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Hypergeometric orthogonal systems of polynomials.

By Dr. *L. Truksa.*

From the relation (16), by means of the values $a_{\lambda+1}$, a_λ, $a_{\lambda-1}$, the functional equation of the polynomials $\mathfrak{J}_\lambda(n, m, x)$ results:

$$\frac{2(n+m+\lambda+1)(n+\lambda+1)}{(n+m+2\lambda+1)(n+m+2\lambda+2)} \mathfrak{J}_{\lambda+1}(x) +$$
$$+\left(x-\frac{(n+m)(s-2\lambda-1)-2\lambda(\lambda+1)}{2(n+m+2\lambda)(n+m+2\lambda+2)}(n-m)\omega\right)\mathfrak{J}_\lambda(x)+ \tag{17}$$
$$+\frac{(s+m+n+\lambda)(m+\lambda)\lambda(s-\lambda)}{2(n+m+2\lambda)(n+m+2\lambda+1)}\omega^2 \mathfrak{J}_{\lambda-1}(x)=0.$$

It is evident that the same fundamental recurrence relation holds good also for the function $\Pi_\lambda(n, m, x)$.

The following analogous recurrence equation is valid also for the system of polynomials $\mathfrak{J}_\lambda(n, m, z)$:

$$\frac{2(n+m+\lambda+1)(n+\lambda+1)}{(n+m+2\lambda+1)(n+m+2\lambda+2)} \mathfrak{J}_{\lambda+1}(z) +$$
$$+\left(z-\frac{s-1}{2}\omega-\frac{(n+m)(s-2\lambda-1)-2\lambda(\lambda+1)}{2(n+m+2\lambda)(n+m+2\lambda+2)}(n-m)\omega\right)\mathfrak{J}_\lambda(z)+$$
$$+\frac{(s+m+n+\lambda)(m+\lambda)\lambda(s-\lambda)}{2(n+m+2\lambda)(n+m+2\lambda+1)}\omega^2 \mathfrak{J}_{\lambda-1}(z)=0.$$

As an example of the expansion of a polynomial in a series

$$a_0\mathfrak{J}_0(x)+a_1\mathfrak{J}_1(x)+\ldots+a_k\mathfrak{J}_k(x)$$

we shall carry through the calculation of the coefficients a_i if the polynomial

$$F_k\left(\frac{s-1}{2}\omega+x\right)=\sum_{i=0}^{k} a_i\,\mathfrak{J}_i(x) \tag{18}$$

is in question.

Multiplying both sides of this equation with the product $\mathfrak{J}_i(x)\,\Phi_0(x)$ and carrying out the summation according to the variable x over the interval $\pm\frac{1}{2}(s-1)\omega$ we obtain:

$$a_i = \frac{\sum_{-a}^{a} F_k\left(\frac{s-1}{2}\omega + x\right) \Phi_0(x)\mathfrak{F}_i(x)\omega}{\sum_{-a}^{a} \mathfrak{F}_i^2(x)\,\Phi_0(x)\,\omega}. \tag{19}$$

Using the relation (6) we can express the sum in the numerator also in the form

$$(-)^i \frac{(m+1,i)}{2^i} \frac{F_{n+m+2i+1}\,(\overline{n+m+s+i\omega})}{F_{n+m+1}\,(\overline{n+m+s\omega})} \times$$

$$\times \sum_{-a}^{a} \Phi_i(x+i\omega) \mathop{\Delta^i}_{\omega} F_k\left(\frac{s-1}{2}\omega + x\right) = (-)^i A_i \times$$

$$\times \sum_{-a}^{a} \binom{\frac{s-1}{2} + n + i + \frac{x}{\omega}}{\frac{s-1}{2} + \frac{x}{\omega}} \binom{\frac{s-1}{2} + m - \frac{x}{\omega}}{\frac{s-1}{2} - i - \frac{x}{\omega}} \binom{\frac{s-1}{2} + \frac{x}{\omega}}{k-i}.$$

With regard to the relation (15), which we apply to the first and third factor after the summation sign, this expression becomes considerably simpler

$$(-)^i A_i \binom{k+n}{k-i} \sum_{-a}^{a} \binom{\frac{s-1}{2} + n + i + \frac{x}{\omega}}{\frac{s-1}{2} - k + i + \frac{x}{\omega}} \binom{\frac{s-1}{2} + m - \frac{x}{\omega}}{\frac{s-1}{2} - i - \frac{x}{\omega}} =$$

$$= (-)^i A_i \binom{k+n}{k-i} \sum_{-a}^{a} \binom{-\overline{k+n+1}}{\frac{s-1}{2} - k + i + \frac{x}{\omega}} \binom{-\overline{m+i+1}}{\frac{s-1}{2} - i - \frac{x}{\omega}} (-)^{s-1-k}.$$

From this expression it is evident that the lower summation limit can be increased by $\overline{k-i\omega}$, and the upper limit can be reduced by $i\omega$. Putting

$$y = \frac{x}{\omega} + \frac{s-1}{2} + i - k$$

the required sum assumes the form

$$(-)^{s-k+i-1} A_i \binom{k+n}{k-i} \sum_{y=0}^{s-k-1} \binom{-\overline{k+n+1}}{y} \binom{-\overline{m+i+1}}{s-k-1-y} =$$

$$= (-)^{s+i-k-1} A_i \binom{k+n}{k-i} \binom{-\overline{n+m+k+i+2}}{s-k-1} =$$

$$= (-)\, A_i \binom{k+n}{k-i} \frac{F_{n+m+k+i+1}\,(\overline{n+m+s+i\,\omega})}{\omega^{n+m+k+i+1}} =$$

$$= (-)^i \frac{(m+1, i)}{2^i} \binom{k+n}{k-i} \frac{F_{n+m+k+i+1}\,\overline{(n+m+s+i\,\omega)}}{F_{n+m+1}\,\overline{(n+m+s\,\omega)}}.$$

Substituting the value (14′) for the denominator of the expression (19), we obtain for the coefficient a_i the expression:

$$a_i = (-)^i \binom{k+n}{k-i} \frac{(n+1, i)\, 2^i}{(1, i)\,(n+m+i+1, i)} \times$$
$$\times \frac{F_{n+m+k+i+1}\,\overline{(n+m+s+i\,\omega)}}{F_{n+m+2i+1}\,\overline{(n+m+s+i\,\omega)}}.$$

If $i = k$, the respective $a_k(1, k)$ must be equal to the reciprocal value of the factor of the power x^k in the polynomial $\mathfrak{J}_k(x)$ whereby the correctness of the calculation can be controlled.

The coefficient a_0 simultaneously represents the value of binomial moments B_k[11]) of the characteristic function $\Phi_0(n, m, x)$:

$$\omega^k a_0 = B_k = \sum_{-a}^{a} \binom{\frac{s-1}{2} + \frac{x}{\omega}}{k} \Phi_0(n, m, x)\,\omega = \sum_{0}^{s-1} \binom{\frac{z}{\omega}}{k} \Phi_0(n, m, z)\,\omega =$$
$$= \binom{k+n}{k} \frac{\binom{n+m+s}{s-k-1}}{\binom{n+m+s}{s-1}}.$$

3. The polynomials $\mathfrak{J}_\lambda(n, m, x)$ satisfy the hypergeometric difference equation of the second order. Also the function $\Pi_\lambda(n, m, x)$ satisfies an analogous hypergeometric difference equation.

We shall first deduce the recurrence relation between $\Pi_\lambda(x)$, $\underset{\omega}{\Delta}\Pi_\lambda(x)$ and $\underset{\omega}{\Delta}\,\Pi_{\lambda+1}(x)$.

Starting from the definition (5) of the function

$$\Pi_\lambda(x) = \frac{(m+1, \lambda)}{2^\lambda} F_{n+m+2\lambda+1}\,\overline{(n+m+\lambda+s\,\omega)}\,\underset{\omega}{\Delta^\lambda}\,\Phi_\lambda(x)$$

we obtain for $\underset{\omega}{\Delta}\,\Pi_{\lambda+1}(x)$ the value

$$\underset{\omega}{\Delta}\Pi_{\lambda+1}(x) =$$
$$= \frac{(m+1, \lambda+1)}{2^{\lambda+1}} F_{n+m+2\lambda+3}\overline{(n+m+\lambda+1+s\omega)}\underset{\omega}{\Delta^{\lambda+1}}(\Delta\Phi_{\lambda+1}(x)).$$

Between the difference ratio $\underset{\omega}{\Delta}\Phi_\lambda(x)$ and the function $\Phi_{\lambda-1}(x)$ a simple relation holds good:

$$\underset{\omega}{\Delta}\Phi_\lambda(x) = \Phi_{\lambda-1}(x)\,[-x\,\overline{n+m+2\lambda} +$$

[11]) Cp. e. g. J. F. Steffensen, Factorial moments and discontinuous frequency-functions, Skandinavisk Aktuarietidskrift 1923.

$$+\frac{s}{2}(n-m)\omega-\frac{n+m+2\lambda}{2}\omega+\lambda\overline{m+\lambda\omega}]\,(n+m+2\lambda+1)\times$$
$$\times(n+m+2\lambda):(n+m+\lambda+s)\,(s-\lambda)\,\omega^2\,(n+\lambda)\,(m+\lambda) \qquad (20)$$

which we can deduce by simply forming the difference ratio

$$\underset{\omega}{\Delta}\left\{F_{n+\lambda}\left(\frac{s-1}{2}\omega+n\omega+x\right)F_{m+\lambda}\left(\frac{s-1}{2}\omega+\overline{m+\lambda}\,\omega-x\right)\right\}=$$
$$=-F_{n+\lambda}\left(\frac{s+1}{2}\omega+n\omega+x\right)F_{m+\lambda-1}\left(\frac{s-3}{2}\omega+\overline{m+\lambda}\,\omega-x\right)+$$
$$+F_{n+\lambda-1}\left(\frac{s-1}{2}\omega+n\omega+x\right).\,F_{m+\lambda}\left(\frac{s-1}{2}\omega+\overline{m+\lambda}\,\omega-x\right)=$$
$$=F_{n+\lambda-1}\left(\frac{s-1}{2}\omega+n\omega+x\right)F_{m+\lambda-1}\left(\frac{s-3}{2}\omega+\overline{m+\lambda}\,\omega-x\right)\times$$
$$\times\left\{-\frac{\frac{s+1}{2}\omega+n\omega+x}{n+\lambda}+\frac{\frac{s-1}{2}\omega+\overline{m+\lambda}\,\omega-x}{m+\lambda}\right\}=$$
$$=\frac{1}{(n+\lambda)(m+\lambda)}F_{n+\lambda-1}\left(\frac{s-1}{2}\omega+n\omega+x\right)F_{m+\lambda-1}\left(\frac{s-3}{2}\omega+\overline{m+\lambda}\omega-x\right)$$
$$\left\{-x\,\overline{n+m+2\lambda}+\frac{s}{2}(n-m)\omega-\frac{n+m+2\lambda}{2}\omega+\lambda(m+\lambda)\omega\right\}.$$

