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STABILITY OF CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS BY LYAPUNOV FUNCTIONS

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Abstract. The stability of the zero solution of a nonlinear nonautonomous Caputo fractional differential equation is studied using Lyapunov-like functions. The novelty of this paper is based on the new definition of the derivative of a Lyapunov-like function along the given fractional equation. Comparison results using this definition for scalar fractional differential equations are presented. Several sufficient conditions for stability, uniform stability and asymptotic uniform stability, based on the new definition of the derivative of Lyapunov functions and the new comparison result, are established.

Keywords: stability; Caputo derivative; Lyapunov function; fractional differential equation

MSC 2010: 34A34, 34A08, 34D20

1. Introduction

One of the main properties in the qualitative theory of differential equations is stability of solutions. Stability enables us to compare the behavior of solutions starting at different points.

The stability of fractional order systems is quite recent. There are several approaches in the literature to the study of stability, one of which is the Lyapunov approach. As is mentioned in [15] there are several difficulties encountered when one applies the Lyapunov technique to fractional differential equations. Results on stability in the literature via Lyapunov functions could be divided into two main groups:

▷ continuously differentiable Lyapunov functions (see, for example, the papers [1], [3], [6], [7], [12], [10]). Different types of stability are discussed using the Caputo derivative of Lyapunov functions.

▷ continuous Lyapunov functions (see, for example, the papers [16], [9], [8]). In these papers the authors use the derivative of a Lyapunov function similar to the Dini derivative of Lyapunov functions in the literature, i.e., the Dini derivative

$$D_+V(t,x) = \limsup_{h\to 0+} \frac{1}{h}(V(t,x) - V(t-h, x-hf(t,x)))$$

is generalized to

$$D_+^q V(t,x) = \limsup_{h\to 0+} \frac{1}{h^q}(V(t,x) - V(t-h, x-h^q f(t,x)) \tag{1.1}$$

where $0 < q < 1$.

Stability for the zero solution of fractional nonlinear equations is studied in [1], [6], which requires differentiability of the applied Lyapunov function. Also, the fractional derivative of the Lyapunov function depends significantly on any solution of the given fractional equation.

In this paper the stability of the zero solution of nonlinear nonautonomous fractional differential equations is studied. We define in an appropriate way the Caputo fractional Dini derivative of a Lyapunov function. Comparison results using this new definition and scalar fractional differential equations are presented and sufficient conditions for stability, uniform stability and asymptotic uniform stability are obtained.

2. Notes on fractional calculus

Fractional calculus generalizes the derivative and the integral of a function to a non-integer order [9], [13], [14] and there are several definitions of fractional derivatives and fractional integrals.

General Case. Let the number $q > 0$, $n-1 < q < n$ be given, where n is a natural number, and $\Gamma(\cdot)$ denotes the Gamma function.

1: The Riemann-Liouville (RL) fractional derivative of order q of $m(t)$ is given by (see, for example, 1.4.1.1 in [4], or [13])

$$ {}_{t_0}D_t^q m(t) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^t (t-s)^{n-q-1} m(s)\,\mathrm{d}s, \quad t \geqslant t_0.$$

2: The Caputo fractional derivative of order q of $m(t)$ is defined by (see, for example, 1.4.1.3 in [4])

$$ {}_{t_0}^{c}D_t^q m(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^t (t-s)^{n-q-1} m^{(n)}(s)\,\mathrm{d}s, \quad t \geqslant t_0.$$

The Caputo and Riemann-Liouville formulations coincide when $m(t_0) = 0$. The properties of the Caputo derivative are quite similar to those of ordinary derivatives. Also, the initial conditions of fractional differential equations with the Caputo derivative has a clear physical meaning and as a result the Caputo derivative is usually used in real applications.

3: The Grunwald-Letnikov fractional derivative of order q of $m(t)$ is given by (see, for example, 1.4.1.2 in [4])

$$D_0^q m(t) = \lim_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[(t-t_0)/h]} (-1)^r (qCr) m(t-rh), \quad t \geqslant t_0,$$

and the Grunwald-Letnikov fractional Dini derivative of order q by

$$D_{0+}^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[(t-t_0)/h]} (-1)^r (qCr) m(t-rh), \quad t \geqslant t_0, \tag{2.1}$$

where qCr are the binomial coefficients and $[(t-t_0)/h]$ denotes the integer part of the fraction $(t-t_0)/h$.

If $m(t) \in C^n([t_0, T])$, $n-1 < q < n$ then the Grunwald-Letnikov fractional derivative is given by (see [11], Definition 1.3)

$$D_0^q m(t) = \sum_{r=0}^{n-1} \frac{m^{(r)}(t_0)(t-t_0)^{-q+r}}{\Gamma(-q+r+1)} + \frac{1}{\Gamma(n-q)} \int_{t_0}^t (t-s)^{n-q-1} m^{(n)}(s)\,\mathrm{d}s, \ \ t \in (t_0, T].$$

Partial Case. In engineering, the fractional order q is often less than 1, so we restrict our attention to $q \in (0,1)$. Then to simplify the notation we will use ${}^cD^q$ instead of ${}_{t_0}^{c}D_t^q$ and the Caputo fractional derivative of order q of the function $m(t)$ then is

$${}^cD^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^t (t-s)^{-q} m'(s)\,\mathrm{d}s, \qquad t \geqslant t_0. \tag{2.2}$$

Also, the RL fractional derivative of order q of $m(t)$ is given by

$$D^q m(t) = \frac{1}{\Gamma(1-q)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_0}^t (t-s)^{-q} m(s)\,\mathrm{d}s, \quad t \geqslant t_0.$$

In this paper, since $0 < q < 1$, we are interested in the Caputo fractional Dini derivative of a function $m(t)$. Since the relation between the Caputo fractional derivative and the Grunwald-Letnikov fractional derivative is given by ${}^{c}D^{q}m(t) = D_{0}^{q}[m(t) - m(t_0)]$, using (2.1) we define the Caputo fractional Dini derivative as

$$ {}^{c}D_{+}^{q}m(t) = D_{0+}^{q}[m(t) - m(t_0)], $$

i.e.

(2.3)

$$ {}^{c}D_{+}^{q}m(t) = \limsup_{h\to 0+} \frac{1}{h^q}\left[m(t) - m(t_0) - \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1}(qCr)(m(t-rh) - m(t_0))\right]. $$

Definition 1 ([16])**.** We say $m \in C^q([t_0,T],\mathbb{R}^n)$ if $m(t)$ is differentiable (i.e. $m'(t)$ exists), and the Caputo derivative ${}^{c}D^{q}m(t)$ exists and satisfies (2.2) for $t \in [t_0,T]$.

