infty \overline{N}_x\,\mathrm{d}x + \frac{1}{\gamma\,D_x}\int_x^\infty \overline{N}_x\,\mathrm{d}\mu_x =$$

$$= A\frac{\bar{S}_x}{D_x} + \frac{1}{\gamma\,D_x}[\overline{N}_x\,\mu_x]_x^\infty + \frac{1}{\gamma\,D_x}\int_x^\infty D_x\,\mu_x\,\mathrm{d}x$$

or finally

$$\bar{a}_{x|x} = A\frac{\overline{S}_x}{D_x} + \frac{1-(\delta+\mu_x)\,\bar{a}_x}{\gamma}. \tag{8}$$

We shall presently make certain numerical applications of this formula, but before doing so, we will examine a transformation of (2).

We find by partial integration

$$\bar{a}_{x|y} = -\frac{1}{l_x}\int_0^\infty e^{-\delta t}\,\bar{a}_{y+t}\,\mathrm{d}l_{x+t} =$$

$$= \bar{a}_y + \frac{1}{D_x}\int_0^\infty D_{x+t}\,(\bar{a}'_{y+t} - \delta\bar{a}_{y+t})\,\mathrm{d}t =$$

$$= \bar{a}_y + \frac{1}{D_x}\int_0^\infty D_{x+t}\,(\mu_{y+t}\,\bar{a}_{y+t} - 1)\,\mathrm{d}t$$

or

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_x + \frac{1}{D_x}\int_0^\infty D_{x+t}\,\mu_{y+t}\,\bar{a}_{y+t}\,\mathrm{d}t. \tag{9}$$

The integral in (9) only differs from (3) in this, that μ_{y+t} has taken the place of μ_{x+t}. Now, if (x) and (y) both follow the same Makeham formula (6), we have

$$\mu_{y+t} = c^{y-x}\,\mu_{x+t} + A\,(1 - c^{y-x}), \tag{10}$$

and introducing this expression in (9), we find

$$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_x + c^{y-x}\,\bar{a}_{x|y} + \frac{A\,(1 - c^{y-x})}{D_x}\int_0^\infty D_{x+t}\,\bar{a}_{y+t}\,\mathrm{d}t$$

whence, if $y \neq x$,

$$\bar{a}_{x|y} = \frac{\bar{a}_y - \bar{a}_x}{1 - c^{y-x}} + \frac{A}{D_x}\int_0^\infty D_{x+t}\,\bar{a}_{y+t}\,\mathrm{d}t. \tag{11}$$

The case $A = 0$ corresponds to Gompertz' formula, and we have, then, the curious and simple expression

$$\bar{a}_{x|y} = \frac{\bar{a}_y - \bar{a}_x}{1 - c^{y-x}} = \mu_x\,\frac{\bar{a}_y - \bar{a}_x}{\mu_x - \mu_y} \tag{12}$$

from which follows, *inter alia,*

$$\bar{a}_{x|y} - \bar{a}_{y\|x} = \bar{a}_y - \bar{a}_x. \tag{13}$$

But if $A \neq 0$, (11) is not much more convenient than (3), although the numerical calculation of the integral is easier. Both (11) and (12) suffer from the inconvenience that if the difference $x - y$ is not large, only few figures in the difference $\bar{a}_y - \bar{a}_x$ remain, unless the annuity values are tabulated with more figures than usual. — It is easy to verify that (8) is obtained from (11) for $y \to x$.

It seems to appear from the preceding considerations that, unless special methods of approximation can be invented, the ordinary method of calculation will be to obtain $\bar{a}_{x\|y}$ by numerical integration as mentioned above. But if this be so, it will, in practice, be preferable to start from the annual death annuity

$$a_{x\|y} = \frac{1}{D_x} \sum_{t=1}^{\infty} C_{x+t-1}\, a_{y+t} \tag{14}$$

instead of making the continuous death annuity the starting point.