Using this relation for calculating $\underset{\omega}{\Delta}\,\Pi_{\lambda+1}(x)$, we obtain

$$\underset{\omega}{\Delta}\Pi_{\lambda+1}(x)=\frac{(m+1,\lambda+1)}{2^{\lambda+1}(n+\lambda+1)\,(m+\lambda+1)}\times$$
$$\times F_{n+m+2\lambda+1}\,\overline{(n+m+s+\lambda\omega)}\,\underset{\omega}{\Delta^{\lambda+1}}\,[\Phi_\lambda(x)\,.\,(-x\,\overline{n+m+2\lambda+2}+$$
$$+\frac{s}{2}(n-m)\omega-\frac{n+m+2\lambda+2}{2}\omega+(\lambda+1)\,(m+\lambda+1)\,\omega]=$$
$$=\frac{(m+1,\lambda)}{2^{\lambda+1}\,(n+\lambda+1)}F_{n+m+2\lambda+1}\,\overline{(n+m+s+\lambda\,\omega)}\times$$
$$\times[\{-\overline{x+\lambda+1}\,\overline{n+m+2\lambda+2}+\frac{s}{2}(n-m)\omega-$$
$$-\frac{n+m+2\lambda+2}{2}\omega+(\lambda+1)\,(m+\lambda+1)\,\omega\}\,\underset{\omega}{\Delta^{\lambda+1}}\,\Phi_\lambda(x)-$$
$$-(n+m+2\lambda+2)\,(\lambda+1)\,\underset{\omega}{\Delta^{\lambda}}\,\Phi_\lambda(x)].$$

The required recurrence relation results immediately from this equation:

$$2(n+\lambda+1)\,\underset{\omega}{\Delta}\Pi_{\lambda+1}(x) = [-x\,\overline{n+m+2\lambda+2}+$$
$$\frac{s}{2}\,\overline{n-m}\,\omega - \frac{n+m}{2}\,\omega - (\lambda+1)(n+\lambda+2)\,\omega].\,\underset{\omega}{\Delta}\Pi_{\lambda}(x)-$$
$$-(\lambda+1)(n+m+2\lambda+2)\,\Pi_{\lambda}(x). \qquad (21)$$

This relation with the functional equation (7) is sufficient to deduce the respective difference equations of the function $\Pi_{\lambda}(x)$. If we simplify the notation of the coefficients, we can write the equation (17) in the form:

$$2(n+\lambda+1)\,\Pi_{\lambda+1}(x)+\varphi(x)\,\Pi_{\lambda}(x)+\beta\,\Pi_{\lambda-1}(x)=0, \qquad (\mathrm{I})$$

then the equation (21)

$$2\,(n+\lambda+1)\,\underset{\omega}{\Delta}\Pi_{\lambda+1}(x)+\psi(x,\lambda)\,\underset{\omega}{\Delta}\Pi_{\lambda}(x)+$$
$$+(\lambda+1)(n+m+2\lambda+2)\,\Pi_{\lambda}(x)=0. \qquad (\mathrm{II})$$

and for λ less a unit

$$2(n+\lambda)\,\underset{\omega}{\Delta}\Pi_{\lambda}(x)+\psi(x,\lambda-1)\,\underset{\omega}{\Delta}\Pi_{\lambda-1}(x)+\lambda\,(n+m+2\lambda)\,\Pi_{\lambda-1}(x)=0. \qquad (\mathrm{III})$$

Let us calculate the value of the difference ratio $\underset{\omega}{\Delta}\Pi_{\lambda+1}(x)$ from the equation (I) and put it into the equation (II). We obtain

$$(n+m+2\lambda+2)\,\frac{(n+m+\lambda)\,\lambda}{n+m+\lambda+1}\,\Pi_{\lambda}+$$
$$+[\psi(x,\lambda)-\varphi(x+\omega)]\,\Delta\Pi_{\lambda}-\beta\Delta\Pi_{\lambda-1}=0. \qquad (\mathrm{I}')$$

Let us form now the difference ratio of this last expression:

$$(n+m+2\lambda+2)\,\frac{(n+m+\lambda-1)\,\lambda}{n+m+\lambda+1}\,\Delta\Pi_{\lambda}+$$
$$+[\psi(x+\omega,\lambda)-\varphi(x+2\omega)]\,\Delta^2\Pi_{\lambda}-\beta\Delta^2\Pi_{\lambda-1}=0 \qquad (\mathrm{II}')$$

and the difference ratio of the value (III):

$$(\lambda+1)(n+m+2\lambda)\,\Delta\Pi_{\lambda-1}+\psi(x+\omega,\lambda-1)\,\Delta^2\Pi_{\lambda-1}+$$
$$+2(n+\lambda)\,\Delta^2\Pi_{\lambda}=0. \qquad (\mathrm{III}')$$

In this way we have gained three equations (I'), (II'), (III') which-having been multiplied by convenient factors and added up together — result in the required difference equation. We multiply

the equation	(I')	by	the	value	$(\lambda+1)(n+m+2\lambda)$
,,	,,	(II')	,,	,,	,,	$\psi(x+\omega,\lambda-1)$
,,	,,	(III')	,,	,,	,,	β

and then we add all the three equations and obtain:

$$\underset{\omega}{\Delta^2}\Pi_\lambda(x)\,\{[\psi(x+\omega,\lambda)-\varphi(x+2\omega)]\,\psi(x+\omega,\lambda-1)+2\beta(n+\lambda)\}+$$
$$+\underset{\omega}{\Delta}\Pi_\lambda(x)\,\{(\lambda+1)\,(n+m+2\lambda)\,[\psi(x,\lambda)-\varphi(x+\omega)]+$$
$$+\psi(x+\omega,\lambda-1)\,\frac{(n+m+\lambda-1)\,\lambda}{n+m+\lambda+1}\,(n+m+2\lambda+2)\}+$$
$$+\Pi_\lambda(x)\,(\lambda+1)\,(n+m+2\lambda)\,\frac{\lambda(n+m+\lambda)\,(n+m+2\lambda+2)}{n+m+\lambda+1}=0$$

After substitution of the original values for φ, ψ, β, i. e.

$$\psi(x+\omega,\lambda)-\varphi(x+2\omega)=\frac{n+m+2\lambda+2}{n+m+\lambda+1}\,\lambda\left[-x-\frac{s(n-m)}{2(n+m+2\lambda)}\,\omega+\right.$$
$$\left.+\frac{(n+m+2\lambda)\,(2m+2\lambda-3)-2\lambda\,(m+\lambda)}{2(n+m+2\lambda)}\,\omega\right]$$
$$\psi(x+\omega,\lambda-1)=x(n+m+2\lambda)-\tfrac{1}{2}\,s(n-m)\,\omega+$$
$$+\tfrac{3}{2}\,(n+m)\,\omega+\lambda\,(n+\lambda+3)\,\omega$$
$$\psi(x,\lambda)-\varphi(x+\omega)=\frac{n+m+2\lambda+2}{n+m+\lambda+1}\,\lambda\left[-x-\frac{s(n-m)}{2(n+m+2\lambda)}\,\omega+\right.$$
$$\left.+\frac{(n+m+2\lambda)\,(2m+\lambda-1)-2\lambda\,(m+\lambda)}{2(n+m+2\lambda)}\,\omega\right]$$

this equation assumes the simple form

$$\left(x+\frac{s+3}{2}\,\omega\right)\left(\frac{s+2m-3}{2}\,\omega-x\right)\underset{\omega}{\Delta^2}\Pi_\lambda(x)+$$
$$+\,[x(n+m-2)-\tfrac{1}{2}\,s(n-m)\,\omega+\lambda\,(n+m+\lambda+1)\,\omega+ \tag{22}$$
$$+\tfrac{3}{2}\,(n+m)\,\omega+(m-2)\,\omega]\,\underset{\omega}{\Delta}\Pi_\lambda(x)+(\lambda+1)\,(n+m+\lambda)\,\Pi_\lambda(x)=0.$$

To deduce the analogous difference equation valid for the polynomials $\mathfrak{J}_\lambda(x)$, it is sufficient to substitute the values Π_λ, $\Delta\Pi_\lambda$, $\Delta^2\Pi_\lambda$ by the values

$$\Pi_\lambda(x)=\mathfrak{J}_\lambda(x)\,\Phi_0(x)\,F_{n+m+1}\,(\overline{n+m+s\,\omega})$$
$$\underset{\omega}{\Delta}\Pi_\lambda(x)=F_{n+m+1}\,(\overline{n+m+s\,\omega})\,[\underset{\omega}{\Delta}\mathfrak{J}_\lambda(x)\,\Phi_0(x+\omega)+$$
$$+\mathfrak{J}_\lambda(x)\,\frac{1}{\omega}\,(\Phi_0(x+\omega)-\Phi_0(x))]$$
$$\underset{\omega}{\Delta^2}\Pi_\lambda(x)=F_{n+m+1}\,(\overline{n+m+s\,\omega})\,\{\underset{\omega}{\Delta^2}\,\mathfrak{J}_\lambda(x)\,\Phi_0(x+2\omega)+$$
$$+2\,\underset{\omega}{\Delta}\mathfrak{J}_\lambda(x)\,\frac{1}{\omega}\,[\Phi_0(x+2\omega)-\Phi_0(x+\omega)]+\mathfrak{J}_\lambda(x)\,\frac{1}{\omega^2}\,[\Phi_0(x+$$
$$+2\omega)-2\Phi_0(x+\omega)+\Phi_0(x)]$$

in the equation (22), and to use the evident relations

$$\left(x + \frac{s+2n+3}{2}\omega\right)\left(\frac{s-3}{2}\omega - x\right)\Phi_0(x+\omega) =$$
$$= \left(x + \frac{s+3}{2}\omega\right)\left(\frac{s-3+2m}{2}\omega - x\right)\Phi_0(x+2\omega)$$
$$\left(x + \frac{s+2n+1}{2}\omega\right)\left(\frac{s-1}{2}\omega - x\right)\Phi_0(x) =$$
$$= \left(x + \frac{s+1}{2}\omega\right)\left(\frac{s-1+2m}{2}\omega - x\right)\Phi_0(x+\omega).$$

With regard to these relations it is possible to divide the equation by the factor $\Phi_0(x+\omega)$ whereby we obtain the difference equation of the polynomials $\mathfrak{J}_\lambda(x)$ in the form

$$\left(x + \frac{s+2n+3}{2}\omega\right)\left(\frac{s-3}{2}\omega - x\right)\underset{\omega}{\Delta^2}\mathfrak{J}_\lambda(x) - [x(n+m+2) -$$
$$- \tfrac{1}{2}s(n-m)\omega - \lambda(n+m+\lambda+1)\omega + \tfrac{3}{2}(n+m)\omega - \tag{23}$$
$$-(m-2)\omega]\underset{\omega}{\Delta}\mathfrak{J}_\lambda(x) + \lambda(n+m+\lambda+1)\mathfrak{J}_\lambda(x) = 0.$$

It is evidently a linear homogeneous difference equation of the 2-nd order with rational coefficients.

For the special form of polynomials $\mathfrak{J}_\lambda(x)$ of the formula (8′), we have a quite analogous difference equation

$$(z + \overline{n+1}\,\omega)(\overline{s-2}\,\omega - z)\underset{\omega}{\Delta^2}\mathfrak{J}_\lambda(z) - [z(n+m+2) -$$
$$-(n+1)(s-1)\omega - \lambda(n+m+\lambda+1)\omega + (n+m+2)\omega]\underset{\omega}{\Delta}\mathfrak{J}_\lambda(z) +$$
$$+\lambda(n+m+\lambda+1)\mathfrak{J}_\lambda(z) = 0. \tag{23′}$$

The equation (23) and (23′) respectively represents a remarkable form of the hypergeometric difference equation of the 2-nd order, which enables us to transform this equation into the Gauss differential equation by a simple limiting process.