Remark 1. If $m \in C^q([t_0,T],\mathbb{R}^n)$ then ${}^{c}D_{+}^{q}m(t) = {}^{c}D^{q}m(t)$.

Example 1. Let $m(t) = t$ and $q = 0.5$. Then using the formula (cf. [13])

$$ D_{0+}^{q}(t-t_0)^m = \frac{\Gamma(m+1)}{\Gamma(m-q+1)}(t-t_0)^{m-q} \tag{2.4} $$

we obtain

$$ {}^{c}D_{+}^{0.5}m(t) = D_{0+}^{0.5}(t-t_0) = \frac{\sqrt{t-t_0}}{\Gamma(2-0.5)} = \frac{\sqrt{t-t_0}}{\Gamma(1.5)} = 2\sqrt{\frac{t-t_0}{\pi}} $$

and

$$ {}^{c}D^{0.5}m(t) = \frac{1}{\Gamma(0.5)}\int_{t_0}^{t}\frac{\mathrm{d}s}{\sqrt{t-s}} = 2\frac{\sqrt{t-t_0}}{\Gamma(0.5)} = 2\sqrt{\frac{t-t_0}{\pi}}. $$

3. Statement of the problem

Consider the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < q < 1$,

$$ {}^{c}D^{q}x = f(t,x), \ t \geqslant t_0, \tag{3.1} $$

where $x \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,0) \equiv 0$.

We will assume in the paper that the function $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ is such that for any initial data $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ the system FrDE (3.1) with the initial condition $x(t_0) = x_0$ has a solution $x(t;t_0,x_0) \in C^q([t_0,\infty),\mathbb{R}^n)$. Note that some sufficient conditions for global existence of solutions of (3.1) are given in [2], [5], [9].

The goal of the paper is to study the stability of the system FrDEs (3.1). In the definition below we assume $x(t;t_0,x_0)$ is any solution of (3.1) with $x(t_0) = x_0$.

Definition 2. The zero solution of (3.1) is said to be

- ▷ *stable* if for every $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ such that for any $x_0 \in \mathbb{R}^n$ the inequality $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \geqslant t_0$;
- ▷ *uniformly stable* if for every $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ such that for $t_0 \in \mathbb{R}_+, x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta$ the inequality $\|x(t; t_0, x_0)\| < \varepsilon$ holds for $t \geqslant t_0$;
- ▷ *uniformly attractive* if for $\beta > 0$: for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that for any $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \beta$ the inequality $\|x(t; t_0, x_0)\| < \varepsilon$ holds for $t \geqslant t_0 + T$;
- ▷ *uniformly asymptotically stable* if the zero solution is uniformly stable and uniformly attractive.

In this paper we will use the followings sets:

$$\begin{aligned}
\mathcal{K} &= \{a \in C[\mathbb{R}_+, \mathbb{R}_+] \colon a \text{ is strictly increasing and } a(0) = 0\},\\
\overline{B}(\lambda) &= \{x \in \mathbb{R}^n \colon \|x\| \leqslant \lambda\},\\
B(\lambda) &= \{x \in \mathbb{R}^n \colon \|x\| < \lambda\},\ \lambda = \text{const} > 0.
\end{aligned}$$

We will use comparison results for scalar fractional differential equations of the type

$$(3.2) \qquad {}^cD^q u = g(t, u), \quad t \in J,$$

where $u \in \mathbb{R}$, $J = [t_0, \infty) \subset \mathbb{R}_+$, $g \colon J \times \mathbb{R} \to \mathbb{R}$, $g(t, 0) \equiv 0$. Note that (3.2) with $u(t_0) = u_0$ is called the initial value problem (3.2). We will assume in the paper that the function $g \colon J \times \mathbb{R} \to \mathbb{R}$ is such that for any initial data $(t_0, u_0) \in J \times \mathbb{R}$ the scalar FrDE (3.2) with $u(t_0) = u_0$ has a solution $u(t; t_0, u_0) \in C^q(J \cap [t_0, \infty), \mathbb{R})$. Also, we assume that for any compact subset $I \subset J$ there exists a small enough number $L_I > 0$ such that the corresponding FrDE ${}^cD^q u = g(t, u) + \eta$ with $\eta \in (0, L_I]$ has a solution $u(t; t_0, u_0, \eta) \in C^q(I \cap [t_0, \infty), \mathbb{R})$ where $(t_0, u_0) \in I \times \mathbb{R}$. Note that some existence results for (3.2) are given in [2], [5], [9].

Example 2. Consider the scalar fractional differential equation (3.2) with $g(t, u) = -2u$, i.e. consider the scalar FrDE

$$(3.3) \qquad {}^cD^q u(t) = -2u.$$

The equation (3.3) with an initial condition $u(t_0) = u_0$ has a solution

$$(3.4) \qquad u(t; t_0, u_0) = u_0 E_q(-2(t - t_0)^q).$$

where the Mittag-Leffler function (with one parameter) is defined by

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}, \quad q > 0.$$

From (3.4) and the inequality $0 < E_q(-2(t-t_0)^q) < 1$ for $t \geqslant t_0$ we obtain

$$|u(t; t_0, u_0)| \leqslant |u_0|. \tag{3.5}$$

Inequality (3.5) guarantees that the zero solution of (3.3) is uniformly stable.

In this paper we will study the connection between the stability of the system FrDE (3.1) and the stability of the scalar FrDE (3.2).

We now introduce the class Λ of Lyapunov-like functions which will be used to investigate the stability of the system FrDE (3.1).

Definition 3. Let $t_0, T \in \mathbb{R}_+ \colon T > t_0$, and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$. We will say that the function $V(t,x) \colon [t_0, T) \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda([t_0,T), \Delta)$ if $V(t,x) \in C([t_0,T) \times \Delta, \mathbb{R}_+)$ is locally Lipschitzian with respect to its second argument and $V(t,0) \equiv 0$.

Lyapunov-like functions used to discuss stability for differential equations require an appropriate definition of the derivative of the Lyapunov function along the studied differential equations. For fractional differential equations some authors ([3], [10], [12]) defined and used the so called Caputo fractional derivative of Lyapunov functions. This approach requires the function to be smooth enough (at least continuously differentiable) and also some conditions involved are quite restrictive. Other authors use the Lakshmikantham et al. derivative of Lyapunov functions ([9], [8]) and this definition requires only the continuity of the Lyapunov function. However, it can be quite restrictive (see Example 4) and can cause some problems (see Example 5).