3. As, in the present paper, I am having principles rather than practical details in view, I confine myself to the case of equal ages, assuming the same Makeham-graduated mortality table for the two lives, so that the simple formula (8) suffices. This enables us, without much numerical work, to form an idea of the premium for a death annuity in comparison with the premium for a reversionary annuity. The table I have employed is the Danish table „Overlevelsesrente-tavlen 1910“, derived from the experience of the Danish State Institution for Life Insurance, concerning men insured by reversionary annuities. The rate of interest is 4%, and the constants are

$$A = ,00431 \qquad \log c = ,045 \qquad \delta = ,039221.$$
$$\log B = 5,5795 - 10 \qquad \gamma = ,103617$$

The adjoined table contains specimens of the functions $\bar{a}_x$, $\bar{a}_{xx}$, $\bar{a}_{x|x}$, $\bar{a}_{x\|x}$, $\overline{P}_{x|x}$ and $\overline{P}_{x\|x}$, the premiums having been calculated by the formulas

$$\overline{P}_{x|x} = \frac{\bar{a}_{x|x}}{\bar{a}_{xx}}, \tag{15}$$

$$\overline{P}_{x\|x} = \frac{\bar{a}_{x\|x}}{\bar{a}_x}. \tag{16}$$

x	$\bar{a}_x$	$\bar{a}_{xx}$	$\bar{a}_{x\|x}$	$\bar{a}_{x\|\|x}$	$\overline{P}_{x\|x}$	$\overline{P}_{x\|\|x}$	Δ in %
20	19,980	17,840	2,140	2,6243	,1200	,1313	9,4
30	18,477	16,194	2,283	2,9016	,1410	,1570	11,3
40	16,353	13,884	2,469	3,2683	,1778	,1999	12,4
50	13,539	10,920	2,619	3,6250	,2398	,2677	11,6
60	10,158	7,5968	2,5612	3,7504	,3371	,3692	9,5
70	6,6595	4,5114	2,1481	3,3694	,4761	,5060	6,3
80	3,7010	2,2519	1,4491	2,4574	,6435	,6640	3,2
90	1,7377	,9652	,7725	1,4068	,8004	,8096	1,1

Since the premiums increase with the age, there is no risk of negative reserves.

The column headed „Δ in %“ contains the difference between $\overline{P}_{x\|x}$ and $\overline{P}_{x|x}$ in per cent of the latter. It is seen that the premium for a death annuity is only some ten or twelve per cent higher than the premium for a reversionary annuity, a negligible difference in view of the great additional advantages of the death annuity. These advantages are so important, and the additional premium so small, that it should always be possible to persuade anybody who wants a reversionary annuity to choose a death annuity instead.

The cheapness of the death annuity in comparison with various other forms of insurance is due to its nature of a decreasing insurance. But the disadvantages of a decreasing insurance are not so great as it might seem, provided that it follows approximately what may be termed the „money value" of the life at any time. An old man does not as a rule want a large insurance, because an insurance decreasing at the proper rate will provide the same annual amount for his widow, while his children become independent as he grows old. Besides, a death annuity may, if desired, be modified in various ways; thus, for example, it may be stipulated that the sum insured shall only decrease to a certain point and thereafter remain constant. Several other questions of a practical nature arise which I do not propose to discuss here, such as the choice of mortality tables and the question of loadings. Policy loans can be admitted, with obvious precautions as to repayment, because the insurance is decreasing. Also the wording of the policy conditions requires careful consideration.

The construction of a set of tables of $a_{x|y}$ for the values of x and y which are required in practice is a somewhat heavy piece of work which I am not prepared to tackle single-handed. I shall be satisfied if I have succeeded in convincing the actuarial world that the new form of insurance suggested above is a useful one, and worth the further consideration of the profession.

Some approximate formulas.

By *Jiřina Frantíková* (Prague).

The following note explains a method of obtaining approximate formulas of some actuarial values. We shall use the mean value theorem of a definite integral:

If the functions $\varphi(x)$, $\varphi(x)\,.\,\psi(x)$ are integrable over the range (a, b), $\varphi(x)$ does not change the sign over (a, b) and never is equal to zéro, then the following equation is valid

$$\int_a^b \psi(x)\,.\,\varphi(x)\,\mathrm{d}x = \psi(c)\,.\int_a^b \varphi(x)\,\mathrm{d}x \tag{1}$$

where

$$c = a + (b - a)\,\Theta,\ 0 < \Theta < 1.$$