The principal solutions of the difference equation (23) can be expressed as a function of ω and of the quantity s. We can transform them into the solutions of the respective differential equation in the limiting case $\omega \to 0$, $s \to \infty$ ($s\omega$ = const.) .Such relations between difference and differential equations mediated by ω are hitherto very little known.[12])

[12]) I refer in this respect to to the paper of A. Walther: „Zum Grenzübergange von Differenzengleichungen in Differenzialgleichungen (Math. Annalen, 95/1925), whence I quote following: „Weitere Untersuchungen über das Problem, die Lösungen von Differenzengleichungen als Funktionen der eingehenden Spannen zu studieren und insbesondere den Grenzfall hinzuschwingender Spannen der Betrachtung zu unterziehen, liegen bisher nicht vor.“

A detailed study of the above-mentioned transformation is contained in the monography of J. Kaucký: O přechodu differenční rovnice hypergeometrické v differenciální rovnici Gaussovu (On the transformation of the hypergeometric difference equation into the Gauss differential equation), published in the series „Spisy přírodovědecké fakulty univ. Mas. v Brně“ č. 80, r. 1927 (Publications of the Faculty of Natural Sciences at the Masaryk University at Brno, No. 80. year 1927).

4. Polynomials $\mathfrak{J}_\lambda(x)$ can be expressed also in form of the determinant

$$\mathfrak{J}_\lambda(x) = d_\lambda \begin{vmatrix} 1 & M_0 & M_1 & M_2 & \dots M_{\lambda-1} \\ x & M_1 & M_2 & M_3 & \dots M_\lambda \\ x^2 & M_2 & M_4 & M_4 & \dots M_{\lambda+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x^{\lambda-1} & M_{\lambda-1} & M_\lambda & M_{\lambda+1} & \dots M_{2\lambda-2} \\ x^\lambda & M_\lambda & M_{\lambda+1} & M_{\lambda+2} & \dots M_{2\lambda-1} \end{vmatrix} \frac{(n+m+\lambda+1, \lambda)}{2^\lambda (n+1, \lambda)} \tag{24}$$

where M_i denotes the i-th moment of the characteristic function

$$M_i = \sum_{-a}^{a} x^i \Phi_0(n, m, x)$$

and d_λ the co-factor of x^λ. This expression follows immediately from the one-valued determination of polynomials $\mathfrak{J}_\lambda(x)$ by the orthogonal quality contained in the equation (12).

An analogous expression holds good also for the functions $\Pi_\lambda(x)$.

Another expression of polynomials $\mathfrak{J}_\lambda(x)$ in form of a determinant can be deduced also from the functional equation (17). If we denote

$$\varphi(x, \lambda) = \left[- x + \frac{(n+m)(s-1-2\lambda) - 2\lambda(\lambda+1)}{2(n+m+2\lambda)(n+m+2\lambda+2)} (n-m)\,\omega\right] \times$$
$$\times \frac{(n+m+2\lambda+1)(m+n+2\lambda+2)}{2(n+m+\lambda+1)(n+\lambda+1)},$$

$$\beta(\lambda) = \frac{(s+n+m+\lambda)(m+\lambda)\lambda(s-\lambda)(n+m+2\lambda+2)}{4(n+m+2\lambda)(n+m+\lambda+1)(n+\lambda+1)} \omega^2$$

the equation (17) assumes the form

$$\mathfrak{J}_{\lambda+1}(x) = \varphi(x, \lambda)\,\mathfrak{J}_\lambda(x) - \beta(\lambda)\,\mathfrak{J}_{\lambda-1}(x). \tag{17'}$$

The above-mentioned expression of the polynomial $\mathfrak{J}_\lambda(x)$ is given then by the determinant:

$$\mathfrak{J}_\lambda(x) = \begin{vmatrix} \varphi(x, \lambda-1) & 1 & 0 & \dots & 0 \\ \beta(\lambda-1) & \varphi(x, \lambda-2) & 1 & \dots & \vdots \\ 0 & \beta(\lambda-2) & \varphi(x, \lambda-3) & \dots & \vdots \\ \vdots & 0 & \beta(\lambda-3) & \dots & \vdots \\ \vdots & \vdots & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & 1 \\ 0 & 0 & 0 & \dots & \varphi(x, 0) \end{vmatrix} \qquad (17'')$$

From this definition of the polynomials $\mathfrak{J}_\lambda(x)$ follows immediately the characteristic quality of the zeros of the polynomials $\mathfrak{J}_\lambda(x)$, which occur also in other orthogonal systems of polynomials: all zeros of polynomials $\mathfrak{J}_\lambda(x)$ are real, for the determinant (24) is a special form of secular equation[13]) which possesses the quality mentioned above.

5. The functional equation (17') enables us further to express the polynomials $\mathfrak{J}_\lambda(x)$ as denominators of successive approximative values of a continued fraction

$$\frac{\frac{n+m+2}{4(n+1)}}{|\varphi(x, 0)} - \frac{\beta(1)|}{|\varphi(x, 1)} - \frac{\beta(2)}{|\varphi(x, 2)} - \dots \qquad (25)$$

6. Approximation and interpolation of numerically given functions by aid of the polynomials $\mathfrak{J}_\lambda(n, m, z)$.

By aid of orthogonal systems of polynomials it is possible to obtain a certain modification of the known approximative expression of a function in the form of a power series

$$f(z) = \alpha_0 + \alpha_1 z + \alpha_2 x^2 + \dots + \alpha_k z^k \qquad (26)$$

from the given $n > k$ values of the function in the equidistant intervals, by using either the method of least squares or the method of moments. The calculation of coefficients α_i is, as a rule, very difficult and must be repeated for all i, if we subsequently increase the number of members of the series (26). If we, on the other hand, express $f(z)$ by the series

$$f(z) = a_0 P_0(z) + a_1 P_1(z) + \dots + a_k P_k(z) \qquad (26')$$

the calculation of coefficients a_i becomes essentially simpler and besides these coefficients remain unchanged, if we subsequently increase the series by further members of a higher degree. These advantages assisted in the wide application of the series (26') in numerical approximative and interpolation calculus. As an example, we shall quote the application of ordinary and summation Legendre polynomials, of polynomials of Hermite, of Charlier-Jordan, etc. In mathematical statistics, the series (26') occurs as a rule in the form

[13]) Cp. e. g. Baltzer, Determinanten, 1881.

$$f(z) = \Phi(z) \sum_{i=0}^{k} a_i \, P_i(z) \tag{27}$$

where $\Phi(z)$ denotes the characteristic function.

The coefficients a_i can be determined either by application of the method of least squares from the condition

$$\sum_{z} [f(z) - \Phi(z) \sum_{i=0}^{k} a_i \mathfrak{P}_i(z)]^2 \, \omega = \min$$

$$\text{or} \int_{\alpha}^{\beta} [f(z) - \Phi(z) \sum_{i=0}^{k} a_i P_i(z)]^2 \, dz = \min,$$

or more frequently by application of the method of moments from the condition

$$\sum_{z} [f(z) - \Phi(z) \sum_{i=0}^{k} a_i \mathfrak{P}_i(z)] \, z^i \omega = 0$$

$$\text{or} \int_{\alpha}^{\beta} (f(z) - \Phi(z) \sum_{i=0}^{k} a_i P_i(z)) \, z^i \, dz = 0.$$

In the first case, the coefficient a_i is given by the expression

$$a_i = \frac{\sum f(z) \, \mathfrak{P}_i(z) \, \Phi(z) \, \omega}{\sum \mathfrak{P}_i^2(z) \, \Phi^2(z) \, \omega} \qquad \text{or } a_i = \frac{\int f(z) P_i(z) \, \Phi(z) \, dz}{\int P_i^2(z) \, \Phi^2(z) \, dz},$$

in the second case by the expression

$$a_i = \frac{\sum f(z) \, \mathfrak{P}_i(z) \, \omega}{\sum \mathfrak{P}_i^2(z) \, \Phi(z) \, \omega} \qquad \text{or } a_i = \frac{\int f(z) \, P_i(z) \, dz}{\int P_i^2(z) \, \Phi(z) \, dz}.$$

If, in particular, the application of Jacobi's generalized polynomials to the expression of the function $f(z)$, s values of which are given in the points $0, \omega, 2\omega, \ldots \overline{s-1}\omega$, by the series

$$f(z) = \Phi_0(z) \sum_{i=0}^{\lambda} a_i \, \mathfrak{J}_i \, (z), \tag{28}$$

is in question, there results the value

$$a_i = \frac{\sum_{0}^{\overline{s-1}\omega} f(z) \, \mathfrak{J}_i(z) \, \omega}{\sum_{}^{\overline{s-1}\omega} \mathfrak{J}_i^2(z) \, \Phi_0(z) \, \omega} = \frac{\sum_{0}^{\overline{s-1}\omega} f(z) \, \mathfrak{J}_i x() \, \omega}{I_i}. \tag{29}$$

from the orthogonality of the polynomials $\mathfrak{J}_\lambda(z)$ by using the method of moments for the coefficient a_i.

While applying the formula (28) we can put $\omega = 1$, s equal to the number of given values. Besides it is necessary to decide upon the choice of parameters n, m of the characteristic function $\Phi_0(n, m, z)$. The calculation of these parameters can be carried through e. g. provided that the first and second moment of the function $f(z)$ is equal to the first and second moment of the function $\Phi_0(n, m, z)$:[14])

$$\Sigma z\, f(z) = \Sigma z\, \Phi_0(z); \quad \Sigma z^2 f(z) = \Sigma z^2 \Phi_0(z)$$

If we determine n and m in this manner, then the coefficients a_1, a_2 in the expansion (28) are evidently equal to zero, so that we have

$$f(z) = \Phi_0(z)\,[a_0 + a_3\,\mathfrak{J}_3(z) + a_4\,\mathfrak{J}_4(z) + \ldots + a_\lambda\,\mathfrak{J}_\lambda(z)].$$

The coefficient

$$a_0 = \Sigma f(z),$$

and the other coefficients can be expressed in the form of a linear function of binomial moments $B_\lambda(\lambda)$ of the given function $f(z)$

$$B_\lambda(\lambda) = \sum_{z=0}^{s-1} \binom{z}{\lambda} f(z).$$

This expression is especially advantageous for practical calculation, as we can very easily obtain the moments B_λ by repeated summation of the function $f(z)$. That is the following relations hold good:

$$B_0(o) = \sum_{z=s-1}^{0} f(z); \qquad B_1(1) = \sum_{z=s-1}^{1} B_0(z) = \sum_{z=0}^{s-1} \binom{z}{1} f(z)$$

and generally

$$B_\lambda(\lambda) = \sum_{z=s-1}^{\lambda} B_{\lambda-1}(z) = \sum_{z=0}^{s-1} \binom{z}{\lambda} f(z),$$

which follows from the application of the formula on partial summation. Schematically the procedure of summation is illustrated in the following table[15])

z	$y(z)$	$B_0(z)$	$B_1(z)$	$B_2(z)$
$s-1$	$y(s-1)$	$y(s-1)=B_0(s-1)$	$B_0(s-1)=B_1(s-1)$	$B_1(s-1)=B_2(s-1)$
$s-2$	$y(s-2)$	$y(s-1)+y(s-2)$	$B_0(s-1)+B_0(s-2)$	$B_1(s-1)+B_1(s-2)$
⋮	⋮	⋮	⋮	⋮
2	$y(2)$	$B_0(2)$	$B_1(2)$	$B_2(2)$
1	$y(1)$	$B_0(1)$	$B_1(1)$	
0	$y(0)$	$B_0(0)$		

[14]) The detailed calculation is contained in the second part of this paper.

[15]) See Četverikov: „The Technics of calculation of statistical parabolic series", Problems of conjuncture, Moscow 1926.