In this paper we introduce the derivative of the Lyapunov function. For this we will use the Caputo fractional Dini derivative of a function $m(t)$ given in (2.3). We define the generalized *Caputo fractional Dini derivative* of the function $V(t,x) \in \Lambda([t_0,T), \Delta)$ along the trajectories of solutions of the system FrDE (3.1) as follows:

$$\begin{aligned} {}_{+}^{c}D^q_{(3.1)} V(t,x) = \limsup_{h \to 0^+} \frac{1}{h^q} \Big\{ & V(t,x) - V(t_0,x_0) \\ & - \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} qCr [V(t-rh, x - h^q f(t,x)) - V(t_0,x_0)] \Big\} \text{ for } t \geqslant t_0, \end{aligned} \tag{3.6}$$

where $t \in (t_0, T)$, $x, x_0 \in \Delta$, and there exists $h_1 > 0$ such that $t - h \in [t_0, T)$, $x - h^q f(t,x) \in \Delta$ for $0 < h \leqslant h_1$.

R e m a r k 2. Let $q = 1$ and $x_0 = 0$ in formula (3.6). Then using $nCr = 0$ for $n < r$, $n, r > 0$ integers, we obtain for any $t \geqslant 0$ the formula ${}_{+}^{c}D^{1}_{(3.1)}V(t,x) = \limsup_{h\to 0^+} h^{-1}\{V(t,x) - V(t-h, x-hf(t,x))\}$ which coincides with the known in the literature $D_+V(t,x)$ used for studying stability of zero solution of ordinary differential equations.

We will give some examples to illustrate the application of the introduced Caputo fractional Dini derivative of the function $V(t,x) \in \Lambda([t_0,T),\Delta)$ along the trajectories of solutions of the initial value problem for the system FrDE (3.1) and make comparisons with other derivatives of Lyapunov functions in the literature.

E x a m p l e 3. Let $V\colon \mathbb{R} \to \mathbb{R}_+$ be given by $V(t,x) = x^2$. Then the Caputo fractional Dini derivative of the function $V(t,x)$ is

$$\begin{aligned}
{}_{+}^{c}D^{q}_{(3.1)}V(t,x) &= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{x^2 - x_0^2 - \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} qCr[(x - h^q f(t,x))^2 - x_0^2]\Big\} \\
&= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{x^2 - x_0^2 + \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r} qCr[(x - h^q f(t,x))^2 - x^2 + x^2 - x_0^2]\Big\} \\
&= (x^2 - x_0^2)\limsup_{h\to 0^+} \frac{1}{h^q}\sum_{r=0}^{[(t-t_0)/h]} (-1)^{r} qCr - 2xf(t,x)\limsup_{h\to 0^+}\sum_{r=1}^{[(t-t_0)/h]} (-1)^{r} qCr \\
&\quad - (f(t,x))^2 \limsup_{h\to 0^+} h^q \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r} qCr.
\end{aligned}$$

Using the equalities $\lim_{N\to\infty}\sum_{r=0}^{N}(-1)^r qCr = 0$, where N is a natural number, and $\lim_{h\to 0^+}[(t-t_0)/h] = \infty$, we obtain

$$\lim_{h\to 0^+} \sum_{r=1}^{[(t-t_0)/h]} (-1)^r qCr = -1 \tag{3.7}$$

and

$$\limsup_{h\to 0^+} \frac{1}{h^q}\sum_{r=0}^{[(t-t_0)/h]} (-1)^r qCr = D_0^q(1) = \frac{(t-t_0)^{-q}}{\Gamma(1-q)}. \tag{3.8}$$

Therefore, the Caputo fractional Dini derivative of V is given by

$$ {}_{+}^{c}D^{q}_{(3.1)}V(t,x) = \frac{x^2 - x_0^2}{(t-t_0)^q\Gamma(1-q)} + 2xf(t,x). \tag{3.9}$$

R e m a r k 3. Consider the Lyapunov function which does not depend on t, $V(x)$. Then the Caputo fractional Dini derivative along the trajectories of solutions of the initial value problem for the system FrDE (3.1) given by formula (3.6) is reduced to

$$
\tag{3.10}
{}_{+}^{c}D^{q}_{(3.1)}V(x;t_0,x_0) = \limsup_{h\to 0^+}\frac{V(x)-V(x-h^q f(t,x))}{h^q} + (V(x)-V(x_0))\frac{(t-t_0)^{-q}}{\Gamma(1-q)}.
$$

Note that in [9] and [8] the derivative of the Lyapunov function from the class Λ is introduced as

$$
\tag{3.11}
D^q V(t,x) = \limsup_{h\to 0}\frac{1}{h^q}[V(t,x)-V(t-h,x-h^q f(t,x)].
$$

This derivative can be quite different from the classical one (see below).

E x a m p l e 4. Let $V\colon \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ be given by $V(t,x) = x^2/(t+1)^2$ and consider the case $t_0=0, x_0=0$.

Apply formula (3.11) to obtain the derivative of V, namely

$$
\tag{3.12}
\begin{aligned}
D^q V(t,x) &= \limsup_{h\to 0}\frac{1}{h^q}\Big[\frac{x^2}{(t+1)^2}-\frac{(x-h^q f(t,x))^2}{(t+1-h)^2}\Big]\\
&= \limsup_{h\to 0}\frac{1}{h^q}\frac{x^2(t+1-h)^2-(x-h^q f(t,x))^2(t+1)^2}{(t+1)^2(t+1-h)^2}\\
&= \limsup_{h\to 0}\frac{1}{h^q}\frac{(-xh+h^q f(t,x)(t+1))(2x(t+1)-xh-h^q f(t,x)(t+1))}{(t+1)^2(t+1-h)^2}\\
&= \limsup_{h\to 0}\frac{(-xh^{1-q}+f(t,x)(t+1))(2x(t+1)-xh-h^q f(t,x)(t+1))}{(t+1)^2(t+1-h)^2}\\
&= \frac{2xf(t,x)}{(t+1)^2}.
\end{aligned}
$$