If s is large enough, the sums $B_\lambda(\lambda)$ increase with increasing λ very rapidly and encumber the calculation. For this reason Frederik Esscher[16]) and independently K. Jordan[17]) proposed the application of mean binomial moments $\overline{B_\lambda}$, which we obtain from the moments B_λ dividing by the value

$$\binom{s}{\lambda+1}$$

of the same order as the moment B_λ. The coefficient a_λ ist then determined by the expression

$$a_\lambda = \sum_{r=0} {}_\lambda\alpha_r \, \overline{B}_r(r),$$

where the factor ${}_\lambda\alpha_r$ is expressed, with regard to (8′) and (14′) by the relation

$${}_\lambda\alpha_r = \frac{(-)^r s 2^\lambda (n+m+2\lambda+1)(n+1,\lambda)(n+m+\lambda+1,r)\,\overline{n+m+\lambda}!\,\overline{n+m+s}!}{\omega^\lambda(\overline{n+m+1})!\,(n+m+s+\lambda!)(m+1,\lambda)(n+1,r)\overline{r+1}!\,\overline{\lambda-r}!}.$$

and therefore

$$a_\lambda = \frac{s\ \overline{n+m+\lambda}!\ 2^\lambda\,\overline{n+m+s}!\ (n+m+2\lambda+1)\ (n+1,\lambda)}{\omega^\lambda\ \overline{n+m+1}!\ \overline{n+m+s+\lambda}!\ (m+1,\lambda)} \times$$

$$\times \sum_{r=0}^{\lambda} (-)^r \frac{(n+m+\lambda+1,r)}{(n+1,r)\,\overline{\lambda-r}!\,\overline{r+1}!}\,\overline{B}_r(r).$$

In the third part of this paper, we shall refer, in a more detailed manner, to an easier way of calculation of coefficients a_λ by means of numerical tables suggested by Esscher and Jordan, if we apply the especially simple case of Jacobi's generalized polynomials, namely the generalized Legendre polynomials ($n = m = 0$).

The degree of approximation in application of the formula (28), if we keep the numbers till the polynomial $\mathfrak{J}_\lambda(z)$ inclusively, can be estimated from the square of standard deviation

$$\sigma_\lambda^2 = \frac{1}{s}\sum[f(z) - \Phi_0(z)\sum a_i\,\mathfrak{J}_i(z)]^2 = \tag{30}$$

$$= \frac{1}{s}\sum f^2(z) - \frac{a_0^2}{s}\sum \Phi_0(z) - \frac{a_3^2}{s}\sum \Phi_0(z)\,\mathfrak{J}_3^2(z) - \ldots - \frac{a_\lambda^2}{s}\sum \Phi_0(z)\,\mathfrak{J}_\lambda^2(z).$$

Successively we can calculate the value $\sigma^2_{\lambda+1}$ from the value σ_λ

[16]) In the paper: „On graduation according to the method of least squares by means of certain polynomials" in anniversary publication of the insurance company Skandia.

[17]) Mitteilungen der ungar. Landeskommission 1930, Studie Nro 1. Berechnung der Trendlinie auf Grund der Methode der kleinsten Quadrate.

using the evident relation

$$\sigma^2_{\lambda+1} = \sigma_\lambda^2 - \frac{a^2_{\lambda+1}}{s}\sum \Phi_0(z)\,\mathfrak{J}_{\lambda+1}^2(z).$$

We can simplify the calculation of σ_λ^2 considerably by applying the normalized form of polynomials $\mathfrak{J}_i(z)$ in the approximative expression of (28), for hence

$$\frac{1}{s}\sum \Phi_0(z)\,\mathring{\mathfrak{J}}_\lambda^2(z) = 1,$$

and hence

$$\sigma_\lambda^2 = \frac{1}{s}\sum f(z)^2 - a_0^2 - a_3^2 - \ldots - a_\lambda^2.$$

7. Let us now transform the expression (28), which then acquires a special importance im some limiting cases of the polynomials $\mathfrak{J}_\lambda(z)$. We shall follow the procedure of Darboux explained in his treatise „Mémoirs sur l'approximation des fonctions de très grand nombres et sur une classe étendue de développements en série".[18])

From the recurrence equations

$$\mathfrak{J}_{\lambda+1}(z) = \varphi(z, \lambda)\,\mathfrak{J}_\lambda(z) - \beta(\lambda)\,\mathfrak{J}_{\lambda-1}(z)$$
$$\mathfrak{J}_{\lambda+1}(y) = \varphi(y, \lambda)\,\mathfrak{J}_\lambda(y) - \beta(\lambda)\,\mathfrak{J}_{\lambda-1}(y)$$

we obtain — multiplying the first of them by $-\mathfrak{J}_\lambda(y)$ and the second by $\mathfrak{J}_\lambda(z)$ and adding them up — the relation

$$\frac{\mathfrak{J}_\lambda(z)\,\mathfrak{J}_\lambda(y)}{I_\lambda} = \frac{\mathfrak{J}_{\lambda+1}(z)\,\mathfrak{J}_\lambda(y) - \mathfrak{J}_\lambda(z)\,\mathfrak{J}_{\lambda+1}(y)}{I_\lambda(z-y)\dfrac{(n+m+2\lambda+1)(n+m+2\lambda+2)}{2(n+m+\lambda+1)(n+\lambda+1)}} - $$
$$- \frac{\mathfrak{J}_\lambda(z)\,\mathfrak{J}_{\lambda-1}(y) - \mathfrak{J}_{\lambda-1}(z)\,\mathfrak{J}_\lambda(y)}{I_{\lambda-1}(z-y)\dfrac{(n+m+2\lambda-1)(n+m+2\lambda)}{2(n+m+\lambda)(n+\lambda)}},$$

from which follows the important relation

$$\frac{\mathfrak{J}_0(z)\,\mathfrak{J}_0(y)}{I_0} + \frac{\mathfrak{J}_1(z)\,\mathfrak{J}_1(y)}{I_1} + \ldots + \frac{\mathfrak{J}_\lambda(z)\,\mathfrak{J}_\lambda(y)}{I_\lambda} = \tag{31}$$
$$= \frac{\mathfrak{J}_{\lambda+1}(z)\,\mathfrak{J}_\lambda(y) - \mathfrak{J}_\lambda(z)\,\mathfrak{J}_{\lambda+1}(y)}{I_\lambda(z-y)} \cdot \frac{2(n+m+\lambda+1)(n+\lambda+1)}{(n+m+2\lambda+1)(n+m+2\lambda+2)}.$$

Expression (28) can be written, with respect to this relation and to the formula (29), in the following form

[18]) Journal de Math. pures et app., 1878.

$$f(z) = \Phi_0(z) \sum_i \frac{\mathfrak{J}_i(z)}{I_i} \sum_y f(y)\, \mathfrak{J}_i(y)\, \omega = \frac{\Phi_0(z)}{I_\lambda} \sum_y f(y) \times \tag{28''}$$
$$\times \frac{\mathfrak{J}_{\lambda+1}(z)\, \mathfrak{J}_\lambda(y) - \mathfrak{J}_\lambda(z)\, \mathfrak{J}_{+1}(y)}{z-y}\, \omega\, \frac{2(n+m+\lambda+1)(n+\lambda+1)}{(n+m+2\lambda+1)(n+m+2\lambda+2)}.$$

If the number of the known values of the function $f(z)$ is s, and if we keep the polynomials till the degree $\lambda = s - 1$ in the series, we get an exact reproduction of the function $f(z)$ by the series (28) in the given s points. In this case the series (28) represents an interpolation formula. If $\lambda < s - 1$, it is a formula of approximation. This case is the most frequent in practice.

The case, in which the number of values s summed up increases beyond all limits and also $\lambda \to \infty$, is especially interesting. The series (28) becomes then infinite and reproduces the given function $f(z)$ in an infinite number of points. The summation interval can be either finite or infinite. We shall refer to the respective limiting process in other passages of this paper. We shall find that it is possible in this manner to deduce e. g. the known expression of an arbitrary function by means of Legendre's, Hermite's, Charlier-Jordan's and other polynomials.

8. Jacobi's generalized polynomials of the second order.

As for the other orthogonal systems of polynomials, the numerators of approximative values of the continued fraction (25) form polynomials of a degree lower by one than the respective denominators. In the following we shall call them Jacobi's generalized polynomials of the second kind: $Q_\lambda(n, m, x)$ of the degree $\lambda - 1$. They can also be deduced from the function $\overline{Q}_\lambda(n, m, x)$ defined by the expression

$$\overline{Q}_\lambda(n, m, x) = \tfrac{1}{2}\omega \sum_{y=-a}^{a} \frac{\mathfrak{J}_\lambda(y)\, \Phi_0(y)}{x-y}. \tag{32}$$

Using the recurrence relation (17′) we obtain for $\overline{Q}_{\lambda+1}(x)$:

$$\overline{Q}_{\lambda+1}(n, m, x) = \tfrac{1}{2}\omega \sum_{y=-a}^{a} \frac{[\varphi(y, \lambda)\, \mathfrak{J}_\lambda(y) - \beta(\lambda)\, \mathfrak{J}_{\lambda-1}(y)]\, \Phi_0(y)}{x-y}.$$

As

$$\frac{\varphi(y, \lambda)\, \mathfrak{J}_\lambda(y)}{x-y} = \frac{\varphi(x, \lambda)\, \mathfrak{J}_\lambda(y)}{x-y} + \mathfrak{J}_\lambda(y) \frac{(n+m+2\lambda+1)(n+m+2\lambda+2)}{2(n+m+\lambda+1)(n+\lambda+1)},$$

there follows from the preceding equation a recurrence relation of the form

$$\overline{Q}_{\lambda+1}(x) = \varphi(x, \lambda)\, \overline{Q}_\lambda(x) - \beta(\lambda)\, \overline{Q}_{\lambda-1}(x) \tag{33}$$

which is identical with the functional equation of the polynomials $\mathfrak{J}_\lambda(x)$.

Let the first two values $\overline{Q}_0(x)$ and $\overline{Q}_1(x)$ be determined directly from the definition (32)

$$\bar{Q}_0(n, m, x) = \tfrac{1}{2}\omega \sum_{-a}^{a} \frac{\Phi_0(y)}{x - y}$$

$$\bar{Q}_1(n, m, x) = \tfrac{1}{2}\omega \sum_{-a}^{a} \frac{\varphi(y, 0)\,\Phi_0(y)}{x - y} =$$

$$= \tfrac{1}{2}\omega \sum_{-a}^{a} \left[\frac{n + m + 2}{2(n + 1)}\,\Phi_0(y) - \frac{2x(n+m+2)-(n-m)(s-1)\omega}{x - y} \times \right.$$

$$\left. \times \frac{\Phi_0(y)}{4(n + 1)}\right] =$$

$$= Q_1(x) - \mathfrak{J}_1(x)\,\bar{Q}_0(x) = \frac{n + m + 2}{4(n + 1)} - \mathfrak{J}_1(x)\,\bar{Q}_0(x).$$

Then we can deduce from the recurrence equation (33) by induction that generally the following relation holds good

$$\bar{Q}_\lambda(x) = Q_\lambda(x) - \mathfrak{J}_\lambda(x)\,\bar{Q}_0(x), \tag{34}$$

where $Q_\lambda(x)$ are the above mentioned Jacobi's generalized polynomials of the second kind of the degree $\overline{\lambda - 1}$. Also these polynomials satisfy evidently the functional equation (17'):

$$Q_{\lambda+1}(n, m, x) = \varphi(x, \lambda)\,Q_\lambda(n, m, x) - \beta(\lambda)\,Q_{\lambda-1}(n, m, x)$$

From the initial members of the system of polynomials $Q_\lambda(x)$

$$\begin{aligned} Q_1(n, m, x) &= \frac{n + m + 2}{4(n + 1)} \\ Q_2(n, m, x) &= \frac{(n + m + 3)(n + m + 4)}{8(n + 1)(n + 2)} \times \\ &\times \left[- x + \frac{(n + m)(s - 3) - 4}{2(n + m + 2)(n + m + 4)}(n - m)\,\omega\right] \end{aligned} \tag{35}$$

the others can be determined by means of the recurrence equation.

9. Application of the polynomials $\mathfrak{J}_\lambda(x)$ and $Q_\lambda(x)$ in the numerical summation.

We have to determine the sum σ of the function $f(x)$ of the form

$$f(x) = \Phi_0(x)\,.\,p(x) \tag{36}$$

in limits $\pm\, a$ by means of λ values of the function $p(x)$ for the arguments $x_1, x_2, \ldots, x_\lambda$ chosen so that the result be exact, if $p(x)$ is a polynomial of degree $i \leqq 2\lambda - 1$. Hence the required sum is to be expressed in the form

$$\sigma = \sum_{-a}^{a} f(x) = \sum_{k=1}^{\lambda} \psi(x_k)\,p(x_k) + R_{2\lambda}, \tag{37}$$

where $\psi(x_k)$ are factors dependant of λ; the rest $R_{2\lambda}$ is equal to zero, if $p(x)$ is a polynomial of degree $i \leqq 2\lambda - 1$. By application of the general Newton interpolation formula $p(x)$ can be expressed in the form

$$p(x) = p(x_1) + (x - x_1)[x_1, x_2] + \dots$$
$$\dots + (x - x_1)\dots(x - x_{\lambda-1})[x_1, x_2, \dots x_\lambda] + (x - x_1)\dots(x - x_\lambda)\frac{p^{(\lambda)}(\xi)}{\lambda!}$$

(ξ is contained between the largest and least value of arguments x_i).

Hence the sum σ in limits $\pm a$ is given by the expression

$$\sigma = \sum_{-a}^{a} \Phi_0(x)\, p(x) = p(x_1)\sum \Phi_0(x) + [x_1, x_2]\sum \Phi_0(x)\,.\,(x - x_1) + \dots +$$
$$+ [x_1, x_2, \dots x_\lambda]\sum \Phi_0(x)\,.\,(x - x_1)\dots(x - x_{\lambda-1}) + \qquad (37')$$
$$\sum \Phi_0(x)\,(x - x_1)\dots(x - x_\lambda)\frac{p^{(\lambda)}(\xi)}{\lambda!}.$$

In order to fulfil the condition stated in the problem, must evidently

$$\begin{aligned} &\sum (x - x_1)(x - x_2)\dots(x - x_\lambda)\,\Phi_0(x) = 0 \\ &\sum x(x - x_1)(x - x_2)\dots(x - x_\lambda)\,\Phi_0(x) = 0 \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &\sum x^{\lambda-1}(x - x_1)(x - x_2)\dots(x - x_\lambda)\,\Phi_0(x) = 0. \end{aligned} \qquad (38)$$

The arguments $x_1, x_2, \dots x_\lambda$ are determined by these relations. From the orthogonal quality of the polynomials $\mathfrak{J}_\lambda(n, m, x)$ follows that these arguments are identical with the arguments of the zeros of the polynomial $\mathfrak{J}_\lambda(n, m, x)$.

To determine the coefficients $\psi(x_k)$ we shall express $p(x)$ by the Lagrange interpolation formula

$$p(x) = \frac{\mathfrak{J}_\lambda(x)}{(x - x_1)\mathfrak{J}'_\lambda(x_1)}\, p(x_1) + \dots + \frac{\mathfrak{J}_\lambda(x)}{(x - x_\lambda)\mathfrak{J}'_\lambda(x_\lambda)}\, p(x_\lambda) + r_\lambda. \quad (39)$$

Hence the sum

$$\sigma = \sum_{k=1}^{\lambda} \frac{p(x_k)}{\mathfrak{J}_\lambda'(x_k)} \sum_{-a}^{a} \frac{\mathfrak{J}_\lambda(x)}{x - x_k}\,\Phi_0(x) + R_{2\lambda}. \qquad (37'')$$

It is evident that

$$\tfrac{1}{2}\omega \sum_{-a}^{a} \frac{\mathfrak{J}_\lambda(x)\,\Phi_0(x)}{x - x_k}$$

represents a special value of the function (32) for the argument x_k of the zero of the polynomial $\mathfrak{J}_\lambda(x)$. This value is reduced with regard to (34) in this case to the value of the Jacobi's generalized polynomial of the second kind $Q_\lambda(x_k)$. The required coefficients $\psi(x_k)$ are therefore expressed by means of the polynomials of second kind and of derivates of the polynomials of the first kind in the form

$$\psi(x_k) = -\frac{2}{\omega}\,\frac{Q_\lambda(x_k)}{\mathfrak{J}'_\lambda(x_k)}. \tag{40}$$

Finally the expression for the rest can be also simplified:

$$R_{2\lambda} = \sum_{-a}^{a} (x - x_1)(x - x_2)\ldots(x - x_{2\lambda})\,\frac{p^{(2\lambda)}(\xi)}{(2\lambda)!}\,\Phi_0(x). \tag{41}$$

If we choose the values $x_{\lambda+1}, x_{\lambda+2}, \ldots x_{2\lambda}$, which were till now quite arbitrary e. g. so that we identify them with the arguments $x_1, x_2, \ldots x_\lambda$, then the rest assumes the form

$$R_{2\lambda} = \sum_{-a}^{a} (x - x_1)^2 \ldots (x - x_\lambda)^2\,\frac{p^{(2\lambda)}(\xi)}{(2\lambda)!}\,\Phi_0(x) =$$

$$= \varkappa_\lambda^2 \sum_{-a}^{a} \mathfrak{J}_\lambda^2(x)\,\frac{p^{(2\lambda)}(\xi)}{(2\lambda)!}\,\Phi_0(x);\quad \varkappa_\lambda = \frac{2(n+1,\lambda)}{(n+m+\lambda+1,\lambda)}.$$

If we apply the mean-value theorem to this sum supposing that the 2λ-th derivate of the function $p(x)$ is in the summation interval $\pm a$ continuous, we obtain

$$R_{2\lambda} = \frac{p^{(2\lambda)}(\zeta)}{(2\lambda)!}\,\varkappa_\lambda^2 \sum_{-a}^{a} \mathfrak{J}_\lambda^2(x)\,\Phi_0(x)$$

(ζ is certain value of the interval $\pm a$).

With regard to (14′) the rest takes the form

$$R_{2\lambda} = \frac{p^{(2\lambda)}(\zeta)\,\lambda!\,(m+1,\lambda)\,(n+1,\lambda)}{(2\lambda)!\,2^{2\lambda}(n+m+\lambda+1,\lambda)\,\omega}\;\frac{F_{n+m+2\lambda+1}\,(\overline{n+m+s+\lambda\,\omega})}{F_{n+m+1}\,(\overline{n+m+s\,\omega})}. \tag{41'}$$

We have to observe that the problem solved in the preceding is in a way a generalization of the known mechanical quadrature of Gauss,[19]) which suggested to many authors the deduction of a whole set of integral systems of orthogonal polynomials. A detailed survey of the literature in question is contained in the „Encyklopädie der Math. Wissenschaften II C 2.[20])

10. As mentioned in the introduction Jacobi's generalized polynomials were for the first time discussed by Tchebychef in his article „Sur l'interpolation des valeurs équidistantes" 1875. This article contains only the definition of these polynomials expressed in the

[19]) See „Methodus nova integralium valores per approximationem inveniendi". Gesam. Werke 3, p. 163—196.

[20]) C. Runge — Fr. A. Willers: Numerische und graphische Quadratur und Integration gewöhnlicher und partieller Differenzialgleichungen.

formula (6) and (6') respectively, the deduction of the orthogonal relation and the value of the sum $\Sigma \Phi_0(x) \mathfrak{J}_\lambda^2(x)$. Tchebychef puts the summation interval into the limits $(0, s-1)$, $\omega = 1$. Only indirectly — under the title of hypergeometric series of the third degree — some qualities of Jacobi's generalized polynomials are discussed in the above mentioned papers of Thomae and Nörlund, further in numerous studies about the solution of the hypergeometric difference equation of the second order.

11. Integral system of orthogonal polynomials corresponding to Jacobi's generalized polynomials.

If the number of values of the argument x in the summation interval $\pm \alpha = \pm \frac{s-1}{2} \omega$ in question, increases beyond all limits and at the same time ω converges to zero so that the product $s\omega$ remains constant e. g. equal to 2, the definition of Jacobi's generalized polynomials expressed by the relation

$$\sum_{-a}^{a} \Phi_0(n, m, x) \mathfrak{J}_\lambda(x) \mathfrak{J}_\mu(x) \omega = 0 \quad \lambda \gtrless \mu$$

is reduced to an analogous definition of a special form of these polynomials expressed by the integral relation

$$\int_{-1}^{1} \overline{\Phi}_0(n, m, x) J_\lambda(x) J_\mu(x) \, dx = 0, \quad \lambda \gtrless \mu, \tag{42}$$

where

$$\overline{\Phi}_0(n, m, x) = \lim_{\substack{s \to \infty \\ \omega \to 0}} \Phi_0(n, m, x)$$

$$J_\lambda(n, m, x) = \lim_{\substack{s \to \infty \\ \omega \to 0}} \mathfrak{J}_\lambda(n, m, x).$$

We shall first find the limiting value of the function $\Phi_\lambda(n, m, x)$, if $s \to \infty$, $\omega \to 0$, $s\omega = 2$. For this purpose we shall use the expression of this function in the form

$$\Phi_\lambda(n, m, x) =$$

$$= \frac{\Gamma\left(\frac{s+1}{2}+n+\frac{x}{\omega}\right) \Gamma\left(\frac{s+1}{2}+m+\lambda-\frac{x}{\omega}\right) \Gamma(n+m+2\lambda+2)\, \Gamma(s-\lambda)}{\omega \Gamma(n+\lambda+1) \Gamma(m+\lambda+1) \Gamma\left(\frac{s+1}{2}-\lambda+\frac{x}{\omega}\right) \Gamma\left(\frac{s+1}{2}-\frac{x}{\omega}\right) \Gamma(n+m+s+\lambda+1)} =$$

$$= \frac{\Gamma(\frac{1}{2}s\,\overline{1+x}+\frac{1}{2}+n)}{\Gamma(\frac{1}{2}s\,\overline{1+x}+\frac{1}{2}-\lambda)} \frac{\Gamma(\frac{1}{2}s\,\overline{1-x}+\frac{1}{2}+m+\lambda)}{\Gamma(\frac{1}{2}s\,\overline{1-x}+\frac{1}{2})} \times$$

$$\times \frac{\Gamma(s-\lambda)}{\Gamma(s+\lambda+n+m+1)} \frac{\Gamma(n+m+2\lambda+2)}{\Gamma(n+\lambda+1)\, \Gamma(m+\lambda+1)} \frac{1}{2}s.$$