Use (3.6) to obtain the derivative of V, namely

$$
\tag{3.13}
\begin{aligned}
{}_{+}^{c}D^{q}_{(3.1)}V(t,x) &= \limsup_{h\to 0^+}\frac{1}{h^q}\Big\{\frac{x^2}{(t+1)^2}-\sum_{r=1}^{[t/h]}(-1)^{r+1}qCr\frac{(x-h^q f(t,x))^2}{(t+1-rh)^2}\Big\}\\
&= x^2\limsup_{h\to 0^+}\frac{1}{h^q}\sum_{r=0}^{[t/h]}\frac{(-1)^r qCr}{(t+1-rh)^2}\\
&\quad - 2xf(t,x)\limsup_{h\to 0^+}\sum_{r=1}^{[t/h]}\frac{(-1)^{r+1}qCr}{(t+1-rh)^2}.
\end{aligned}
$$

Applying (3.7), (3.8) and simplifying (3.13), we obtain

$$ {}^{c}_{+}D^{q}_{(3.1)}V(t,x) = \frac{2xf(t,x)}{(t+1)^2} + x^2 D^q_0\Big(\frac{1}{(t+1)^2}\Big). \tag{3.14}$$

Now consider the well known case $q = 1$. The derivative of the Lyapunov function with respect to an ordinary differential equation is

$$ DV(t,x) = \frac{2xf(t,x)}{(t+1)^2} + x^2\Big(\frac{1}{(t+1)^2}\Big)'. \tag{3.15}$$

In formula (3.11), we obtain only the first term of $DV(t,x)$ in (3.15). Formulas (3.14) and (3.15) are quite similar.

The formula (3.11) also leads to some problems.

Example 5. Let $V\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t,x) = \sin^2 tx^2$. It is locally Lipschitz with respect to its second argument x.

Apply formula (3.11) to obtain the derivative of V, namely

$$\begin{aligned}
D^qV(t,x) &= \limsup_{h\to 0} \frac{1}{h^q}[\sin^2 tx^2 - \sin^2(t-h)(x - h^q f(t,x))^2] \\
&= \limsup_{h\to 0} \frac{1}{h^q}\{(\sin^2 t - \sin^2(t-h))x^2 + \sin^2(t-h)h^q f(t,x)(2x - h^q f(t,x))\} \\
&= x^2 \limsup_{h\to 0} \frac{\sin^2 t - \sin^2(t-h)}{h^q} + \limsup_{h\to 0} \sin^2(t-h) f(t,x)(2x - h^q f(t,x)) \\
&= 2x\sin^2(t) f(t,x).
\end{aligned}$$

Let $f(t,x) \equiv 0$. Then the solution of (3.1) for $n = 1$ and $t_0 = 0$ is $x(t) \equiv x_0$, $t \geqslant 0$ and $V(t,x(t)) = x_0^2 \sin^2 t$. All the conditions of [8], Corollary 2.2, are satisfied so the inequality $V(t,x(t)) \leqslant V(t_0,x_0)$, $t \geqslant t_0$ has to hold. However, in this case the inequality $x_0^2\sin^2 t \leqslant x_0^2 \sin^2 0 = 0$ is not satisfied for all $t \geqslant t_0$.

4. Fractional differential inequalities and comparison results for scalar FrDE

Again in this section we assume $0 < q < 1$.

Lemma 1. *Let $m \in C([t_0,T],\mathbb{R})$ and suppose that there exists $t^* \in (t_0,T]$ such that $m(t^*) = 0$ and $m(t) < 0$ for $t_0 \leqslant t < t^*$. Then if the Caputo fractional Dini derivative (2.3) of m exists at t^* then the inequality ${}^{c}D^q_+ m(t^*) > 0$ holds.*

P r o o f. From (2.1) (note that $m(t^*) = 0$, $r - q > 0$ for $r = 1, 2, \ldots$, and $0 < q < 1$) we obtain

$$
\begin{aligned}
D^q_{0+} m(t^*) &= \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[(t^*-t_0)/h]} (-1)^r (qCr) m(t^* - rh) \\
&= m(t^*) + \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=1}^{[(t^*-t_0)/h]} (-1)^r \frac{q(q-1)\ldots(q-r+1)}{r!} m(t^* - rh) \\
&= \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=1}^{[(t^*-t_0)/h]} \frac{q(1-q)\ldots(r-1-q)}{r!} (-m(t^* - rh)).
\end{aligned}
$$

Since all the terms of the above series are positive we obtain $D^q_{0+} m(t^*) \geqslant 0$. From the relation ${}^cD^q_+ m(t) = D^q_{0+}[m(t) - m(t_0)]$ we get

$$
{}^cD^q_+ m(t^*) = D^q_{0+} m(t^*) - \frac{m(t_0)(t^* - t_0)^{-q}}{\Gamma(1-q)}. \tag{4.1}
$$

Now $m(t_0) < 0$, $t^* > t_0$, $\Gamma(1-q) > 0$, and (4.1) completes the proof. $\square$

Now we present a comparison result.

Lemma 2 (Comparison result)**.** *Assume the following conditions are satisfied:*

1. *The function $x^*(t) = x(t; t_0, x_0)$, $x^* \in C^q([t_0, T], \Delta)$, is a solution of the FrDE* (3.1), *where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$, t_0, $T \in \mathbb{R}_+ :$ $t_0 < T$ are given constants, $x_0 \in \Delta$.*
2. *The function $g \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$.*
3. *The function $V \in \Lambda([t_0, T], \Delta)$ and for any points $t \in [t_0, T]$ and $x \in \Delta$ the inequality*
$$ {}^c_+D^q_{(3.1)} V(t, x) \leqslant g(t, V(t, x)) $$
holds.
4. *The function $u^*(t) = u(t; t_0, u_0)$, $u^* \in C^q([t_0, T], \mathbb{R})$, is the maximal solution of the initial value problem* (3.2).

Then the inequality $V(t_0, x_0) \leqslant u_0$ implies $V(t, x^*(t)) \leqslant u^*(t)$ for $t \in [t_0, T]$.