Using the well-known limiting relation

$$\frac{\Gamma(ax+\alpha)}{\Gamma(ax+\beta)} \sim \frac{(ax+\alpha)^{ax+\alpha} e^{-\alpha}}{(ax+\beta)^{ax+\beta} e^{-\beta}} \sim (ax)^{\alpha-\beta}, \tag{43}$$

for large x, there follows

$$\Phi_\lambda(n, m, x) \sim \left(\frac{s}{2}\,\overline{1+x}\right)^{n+\lambda} \left(\frac{s}{2}\,\overline{1-x}\right)^{m+\lambda} s^{-n-m-2\lambda} \times$$
$$\times \frac{\Gamma(n+m+2\lambda+2)}{2\Gamma(n+\lambda+1)\,\Gamma(m+\lambda+1)}.$$

and hence

$$\lim \Phi_\lambda(n, m, x) = \overline{\Phi}_\lambda(n, m, x) = \frac{1}{2^{n+m+2\lambda+1}} (1+x)^{n+\lambda} \times$$
$$\times (1-x)^{m+\lambda} \frac{\Gamma(n+m+2\lambda+2)}{\Gamma(n+\lambda+1)\,\Gamma(m+\lambda+1)}. \tag{44}$$

According to the inequality (2) in this case must of course

$$n > -1 \text{ and } m > -1$$

In the same way we can deduce the relation

$$\lim \frac{F_{n+m+2\lambda+1}\,(\overline{n+m+\lambda+s\,\omega})}{F_{m+n+1}\,(\overline{n+m+s\,\omega})} = \lim \frac{\Gamma(n+m+s+\lambda+1)}{\Gamma(s-\lambda)\,\Gamma(n+m+2\lambda+2)} \times$$
$$\times \frac{\Gamma(n+m+2)\,\Gamma(s)}{\Gamma(n+m+s+1)} \left(\frac{2}{s}\right)^{2\lambda} = \frac{\Gamma(n+m+2)}{\Gamma(n+m+2\lambda+2)} 2^{2\lambda}. \tag{45}$$

If $\lambda = 0$, it is evidently

$$\lim \Phi_0(n, m, x) = \overline{\Phi}_0(n, m, x) =$$
$$= \frac{\Gamma(n+m+2)}{2^{n+m+1}\,\Gamma(n+1)\,\Gamma(m+1)} (1+x)^n (1-x)^m \tag{46}$$

and

$$\lim \mathfrak{J}_\lambda(n, m, x) = J_\lambda(n, m, x) =$$
$$= \frac{\Gamma(n+1)}{2^\lambda . \Gamma(n+\lambda+1)} (1+x)^{-n} (1-x)^{-m} \frac{d^\lambda}{dx^\lambda} [(1+x)^{n+\lambda} (1-x)^{m+\lambda}]. \tag{47}$$

These are the well-known polynomials of Jacobi which form the orthogonal integral system corresponding to the characteristic function

$$\overline{\Phi}_0(n, m, x) = (1+x)^n (1-x)^m$$

and to the integration interval ± 1.

Putting

$$s\omega = 1, \quad n = q-1, \quad m = p-q, \tag{48}$$

we obtain their more usual form from (4′) in the limit

$$J_\lambda(z) = \frac{1}{2^\lambda(q, \lambda)} z^{1-q}(1-z)^{q-p} \frac{d^\lambda}{dz^\lambda}[z^{q+\lambda-1}(1-z)^{p-q+\lambda}] \qquad (47')$$

corresponding to the integration interval (0,1).

We do not intend to discuss in detail the properties of Jacobi's polynomials, since there exists a great deal of research work about them. We confine ourselves to the deduction of some principal properties by a limiting process from the formulas valid for Jacobi's generalized polynomials.

E. g. the following expression can be deduced from the expression (8′) for Jacobi's polynomials in the limit:

$$J_\lambda(p, q, z) = \frac{(-)^\lambda}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \binom{\lambda}{k} \frac{(n+m+\lambda+1, \lambda-k)}{(n+1, \lambda-k)} z^{\lambda-k} =$$
$$= \frac{1}{2^\lambda} F(n+m+\lambda+1, -\lambda, n+1, z), \qquad (47'')$$

from which at the same time their relation to the hypergeometric Gauss series is evident.

The sum (14′) changes into the following value in the limit:

$$I_\lambda = \int_0^1 \overline{\Phi}_0(n, m, z) J_\lambda^2(n, m, z)\, dz = \frac{(1, \lambda)(m+1, \lambda)(n+m+\lambda+1, \lambda)}{2^{2\lambda}(n+1, \lambda)(n+m+2, 2\lambda)},$$

the functional equation (17′) into the recurrence relation

$$\frac{2(n+m+\lambda+1)(n+\lambda+1)}{(n+m+2\lambda+1)(n+m+2\lambda+2)} J_{\lambda+1}(z) +$$
$$+ \left(z - \frac{2\lambda^2 + 2\lambda(n+m+1) + (n+m)(1+n)}{2(n+m+2\lambda)(n+m+2\lambda+2)}\right) J_\lambda(z) +$$
$$+ \frac{\lambda(m+\lambda)}{2(n+m+2\lambda)(n+m+2\lambda+1)} J_{\lambda-1}(z) = 0. \qquad (49)$$

As we have pointed out before the limiting process of the difference equation (22′) into the Gauss differential equation

$$z(1-z) J_\lambda''(z) + (n+1-z\,\overline{n+m+2}) J_\lambda'(z) +$$
$$+ \lambda(n+m+\lambda+1) J_\lambda(z) = 0, \qquad (50)$$

is especially important.

To check these results, we can use e. g. the above mentioned monography of Abramescu: „Résumé de principales propriétés des polynomes orthogonaux"[21]) and an older paper of J. Darboux which we have

[21]) Nouvelles Annales de Math., 1923.

quoted already: „Mémoire sur l'approximation des fonctions de très grand nombres et sur une classe étendue de développements en série".[22]

In the last quoted paper there is solved for the first time the problem of expanding an arbitrary function in a series proceeding according to Jacobi's polynomials, which we obtain from the formula (28), if $s \to \infty$, $\omega \to 0$, $s\omega = 1$.

In this limiting case, we have

$$f(x) = \overline{\Phi}_0(x) \sum_i \frac{J_i(x)}{I_i} \int_0^1 f(y)\, J_i(y)\, dy. \tag{51}$$

The sum of the first k members of this series is given by the expression

$$S_k = \frac{2(n+m+\lambda+1)(n+\lambda+1)\overline{\Phi}_0(x)}{(n+m+2\lambda+1)(n+m+2\lambda+2)\, I_k} \int_0^1 f(y) \times \\ \times \frac{J_{k+1}(x)\, J_k(y) - J_k(x)\, J_{k+1}(y)}{x-y}\, dy. \tag{52}$$

In regard of the conditions on which this series converges and represents the function $f(x)$, I refer to the original paper of J. Darboux.

The practical application of the series (51) reduced to some initial members to the approximative expression of frequency curves was proposed by Professor V. Romanovsky in his treatise: „Generalization of some types of the frequency curves of Professor K. Pearson."[23]

Pearson's frequency curve of type I

$$y = y_0 \left(1 + \frac{x}{a}\right)^{\nu a} \left(1 - \frac{x}{b}\right)^{\nu b}$$

can be written also in the general form of the characteristic function $\overline{\Phi}_0(x)$

$$\overline{\Phi}_0(x) = \varkappa(a+x)^n\, (b-x)^m. \tag{53}$$

If we determine the constants $\varkappa$, a, b, n, m so that the first 5 moments of this function are identical with the moments of the function $y(x)$ given numerically, then it is obvious from the condition

$$\int_a^b x^i \overline{\Phi}_0(x)\, dx = \int_a^b x^i y(x)\, dx, \quad i = 0, 1, 2, 3, 4 \tag{54}$$

that the coefficients

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4$$

in the series

$$y(x) = \overline{\Phi}_0(x)\, [\alpha_0 J_0 + \alpha_1 J_1 + \ldots] \tag{55}$$

[22]) Journal de Math. p. et app. 1878.
[23]) Biometrika XVI (1924).

are equal to zero, Hence

$$y(x) = \overline{\Phi}_0(x)\,[1 + \alpha_5 J_5 + \alpha_6 J_6 + \ldots]. \qquad (55')$$

Professor Pearson has a sceptical opinion of the applicability of this series for the approximative expression of frequency curves, especially with regard to the high probable errors of the higher moments which are contained in the coefficients α_i.[24])

The literature on Jacobi's polynomials is quoted in full in the treatise: P. Appel A. Lambert: Généralisations diverses des fonctions sphériques, Encyclopédie des Sciences mathém. A II. vol. V fasc. From the older original work on this question we may single out the paper of C. G. J. Jacobi: Untersuchungen über die Differenzialgleichung der hypergeom. Reihe, published in volume 56 of the Journal de Crelles.

PART II.

The relation of the characteristic function Φ_0 to the hypergeometric frequency function. Connection with the Pearson frequency curves. The course of the characteristic function. Numerical computation. The approximative Pearson expression. The approximation of empirical frequency functions. Simple special cases of the function Φ_λ: a) $n=m$, b) $n=m=-\frac{1}{2}$, c) $n=m=0$. Degeneration of the function Φ_λ: a) the generalized Laplace frequency function, b) the generalized Poisson function, c) the Poisson function. The limiting forms of the function Φ_λ in case of $s \to \infty$, $\omega \to 0$.

Before we approach the deduction of several important special cases of Jacobi's generalized polynomials it will be advisable to discuss the characteristic function $\Phi_0(x)$ in some detail and to point to the special importance of this function in mathematical statistics.

1. The relation of the characteristic function $\Phi_0(x)$ to the hypergeometric frequency function.

The hypergeometric function

$$f(x) = \frac{\binom{k}{x}\binom{h}{m-x}}{\binom{k+h}{m}}$$

belongs to a very general type of frequency curves, a special case of which is among others especially the classical normal function of Laplace-Gauss and the Poisson function.

Among the different modifications of the corresponding problem of the theory of probability, the solution of which is given by this

[24]) See a note added to the above mentioned treatise of Romanovsky.

function, we shall refer in this section to the problem of Pólya,[25]) and in section 4 to Pearson's modification.

In the first case we have the following urn scheme:

Let us put into a bag which, at the beginning, contains R white balls, S black balls, $1 + \Delta$ balls of the colour of the ball just drawn after each drawing of a single ball. If we perform in this way k drawings, what is the probability that we shall draw x white balls, and $(k - x)$ black balls respectively?

If we denote the initial probabilities of drawing the white ball and the black ball by

$$p = \frac{R}{N}, \quad \text{and } q = \frac{S}{N} \text{ respectively}, \quad (R + S = N)$$

further the quotient Δ/N by δ which satisfies the inequalities

$$\delta > -\frac{p}{k-1}, \text{ or } \delta > -\frac{q}{k-1}$$

according to whether $p < q$ or $p > q$, then the required probability is given by the expression

$$p(x) = \binom{k}{x} \frac{R(R+\Delta)(R+2\Delta)\ldots(R+\overline{x-1}\Delta)S(S+\Delta)+\ldots(S+\overline{k-x-1}\Delta)}{N(N+\Delta)(N+2\Delta)\ldots(N+\overline{k-1}\,\Delta)} =$$

$$= \frac{\binom{\frac{p}{\delta}+x-1}{k}\binom{\frac{q}{\delta}+k-1-x}{k-x}}{\binom{\frac{1}{\delta}+k-1}{k}}. \tag{56}$$

The close connection of the characteristic function $\Phi_0(n, m, x)$ with the probability $p(x)$ is apparent especially from the comparison of the difference equation of the first order of the function $p(x)$

$$\frac{p(x+1)}{p(x)} = \frac{(k-x)(p+x\delta)}{(x+1)(q+\overline{k-1}-x\delta)} = \frac{(k-x)\left(\frac{p}{\delta}+x\right)}{(x+1)\left(\frac{q}{\delta}+k-1-x\right)}$$

with the analogous difference equation of the function $\Phi_0(n, m, z) =$ $= \Phi_0(n, m, x + \frac{1}{2}\,\overline{s-1}\,\omega)$

$$\frac{\Phi_0(n, m, z+\omega)}{\Phi_0(n, m, z)} = \frac{(\overline{n+1}\,\omega + z)(\overline{s-1}\,\omega - z)}{(z+\omega)(\overline{s-1-m}\,\omega - z)} \tag{57}$$

[25]) F. Eggenberger-G. Pólya; Ueber die Statistik verketteter Vorgänge, Zeit. f. angew. Math. u. Mech., 1923; c. p. also F. Eggenberger, Die Wahrscheinlichkeitsansteckung, Mitteilungen der Verein. schweiz. Versich.-Math., No 19, 1924.

Putting in this equation

$$n+1=\frac{p}{\delta}, \quad m+1=\frac{q}{\delta}, \quad s-1=k, \quad \omega=1 \qquad (58)$$

the two difference equations will entirely coincide. Therefore the function $\Phi_0(n, m, z)$ does not differ — except perhaps by a constant factor — from the frequency function $p(x)$. At the same time it is obvious, that the parameters n, m, s can be considered as characteristic constants, which possess a special importance in the theory of frequency functions.

2. The connection with the Pearson frequency curves.

The characteristic function $\Phi_0(n, m, x)$ is essentially a very general form of the Pearson frequency functions, as follows from the difference equation (57), which can be written in a form analogous to the Pearson differential equation:

$$\frac{\Phi_0(n, m, z+\omega)-\Phi_0(n, m, z)}{\omega\,\Phi_0(n, m, z)}=\frac{\underset{\omega}{\Delta}\Phi_0(n, m, z)}{\Phi_0(n, m, z)}=$$

$$=\frac{z(n+m)+\omega\,\overline{s-1}\,n-m\omega}{(z+\omega)(z-\overline{s-1+m}\,\omega)} \qquad (57')$$

If we transfer the origin of the coordinates to the point $z=b$, this difference equation is reduced to the differential equation

$$\frac{y'}{y}=\frac{(z+b)(n+m)+n\,a}{(z+b)(z+b-a)}=\frac{dz+\beta}{z^2+\gamma z+\delta},$$

in the limit $s \to \infty$, $\omega \to 0$, $s\omega=$const., provided the parameters n, m are finite. Professor Pearson used this differential equation as his starting-point in deducing the well-known 7 types of frequency curves.[26])

3. The course of the characteristic function $\Phi_0(n, m, z)$.

Resuming the thread of Eggenberger's discussions about the probability $p(x)$, we first form the value of the quotient

$$\frac{\Phi_0(n, m, z)}{\Phi_0(n, m, z-\omega)}=\frac{(n\omega+z)(s\omega-z)}{z(\overline{s+m}\,\omega-z)}=f(z). \qquad (59)$$

This quotient can assume the value 1 only in the point

$$z_0=\frac{sn\omega}{n+m}=\frac{s\omega}{1+\frac{m}{n}}$$

which is the root of the equation

$$f(z)=1.$$

If both extreme values $f(\omega)$ and $f(\overline{s-1}\,\omega)$ are at the same time either larger or smaller than 1, the point z_0 is situated outside of the

[26]) The fundamental Problem of practical Statistics, Biom. XIII.

interval $(\omega, \overline{s-1}\,\omega)$ and the function $\Phi_0(n, m, z)$ is monotonous in the interval under consideration. This example is characterized by the inequality

$$\frac{1}{s-1} > \frac{m}{n} > s-1$$

If, on the other hand, the inequality

$$\frac{1}{s-1} < \frac{m}{n} < s-1$$

is satisfied, z_0 comes inside of the summation interval and the function $\Phi_0(n, m, z)$ attains either maximum or minimum in this interval, according to, whether

$$f(\omega) - 1 > 0, \quad f(\overline{s-1}\,\omega) - 1 < 0$$

or

$$f(\omega) - 1 < 0, \quad f(\overline{s-1}\,\omega) - 1 > 0$$

F. Eggenberger expressed all these possibilities very clearly in dependence on the parameter

$$\delta = \frac{1}{n+m+2}$$

From the values of the differences

$$f(\omega) - 1 = \frac{s - 1 - \dfrac{m}{n}}{\dfrac{s-1}{n} + \dfrac{m}{n}} = \frac{\overline{s-1}\,p - q - \delta s}{\overline{s-1}\,\delta - \delta + q}$$

$$f(\overline{s-1}\,\omega) - 1 = \frac{1 - \dfrac{m}{n}\,\overline{s-1}}{\overline{s-1}\left(\dfrac{1}{n} + \dfrac{m}{n}\right)} = \frac{p + \overline{s-2}\,\delta - \overline{s-1}\,q}{\overline{s-1}\,q}$$

he deduced the following table expressing the course of the function $\Phi_0(n, m, z)$ in dependence on the parameter δ:

δ in the limits	$f(\omega)-1$	$f(\overline{s-1}\,\omega)-1$	the form of $\Phi_0(n, m, z)$ in the summation interval
$-\dfrac{p}{s-2},\ p \mp \dfrac{q-p}{s-2}$	+	—	bell-shaped, the maximum inside of the interval
$p \mp \dfrac{q-p}{s-2},\ q \pm \dfrac{q-p}{s-2}$	—	—	monotonous
$q \pm \dfrac{q-p}{s-2},\ \infty$	—	+	U-shaped, the minimum inside of the interval.

It is obvious from this table, that the more the values p, q, approach one the other the more decreases the interval of the values δ, for which $\Phi_0(z)$ is monotonous; if $p = q = \frac{1}{2}$, $\Phi_0(z)$ cannot be monotonous for any value δ in the interval under consideration.

4. The approximative Pearson expression.

In the paper „On a method of ascertaining limits to the actual number of marked members in a population of a given size from a sample"[27]) Professor K. Pearson discussed the approximative expression of the special case of the characteristic function

$$y(x) = C_0 \binom{x}{\varrho} \binom{N-x}{\gamma-\varrho}, \tag{60}$$

which we obtain from the expression $\Phi_0(z)$, if we put

$$z = x - n\omega, \quad n = \varrho, \quad m = \gamma - \varrho, \quad s = N - \gamma + 1, \quad \omega = 1.$$

The function $y(x)$ follows from the solution of the following problem, which is inverse to the problem of Pólya quoted above: Suppose a population of N individuals, of whom an unknown number x are marked by a special characteristic, and $N - x$ not so. Now let us draw at random a sample of γ individuals and find that ϱ of them are marked and $\sigma = \gamma - \varrho$ not so. We have to compute the probability that in the given population there are x marked individuals while $\varrho < x < N - \sigma$.

Professor Pearson attempted to replace the function $y(x)$, the values of which are given for the arguments $x = \varrho, \varrho + 1, \ldots N - \sigma$ — except the common factor — by the members of the hypergeometric series

$$c_0 \left(1 + \frac{\varrho+1}{1} \frac{N-\gamma}{N-\varrho} + \frac{(\varrho+1)(\varrho+2)}{1\,.\,2} \frac{(N-\gamma)(N-\gamma-1)}{(N-\varrho)(N-\varrho-1)} + \ldots\right),$$

in succession, by the well-known Pearson frequency curve of type I.

$$y(x) = y_0 \left(1 + \frac{x}{a_1}\right)^{m_1} \left(1 - \frac{x}{a_2}\right)^{m_2} \tag{61}$$

For this purpose he computed the necessary moments of the function $y(x)$ about the arithmetical mean: μ_2, μ_3, μ_4 using the results deduced by him for the moments of a hypergeometric series in the paper: „On certain properties of the hypergeometrical series" (1899).[28]) From these moments then follow the fundamental constants of the frequency curve of type I

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}; \quad \beta_2 = \frac{\mu_4}{\mu_2^2}.$$

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[27]) Biom. XX A, 1928.
[28]) Phil. Magaz., 1899.

The agreement of such an approximative expression with the exact values of the function $y(x)$ is considered very good by Professor Pearson, but the practical evaluation of the constants of the frequency curve of type I, is considerably difficult.

By performance of a greater number of concrete examples in collaboration with Miss Margaret Moul and Miss E. Worthington, Professor Pearson ascertained that the corresponding frequency curve of type I differs only slightly from the function

$$y = y_0 \left(\frac{\varrho b}{\gamma} + x\right)^{\varrho} \left(\frac{\sigma b}{\gamma} - x\right)^{\sigma} \tag{62}$$

where the constants $\varrho, \sigma, \gamma = \varrho + \sigma$ are, as obvious, known beforehand and the value $b = N - \gamma + 2$ denotes the length of the interval between the extreme zero values of the function $y(x)$. E. g. the Pearson curve of type I

$$10{\cdot}5606 \left(1 + \frac{x}{189{\cdot}5250}\right)^{19\,9076} \left(1 - \frac{x}{756{\cdot}9678}\right)^{79{\cdot}5115}$$

corresponds to the values $\varrho = 20$, $\sigma = 80$, $N = 1000$. The curve can be replaced by the function

$$y_0 \left(1 + \frac{x}{180}\right)^{20} \left(1 - \frac{x}{720}\right)^{80}$$

which can be deduced in an incomparatively easier way.

This knowledge gained empirically enables us to use practically the approximative expression of the function $y(x)$ by the Pearson curve of type I. From the general expression of the characteristic function containing ω, follows this relation as a limiting case, if $s \to \infty$, $\omega \to 0$, $\overline{s-1}\,\omega = b$.[29])

$$\lim_{\substack{s\to\infty \\ \omega\to 0}} \Phi_0(n, m, x) = \frac{\Gamma(n + m + 2)}{b\Gamma(n + 1)\,\Gamma(m + 1)} \cdot \left(\tfrac{1}{2} + \frac{x}{b}\right)^{n} \left(\tfrac{1}{2} - \frac{x}{b}\right)^{m}. \tag{63}$$

If we translate the origin of the coordinates to the argument of the maximal value, which is situated in the point

$$x_{\max} = \frac{n - m}{2(n + m)}\, b,$$

we obtain an expression identical with the approximative expression of Pearson

$$\lim \Phi_0(n, m, x) = y_0 \left(\frac{nb}{n + m} + x\right)^{n} \left(\frac{mb}{n + m} - x\right)^{m}. \tag{62'}$$

It is obvious from the process, by which we arrived at this result, that the difference between the approximative values of the function

[29]) Cp. Part I, sec. 11.

$y(x)$ given by the formulas (61) and (62) respectively and the exact values, decreases with the increasing s and N respectively. Pearson checked this fact on several examples.

In order to get a good approximation especially in the middle of the interval b also for minor values of N, Professor Pearson proposed two variants of this expression differing only in the determination of the value b. In the first case we determine b form the supposition, that the distance between the abscissa of the maximal value and the abscissa of the arithmetic mean of the functions (60) i. e.

$$\frac{\varrho}{\gamma}(N-\gamma) - \frac{(\varrho+1)(N-\gamma)}{\gamma+2} = \frac{(N-\gamma)(\sigma-\varrho)}{\gamma(\gamma+2)}$$

and the analogous distance of the function (62)

$$\frac{b(\sigma-\varrho)}{\gamma(\gamma+2)}$$

are the same, whence

$$b = N - \gamma.$$

Thereby, as it is apparent, the original choice $b = N - \gamma + 2$ changes only very slightly.