P r o o f. Let $\eta > 0$ be an arbitrary number and consider the initial value problem for the scalar FrDE

$$
{}^cD^q u = g(t, u) + \eta \quad \text{for } t \in [t_0, T],\ u(t_0) = u_0 + \eta, \tag{4.2}
$$

where η is small enough (i.e. $\eta \leqslant L_{[t_0,T]}$ as described after (3.2)). The function $u(t, \eta)$ is a solution of the scalar fractional differential equation (4.2) if and only if it satisfies the Volterra fractional integral equation ([5], Lemma 6.2)

$$u(t,\eta) = u_0 + \eta + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} (g(s,u(s,\eta)) + \eta)\,\mathrm{d}s \quad \text{for } t \in [t_0, T]. \tag{4.3}$$

Let the function $m(t) \in C([t_0,T],\mathbb{R}_+)$ be $m(t) = V(t, x^*(t))$. We now prove that

$$m(t) < u(t,\eta) \quad \text{for } t \in [t_0, T]. \tag{4.4}$$

Note that the inequality (4.4) holds for $t = t_0$, since $m(t_0) = V(t_0, x_0) \leqslant u_0 < u(t_0,\eta)$. Assume that inequality (4.4) is not true. Then there exists a point t^* such that $m(t^*) = u(t^*,\eta)$, $m(t) < u(t,\eta)$ for $t \in [t_0, t^*)$. Now Lemma 1 (applied to $m(t) - u(t,\eta)$) yields ${}^{c}D^{q}_{+}(m(t^*) - u(t^*,\eta)) > 0$, i.e.

$${}^{c}D^{q}_{+} m(t^*) > g(t^*, u(t^*,\eta)) + \eta > g(t^*, m(t^*)). \tag{4.5}$$

Due to condition 1 of Lemma 2 the function $x^*(t)$ satisfies the following initial value problem for the system of FrDE:

$${}^{c}D^{q}_{+} x = f(t,x), \quad x(t_0) = x_0, \quad t \in [t_0, T]. \tag{4.6}$$

Then for $t \in (t_0, T)$ the equality

$$\limsup_{h \to 0+} \frac{1}{h^q} [x^*(t) - x_0 - S(x^*(t), h)] = f(t, x^*(t))$$

holds, where

$$S(x^*(t), h) = \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} qCr [x^*(t - rh) - x_0].$$

Therefore,

$$S(x^*(t), h) = x^*(t) - x_0 - h^q f(t, x^*(t)) - \Lambda(h^q)$$

or

$$x^*(t) - h^q f(t, x^*(t)) = S(x^*(t), h) + x_0 + \Lambda(h^q) \tag{4.7}$$

with $\Lambda(h^q)/h^q \to 0$ as $h \to 0$. Then for any $t \in (t_0, T)$ we obtain

$$\begin{aligned}
(4.8)\quad & m(t) - m(t_0) - \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} qCr[m(t-rh) - m(t_0)] \\
&= \Big\{ V(t, x^*(t)) - V(t_0, x_0) \\
&\quad - \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} qCr[V(t-rh, x^*(t) - h^q f(t, x^*(t)) - V(t_0, x_0)] \Big\} \\
&\quad + \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} qCr\{[V(t-rh, S(x^*(t), h) + x_0 + \Lambda(h^q)) - V(t_0, x_0)] \\
&\quad - [V(t-rh, x^*(t-rh)) - V(t_0, x_0)]\}.
\end{aligned}$$

Since V is locally Lipschitzian in its second argument with a Lipschitz constant $L > 0$, we obtain

$$\begin{aligned}
(4.9)\quad & \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} qCr\{V(t-rh, S(x^*(t), h) + x_0 + \Lambda(h^q)) \\
&\quad - V(t-rh, x^*(t-rh))\} \\
&\leqslant L \Big\| \sum_{r=1}^{[(t-t_0)/h]} qCr(S(x^*(t), h) + \Lambda(h^q) - (x^*(t-rh) - x_0)) \Big\| \\
&\leqslant L \Big\| \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} qCr \sum_{j=1}^{[(t-t_0)/h]} qCj(x^*(t-jh) - x_0) \\
&\quad - \sum_{r=1}^{[(t-t_0)/h]} qCr((x^*(t-rh) - x_0)) \Big\| + L\Lambda(h^q) \sum_{r=1}^{[(t-t_0)/h]} qCr \\
&= L \Big\| \Big(\sum_{r=0}^{[t-t_0/h]} (-1)^{r+1} qCr \Big) \Big(\sum_{j=1}^{[(t-t_0)/h]} qCj(x^*(t-jh) - x_0) \Big) \Big\| \\
&\quad + L\, \Lambda(h^q) \sum_{r=1}^{[(t-t_0)/h]} qCr.
\end{aligned}$$

Substituting (4.9) in (4.8), dividing both sides by h^q, taking the limit as $h \to 0^+$, using (3.7) and $\sum_{r=0}^{\infty} qCrz^r = (1+z)^q$ if $|z| \leqslant 1$, we obtain for any $t \in (t_0, T)$ the inequality (note (2.3) and (3.6) and condition 3 of Lemma 2)

$$
\begin{aligned}
(4.10)\quad {}^{c}D_{+}^{q}m(t) &\leqslant {}^{c}_{(3.1)}D_{+}^{q}V(t,x^{*}(t)) + L\lim_{h\to 0+}\frac{\Lambda(h^{q})}{h^{q}}\lim_{h\to 0+}\sum_{r=1}^{[(t-t_0)/h]} qCr \\
&\quad + L\lim_{h\to 0^{+}}\sup\biggl\|\biggl(\sum_{r=0}^{[(t-t_0)/h]}(-1)^{r+1}qCr\biggr)\biggl(\frac{1}{h^{q}}\sum_{j=1}^{[t-t_0/h]} qCj(x^{*}(t-jh)-x_0)\biggr)\biggr\| \\
&= {}^{c}_{(3.1)}D_{+}^{q}V(t,x^{*}(t)) \leqslant g(t,V(t,x^{*}(t))) = g(t,m(t)).
\end{aligned}
$$

Now (4.10) with $t=t^{*}$ contradicts (4.5). Therefore, (4.4) holds.

We now show that if $\eta_2<\eta_1$ then

$$
(4.11)\qquad u(t,\eta_2)<u(t,\eta_1) \text{ for } t\in[t_0,T].
$$

Note that the inequality (4.11) holds for $t=t_0$. Assume that inequality (4.11) is not true. Then there exists a point t^{*} such that $u(t^{*},\eta_2)=u(t^{*},\eta_1)$ and $u(t,\eta_2)<u(t,\eta_1)$ for $t\in[t_0,t^{*})$. Now Lemma 1 (applied to $u(t,\eta_2)-u(t,\eta_1)$) yields ${}^{c}D_{+}^{q}(u(t^{*},\eta_2)-u(t^{*},\eta_1)))>0$. However,

$$
{}^{c}D_{+}^{q}(u(t^{*},\eta_2)-u(t^{*},\eta_1))) = g(t^{*},u(t^{*},\eta_2))+\eta_2-[g(t^{*},u(t^{*},\eta_1))+\eta_1] = \eta_2-\eta_1<0,
$$

a contradiction. Thus (4.11) is true.

Recall $0<\eta\leqslant L_{[t_0,T]}$. Now (4.4) and (4.11) guarantee that the family of solutions $\{u(t,\eta)\}$, $t\in[t_0,T]$ of (4.2) is uniformly bounded, i.e. there exists $K>0$ with $|u(t,\eta)|\leqslant K$ for $(t,\eta)\in[t_0,T]\times[0,L_{[t_0,T]}]$. Let $M=\sup\{|g(t,x)|\colon (t,x)\in[t_0,T]\times[-K,K]\}$. Take a decreasing sequence of positive numbers $\{\eta_j\}_{j=0}^{\infty}$, $\eta_0\leqslant L_{[t_0,T]}$, such that $\lim\limits_{j\to\infty}\eta_j=0$ and consider the sequence of functions $u(t;\eta_j)$. Now for $t_1,t_2\in[t_0,T]$, $t_1<t_2$, we have

$$
\begin{aligned}
(4.12)\quad &|u(t_2,\eta_j)-u(t_1,\eta_j)| \\
&\leqslant \frac{1}{\Gamma(q)}\biggl|\int_{t_0}^{t_1}((t_2-s)^{q-1}-(t_1-s)^{q-1})(g(s,u(s,\eta_j))+\eta_j)\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2}(t_2-s)^{q-1}(g(s,u(s,\eta_j))+\eta_j)\,\mathrm{d}s\biggr| \leqslant 2\frac{[M+1]}{q\Gamma(q)}(t_2-t_1)^{q}.
\end{aligned}
$$

Thus the family $\{u(t;\eta_j)\}$ is equicontinuous on $[t_0,T]$. The Arzela-Ascoli Theorem guarantees that there exists a subsequence $\{u(t;\eta_{j_k})\}$ and a $w\in C[t_0,T]$ with $u(t;\eta_{j_k})\to w$ in $C[t_0,T]$ as $k\to\infty$. Taking the limit in (4.3) as $k\to\infty$ we see that $w(t)$ satisfies the initial value problem (3.2) for $t\in[t_0,T]$. Now from (4.4) we have $m(t)\leqslant w(t)\leqslant u^{*}(t)$ on $[t_0,T]$. $\square$

If $g(t,x)\equiv 0$ in Lemma 2 we obtain the following result:

Corollary 1. *Assume the following conditions are satisfied:*

1. *The function $x^*(t) = x(t; t_0, x_0)$, $x^* \in C^q([t_0, T], \Delta)$, is a solution of the FrDE (3.1) where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$.*
2. *The function $V \in \Lambda([t_0, T], \Delta)$ and for any points $t \in [t_0, T]$ and $x \in \delta$ the inequality*
$$ {}_{+}^{c}D^q_{(3.1)} V(t, x) \leqslant 0 $$
holds.

Then for $t \in [t_0, T]$ the inequality $V(t, x^*(t)) \leqslant V(t_0, x_0)$ holds.

P r o o f. The proof follows from the fact that the Caputo fractional differential equation ${}^cD^q x = 0$ has a constant solution. Apply Lemma 2 with $u_0 = V(t_0, x_0)$. $\square$

R e m a r k 4. Corollary 1 is similar to [8], Corollary 2.2, but in the case we use (3.6).

E x a m p l e 6. Let $V\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t, x) = \sin^2 t x^2$ as in Example 5. Now use (3.6), (3.7), and (3.8) to obtain the Caputo fractional Dini derivative of V, namely

$$
\begin{aligned}
(4.13)\quad & {}_{+}^{c}D^q_{(3.1)} V(t, x) \\
&= (\sin t_0)^2 x_0^2 \frac{(t - t_0)^{-q}}{\Gamma(1 - q)} + x^2 \limsup_{h \to 0^+} \frac{1}{h^q} \sum_{r=0}^{[(t-t_0)/h]} (-1)^r qCr (\sin(t - rh))^2 \\
&\quad - 2x f(t, x) \limsup_{h \to 0^+} \sum_{r=1}^{[(t-t_0)/h]} (-1)^r qCr (\sin(t - rh))^2 \\
&= (\sin t_0)^2 x_0^2 \frac{(t - t_0)^{-q}}{\Gamma(1 - q)} + x^2 D_0^q (\sin(t))^2 + 2x f(t, x)(\sin(t))^2.
\end{aligned}
$$

Use $(\sin(t))^2 = 0.5 - 0.5\cos(2t)$ and $D_0^q \cos(2t) = 2^q \cos(2t + q\pi/2)$ and obtain

$$
\begin{aligned}
(4.14)\quad & {}_{+}^{c}D^q_{(3.1)} V(t, x) \\
&= (\sin t_0)^2 x_0^2 \frac{(t - t_0)^{-q}}{\Gamma(1 - q)} + x^2 \Big(0.5 \frac{(t - t_0)^{-q}}{\Gamma(1 - q)} + 2^{q-1} \cos\Big(2t + \frac{q\pi}{2}\Big)\Big) \\
&\quad + 2x f(t, x)(\sin(t))^2.
\end{aligned}
$$

Let $f(t, x) \equiv 0$. The solution of (3.1) for $n = 1$ and $t_0 = 0$ is $x(t) \equiv x_0$, $t \geqslant 0$ and $V(t, x(t)) = x_0^2 \sin^2 t$. Also, ${}_{+}^{c}D^q_{(3.1)} V(t, x) = x_0^2 (0.5\, t^{-q}/\Gamma(1 - q) + 2^{q-1} \cos(2t + q\pi/2))$ for $t \geqslant 0$. The sign of the fractional derivative of V changes (see Fig. 1 for values of $q = 0.2, 0.5, 0.8$). Therefore, the conditions of Corollary 1 are not satisfied (compare with Example 5).

Now let $V\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ be given by $V(t,x) = x^2$ as in Example 3. According to (3.9), ${}_{+}^{c}D_{(3.1)}^{q}V(t,x) = (x^2 - x_0^2)/(t^q\Gamma(1-q)) + 2xf(t,x)$. Let $f(t,x) = -x/(t^q\Gamma(1-q))$. Then ${}_{+}^{c}D_{(3.1)}^{q}V(t,x;t_0,x_0) \leqslant 0$ and according to Corollary 1 the inequality $|x(t)| \leqslant |x_0|$, $t \geqslant 0$, holds for any solution of (3.1).