In the second case we determine b from the supposition that the dispersion of both functions is equal. From this condition follows for the value b

$$b = \sqrt{(N+2)(N-\gamma)} = (N-\gamma)\sqrt{1+\frac{\gamma+2}{N-\gamma}}.$$

This exemple, quoted last, offers an especially good approximative expression. In the first modification of the approximative expression (62) the length of the interval between the extreme zero values is smaller than the corresponding interval of the original function (60), while in the second modification it is larger.

To illustrate the second variant of the approximative expression, let us reproduce the corresponding formula

$$y(x) = y_0\,(189{\cdot}9263 + x)^{20}\,(759{\cdot}7052 - x)^{80}$$

applied to the concrete example quoted above, in which

$$N = 1000,\ \varrho = 20,\ \sigma = 80$$

from the paper of Professor Pearson.

If the number of individuals N is small, e. g. if $N = 30$, $\varrho = 2$, $\sigma = 10$ the difference between the expression of the frequency functions of type I by aid of the values β_1 and β_2

$$y = 131{\cdot}0512\left(1-\frac{x}{4{\cdot}7158}\right)^{1{\cdot}7111}\left(1-\frac{x}{19{\cdot}1170}\right)^{6{\cdot}9367}$$

and the expression (62), in which $b = \sqrt{(N+2)(N-\gamma)}$

$$y = 131{\cdot}3094\left(1+\frac{x}{5{\cdot}0596}\right)^{2}\left(1-\frac{x}{20{\cdot}2386}\right)^{8}$$

becomes much more considerable.

5. The approximation of the empirical frequency functions by application of the characteristic function $\Phi_0(n, m, z)$.

We can express the relative frequencies $y(z)$ of a certain statistical event, the course of which we can deduce in general outline from the hypothesis of the Pólya urn scheme, with sufficient accuracy by application of the function $\Phi_0(n, m, z)$ in the form

$$y(z) = y_0 \Phi_0(n, m, z) \tag{64}$$

The values y_0, n, m and p, δ respectively corresponding to the given statistical series can be determined by equating the moments of the zero, first and second order of the functions on both sides of the equation (64) (the method of moments). Since the sum of the relative frequencies for all values of the attribute z is equal to 1, the relation

$$y_0 = 1 = 1 : \sum_0^{\overline{s-1}\omega} \Phi_0(n, m, z)\, \omega \tag{65}$$

follows for y_0.

The further conditions expressing the equality of the moments of the first and second order can be replaced by the identical conditions of the equality of the fundamental constants of the frequency functions, i. e. the mean value h and the dispersion σ of the given empirical function and of the analogous values of the function $\Phi_0(n, m, z)$

$$\begin{aligned} h &= \sum_0^{s-1} z\, \Phi_0(n, m, z)\, \omega \\ \sigma^2 &= \sum_0^{s-1} z^2\, \Phi_0(n, m, z)\, \omega - h^2. \end{aligned} \tag{66}$$

By application of the polynomials $\mathfrak{J}_\lambda(n, m, z)$ we can easily compute first the mean value

$$h = \sum_0^{s-1} (-2p\mathfrak{J}_1(z) + kp\omega\mathfrak{J}_0{}^2(z))\, \Phi_0(n, m, z)\, \omega = k \,.\, p \,.\, \omega,$$

whence:

$$p = \frac{h}{k \,.\, \omega} \tag{67}$$

In an analogous manner we deduce for the dispersion σ:

$$\sigma^2 = \sum_0^{s-1} (4p^2\mathfrak{J}_1{}^2(z) - 4p^2k\omega\mathfrak{J}_1(z) + k^2p^2\omega^2\mathfrak{J}_0{}^2(z))\, \Phi_0(z)\omega -$$

$$- k^2p^2\omega^2 = \sum_0^{s-1} 4p^2\mathfrak{J}_1{}^2(z)\, \Phi_0(z)\omega = kp \,.\, q\omega^2 \frac{1 + k\delta}{1 + \delta}$$

and hence

$$\delta = \frac{k \cdot p \cdot q\omega^2 - \sigma^2}{\sigma^2 - k^2\, pq\omega^2}. \tag{68}$$

The results coincide completely with the values computed e. g. by F. Eggenberger and G. Pólya without the use of the polynomials $\mathfrak{J}_\lambda(n, m, z)$. For the constants n, m we get the following values:

$$n + 1 = \frac{h}{\omega} \, \frac{\sigma^2 - \overline{s-1}\, h\omega + h^2}{h\, \overline{s-1}\, \omega - h^2 - \sigma^2\, \overline{s-1}},$$

$$m + 1 = \left(s - 1 - \frac{h}{\omega}\right) \frac{\sigma^2 - \overline{s-1}\, h\omega + h^2}{h\, \overline{s-1}\, \omega - h^2 - \sigma^2\, \overline{s-1}}. \tag{69}$$

If we use the factorial moments of the given empirical function $\mathfrak{M}_i$ instead of its mean value h and dispersion σ

$$\Sigma zy\omega = \mathfrak{M}_1 = h$$

$$\Sigma z(z - \omega)\, y\omega + \omega^2 \Sigma zy - (\Sigma zy)^2 \omega = \sigma^2 = \mathfrak{M}_2 + \omega\, \mathfrak{M}_1 - \mathfrak{M}_1^2,$$

we obtain the expression of the parametres p, δ by these moments in the form:

$$p = \frac{\mathfrak{M}_1}{\overline{s-1}\, \omega}, \qquad \delta = \frac{-\mathfrak{M}_2 + \mathfrak{M}_1^2 \dfrac{s-2}{s-1}}{\mathfrak{M}_2 - \overline{s-2}\, \omega \mathfrak{M}_1}. \tag{70}$$

Professor Steffensen discusses the computation of constants of a hypergeometric frequency function by application of the moments of the function about the mean in his paper „Factorial Moments and Discontinuous Frequency Functions“.[30])

6. Simple special cases of the function $\Phi_\lambda(n, m, x)$.

The function $\Phi_\lambda(x)$ can be expressed by the product of the characteristic function $\Phi_0(x)$ and the function

$$\varphi_\lambda(n, m, x) = \frac{\left(\frac{s-1}{2} + m + 1 - \frac{x}{\omega}, \lambda\right) \left(\frac{s-1}{2} - \lambda + 1 + \frac{x}{\omega}, \lambda\right)}{(n + 1, \lambda)\, (m + 1, \lambda)\, (n + m + s + 1, \lambda)} \times$$

$$\times \frac{(n + m + 2, 2\lambda)}{(s - \lambda, \lambda)}. \tag{71}$$

Those cases, in which the characteristic function assumes a symetrical form, belong to the most important special forms of the function Φ_λ corresponding to certain specially chosen values of the parameters n, m. This happens, if

a) $n = m$.

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[30]) Skand. Aktuarietidskriff, 1923.

As we have said above, the characteristic function in the summation interval $\pm \frac{1}{2}(s-1)\,\omega$ cannot be monotonous in this case — provided $n > 0$ — as it has either a convex or a concave form symmetrical to the axis of coordinates. If n and m are finite quantities, the characteristic function $\Phi_0(n, n, x)$ is a certain generalization of the Pearson frequency function of type II.

By a further specialisation

$$\text{b) } n = m = -\tfrac{1}{2}$$

we obtain the characteristic function $\Phi_0(-\frac{1}{2}, -\frac{1}{2}, x)$ to the limiting form of which in case that $\omega \to 0$, $s \to \infty$ corresponds the well-known integral system of Tchebychef's orthogonal polynomials, important for expressing arbitrary functions by the method of best approximation.

If

$$\text{c) } n = m = 0,$$

the function $\Phi_0(0, 0, x)$ is reduced to a constant equal $\frac{1}{s\omega}$, and the function $\varphi_\lambda(n, m, x)$ to a function

$$\varphi_\lambda(0, 0, x) = \frac{\left(\frac{s+1}{2} - \lambda + \frac{x}{\omega}, \lambda\right)\left(\frac{s+1}{2} - \frac{x}{\omega}, \lambda\right)(2, 2\lambda)}{(1, \lambda)^2\,(s+1, \lambda)\,(s-\lambda, \lambda)}\,. \tag{72}$$

7. Degeneration of the function $\Phi_\lambda(n, m, x)$.

a) The generalized Laplace frequency function.

If we transfer the origin of the coordinates using the substitution

$$x = \bar{x} - \frac{s-1}{2}\,\omega + \frac{n+1}{n+m+2}\,\overline{s-1}\,\omega,$$

we obtain

$$\Phi_0(n, m, x) = \frac{\Gamma\left(n + 1 + \frac{n+1}{n+m+2}(s-1) + \frac{x}{\omega}\right)}{\omega\Gamma(n+1)\,\Gamma(m+1)} \times$$

$$\times \frac{\Gamma\left(s - 1 - \frac{n+1}{n+m+2}\,\overline{s-1} + m + 1 - \frac{x}{\omega}\right)}{\Gamma\left(\frac{n+1}{n+m+2}\,\overline{s-1} + 1 + \frac{x}{\omega}\right)} \times \tag{73}$$

$$\times \frac{\Gamma(n+m+2)\,\Gamma(s)}{\Gamma\left(\overline{s-1} - \frac{n+1}{n+m+2}\,\overline{s-1} + 1 - \frac{x}{\omega}\right)\Gamma(n+m+s+1)}$$

$$\varphi_\lambda(n, m, x) = \frac{\left(\overline{s-1} - \frac{n+1}{n+m+2}\,\overline{s-1} + m + 1 - \frac{x}{\omega}, \lambda\right)}{(n+1, \lambda)\,(m+1, \lambda)} \times$$

$$\times \frac{\left(\frac{n+1}{n+m+2}\,\overline{s-1} - \lambda + 1 + \frac{x}{\omega}, \lambda\right)(n+m+2, 2\lambda)}{(n+m+s+1, \lambda)\,(s-\lambda, \lambda)}.$$

(To be continued.)

Anwendung einiger Sätze aus der Wahrscheinlichkeitsrechnung auf die Berechnung der Prämien mehrerer Versicherungskombinationen.

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Einleitung.

Die Nettoprämie für eine gemischte Versicherung wird bekanntlich so berechnet, dass die Prämienzahlungen einer bestimmten Anzahl von Versicherten gerade ausreichen, um alle Auszahlungen der Gesellschaft zu decken, wenn in jedem Versicherungsjahre gerade die aus der Sterbetafel entnommene wahrscheinlichste Anzahl von Todesfällen eintritt und die Verzinsung den Annahmen entspricht.

Es ist nun naheliegend, nach der Prämie zu fragen, die sich ergibt, wenn wir nicht nur für jedes Jahr die wahrscheinlichste Anzahl von Todesfällen betrachten, sondern alle überhaupt möglichen Verteilungen der Todesfälle auf die einzelnen Jahre berücksichtigen, wobei jede Kombination mit ihrer Wahrscheinlichkeit in die Rechnung eingeführt wird. Es zeigt sich für die gemischte Versicherung, dass sich auf beide Arten dieselbe Prämie ergibt. (Den Beweis hiefür wiederholen wir kurz in unserem ersten Kapitel.)

Fassen wir die einzelnen Kombinationen mit ihren Wahrscheinlichkeiten als Abweichungen von derjenigen Kombination auf, die wir bei der ersten Art der Berechnung als einzige berücksichtigen, so sehen wir in der zweiten Art der Berechnung den Beginn einer risikotheoretischen Betrachtungsweise.*)

*) Vgl. hiezu O. Gruder, Zur Theorie des Risikos. 9. intern. Kongress. D III (1), besonders S. 228, 2. Absatz.