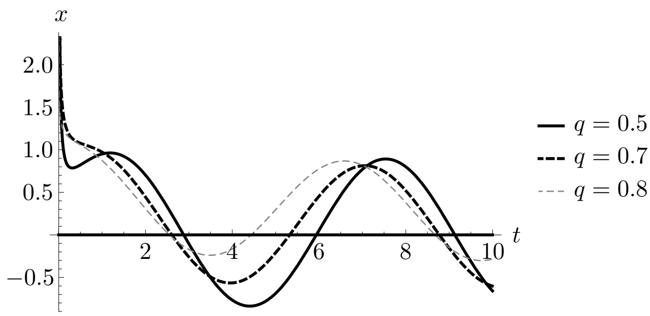

Figure 1. Example 6: $q = 0.2, 0.5$, and 0.8.

The result of Lemma 2 is also true on the half line. The idea is to fix $T > t_0$; then once again we have (4.3) and (4.4). Taking the limit in (4.3) as $k \to \infty$, we see that $\lim\limits_{k\to\infty} u(t;\eta_{j_k}))$ satisfies the initial value problem (3.2) for $t \in [t_0, T]$. We can use this argument for each $T < \infty$. This yields the following result.

Corollary 2. *Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t;t_0,x_0)$, $x^* \in C^q([t_0,\infty),\Delta)$, *is a solution of the FrDE* (3.1), *where* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$.
2. *The function* $g \in C([t_0,\infty) \times \mathbb{R}, \mathbb{R})$.
3. *The function* $V \in \Lambda([t_0,\infty),\Delta)$ *and for any points* $t \geqslant t_0$ *and* $x \in \Delta$ *the inequality*
$$ {}_{+}^{c}D_{(3.1)}^{q}V(t,x) \leqslant g(t,V(t,x)) $$
holds.
4. *The function* $u^*(t) = u(t;t_0,u_0)$, $u^* \in C^q([t_0,\infty),\mathbb{R})$ *is the maximal solution of the initial value problem* (3.2).

Then the inequality $V(t_0,x_0) \leqslant u_0$ *implies* $V(t,x^*(t)) \leqslant u^*(t)$ *for* $t \geqslant t_0$.

If the derivative of the Lyapunov function is negative, the following result is true.

Lemma 3. *Assume the following conditions are satisfied:*

1. *The function* $x^*(t) = x(t;t_0,x_0)$, $x^* \in C^q([t_0,T],\Delta)$ *is a solution of the FrDE* (3.1), *where* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$.

2. *The function* $V \in \Lambda([t_0, T], \Delta)$ *and for any points* $t \in [t_0, T]$, $x \in \Delta$ *the inequality*

$$ {}_{+}^{c}D^{q}_{(3.1)} V(t, x) \leqslant -c(\|x\|) $$

holds where $c \in \mathcal{K}$.

Then for $t \in [t_0, T]$ *the inequality*

$$ V(t, x^*(t)) \leqslant V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} c(\|x^*(s)\|)\,\mathrm{d}s \tag{4.15} $$

holds.

P r o o f. Define the function $m(t) \in C([t_0, T], \mathbb{R}_+)$ by $m(t) = V(t, x^*(t))$ and the function $p \in C([t_0, T], \mathbb{R}_+)$ by $p(t) = c(\|x^*(t)\|)$. As in the proof of (4.10) we have

$$ {}^{c}D^{q}_{+} m(t) \leqslant {}_{+}^{c}D^{q}_{(3.1)} V(t, x^*(t)) \leqslant -c(\|x^*(t)\|) = -p(t), \quad t \in [t_0, T]. \tag{4.16} $$

Let $\eta > 0$ be arbitrary. Consider the following initial value problem for the scalar FrDE:

$$ {}^{c}D^{q} u(t) = -p(t), \quad t \geqslant t_0,\ u(t_0) = m(t_0) + \eta. $$

Its solution satisfies the fractional integral equation

$$ u(t) = m(t_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} p(s)\,\mathrm{d}s + \eta. \tag{4.17} $$

We now prove that

$$ m(t) < u(t), \quad t \in [t_0, T]. \tag{4.18} $$

Assume the contrary and let $t^* \in (t_0, T]$ be such that

$$ m(t^*) = u(t^*) \quad \text{and} \quad m(t) < u(t) \quad \text{for } t \in [t_0, t^*). $$

From Lemma 1 (applied to $m(t) - u(t)$) we obtain

$$ {}^{c}D^{q}_{+} m(t^*) >^{c} D^{q}_{+} u(t^*) =^{c} D^{q} u(t^*) = p(t^*), \tag{4.19} $$

and this contradicts (4.16). Therefore, (4.18) is satisfied. From (4.17) and (4.18) since $\eta > 0$ is arbitrary we obtain (4.15). □

5. Main result

We will obtain sufficient conditions for stability of the system FrDE (3.1). Again we assume $0 < q < 1$.

Theorem 1. *Let the following conditions be satisfied:*

1. *The function $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ satisfies $g(t, 0) \equiv 0$.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ such that $V(t, 0) = 0$ and*
   (i) *for any points $t, t_0 \geqslant 0$ and $x, x_0 \in \mathbb{R}^n$ the inequality*

$$ {}_{+}^{c}D^{q}_{(3.1)}V(t,x) \leqslant g(t, V(t,x)) \tag{5.1}$$

   holds;
   (ii) *$b(\|x\|) \leqslant V(t,x)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, where $b \in \mathcal{K}$.*
3. *The zero solution of the scalar FrDE* (3.2) *is stable.*

Then the zero solution of the FrDE (3.1) *is stable.*

P r o o f. Let $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ be given. Then there exists $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ such that the inequality $|u_0| < \delta_1$ implies

$$ |u(t; t_0, u_0)| < b(\varepsilon), \quad t \geqslant t_0, \tag{5.2}$$

where $u(t; t_0, u_0)$ is any solution of the scalar FrDE (3.2). Since $V(t_0, 0) = 0$ there exists $\delta_2 = \delta_2(t_0, \delta_1) > 0$ such that $V(t_0, x) < \delta_1$ for $\|x\| < \delta_2$. Let $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta_2$. Then $V(t_0, x_0) < \delta_1$.

Consider any solution $x^*(t) = x(t; t_0, x_0)$, $t \geqslant t_0$, $x^* \in C^q([t_0, \infty), \mathbb{R}^n)$ of the FrDE (3.1). Now let $u_0^* = V(t_0, x_0)$. Then $u_0^* < \delta_1$ and inequality (5.2) holds for any solution $u(t; t_0, u_0^*)$ of the scalar FrDE (3.2). Then from Corollary 2 and (5.2) we have

$$ V(t, x^*(t)) \leqslant \overline{u}(t; t_0, u_0^*) < b(\varepsilon), \quad t \geqslant t_0; $$

here $\overline{u}(t; t_0, u_0^*)$ is the maximal solution of the initial value problem (3.2) (with the initial point (t_0, u_0^*)). Then for any $t \geqslant t_0$ condition 2(ii) yields

$$ b(\|x^*(t)\|) \leqslant V(t, x^*(t)) < b(\varepsilon), $$

so the result follows. □

Corollary 3. *Let condition* 2 *of Theorem* 1 *be satisfied where the inequality* (5.1) *is replaced by ${}_{+}^{c}D^{q}_{(3.1)}V(t,x) \leqslant 0$.*

Then the zero solution of the FrDE (3.1) *is stable.*

P r o o f. The proof follows from the fact that the Caputo fractional differential equation ${}^{c}D^{q}x = 0$ has a constant solution which is stable. □

Now we present some sufficient conditions for stability of the zero solution of the FrDE in the case when the condition for the Caputo fractional Dini derivative of the Lyapunov function is satisfied only on a ball.

Theorem 2. *Assume:*

1. *The function* $g \in C([\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ *satisfies* $g(t,0) \equiv 0$.
2. *There exists a function* $V \in \Lambda(\mathbb{R}_+, \overline{B}(\lambda))$ *such that*
   (i) *for any points* $t, t_0 \geqslant 0$ *and any* $x, x_0 \in \overline{B}(\lambda)$ *the inequality*

$$
{}_{+}^{c}D^{q}_{(3.1)} V(t,x) \leqslant g(t, V(t,x)) \tag{5.3}
$$

   holds, where $\lambda > 0$ *is a given number;*
   (ii) $b(\|x\|) \leqslant V(t,x) \leqslant a(\|x\|)$ *for* $t \in \mathbb{R}_+$, $x \in \overline{B}(\lambda)$, *where* $a, b \in \mathcal{K}$.
3. *The zero solution of the scalar FrDE* (3.2) *is uniformly stable.*

Then the zero solution of (3.1) *is uniformly stable.*

Proof. Let $\varepsilon \in (0, \lambda]$ be a positive number. Due to condition 3 of Theorem 2 there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that for any $\tau_0 \geqslant 0$ the inequality $|u_0| < \delta_1$ implies

$$
|u(t; \tau_0, u_0)| < b(\varepsilon), \quad t \geqslant \tau_0, \tag{5.4}
$$

where $u(t; \tau_0, u_0)$ is any solution of (3.2). Let $\delta_1 < \min\{\varepsilon, b(\varepsilon)\}$. Since $a \in \mathcal{K}$ there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that if $s < \delta_2$ then $a(s) < \delta_1$. Let $\delta = \min(\varepsilon, \delta_2)$. Choose the initial value $x_0 \in \mathbb{R}^n$ such that $\|x_0\| < \delta$. Let $x^*(t) = x(t; t_0, x_0)$, $t \geqslant t_0$ be any solution of the FrDE (3.1). We now prove that

$$
\|x^*(t)\| < \varepsilon, \quad t \geqslant t_0. \tag{5.5}
$$

The inequality (5.5) holds for $t = t_0$. Assume inequality (5.5) is not true for all $t > t_0$. Consequently, there exists a point $t^* > t_0$ such that

$$
\|x^*(t^*)\| = \varepsilon, \quad \text{and} \quad \|x^*(t)\| < \varepsilon, \quad t \in [t_0, t^*). \tag{5.6}
$$

Now let $u_0^* = V(t_0, x_0)$. Then from 2(ii) we get $u_0^* \leqslant a(\|x_0\|) < a(\delta_2) < \delta_1$. Consider any solution $u(t; t_0, u_0^*)$ of the scalar FrDE (3.2). Therefore, the inequality (5.4) holds for $\tau_0 = t_0$ and $u_0 = u_0^*$. From the choice of the point t^* it follows that $x^*(t) \in \overline{B}(\lambda)$ for $t \in [t_0, t^*]$. Then from the reasoning in Lemma 2 for the interval $[t_0, t^*]$ we have

$$
V(t, x^*(t)) \leqslant u^*(t; t_0, u_0^*), \quad t \in [t_0, t^*], \tag{5.7}
$$

where $u^*(t; t_0, u_0^*)$, $t \geqslant t_0$, is the maximal solution of the scalar FrDE (3.2) (for the initial point (t_0, u_0^*)). From inequalities (5.4), (5.7), the choice of t^*, and condition 2(ii) of Theorem 2 we obtain $b(\varepsilon) > u^*(t^*; t_0, u_0^*) \geqslant V(t^*, x^*(t^*)) \geqslant b(\|x^*(t^*)\|) = b(\varepsilon)$. The contradiction proves (5.5) and therefore, the zero solution of the FrDE (3.1) is uniformly stable. $\square$

Corollary 4. *Let condition* 2 *of Theorem* 2 *be satisfied, where the inequality* (5.3) *is replaced by* ${}_{+}^{c}D^{q}_{(3.1)}V(t,x) \leqslant 0$. *Then the zero solution of the FrDE* (3.1) *is uniformly stable.*

Now we present some sufficient conditions for the uniform asymptotic stability of the zero solution of the FrDE.

Theorem 3. *Assume:*

1. *There exists a function* $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that*
   (i) *for any points* $t, t_0 \geqslant 0$ *and* $x, x_0 \in \mathbb{R}^n$ *the inequality*

$$ {}_{+}^{c}D^{q}_{(3.1)}V(t,x) \leqslant -c(\|x\|)$$

   holds, where $c \in \mathcal{K}$;
   (ii) $b(\|x\|) \leqslant V(t,x) \leqslant a(\|x\|)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $a, b \in \mathcal{K}$.

Then the zero solution of the FrDE (3.1) *is uniformly asymptotically stable.*

P r o o f. According to Corollary 4, the zero solution of the FrDE (3.1) is uniformly stable. Therefore, for the number λ there exists $\alpha = \alpha(\lambda) \in (0, \lambda)$ such that for any $\tilde{t}_0 \in \mathbb{R}_+$ and $\tilde{x}_0 \in \mathbb{R}^n$ the inequality $\|\tilde{x}_0\| < \alpha$ implies

$$\|x(t; \tilde{t}_0, \tilde{x}_0)\| < \lambda \text{ for } t \geqslant \tilde{t}_0, \tag{5.8}$$

where $x(t; \tilde{t}_0, \tilde{x}_0)$ is any solution of the FrDE (3.1) (with initial data $(\tilde{t}_0, \tilde{x}_0)$).

Now we will prove that the zero solution of the fractional differential equation (3.1) is uniformly attractive. Consider a constant $\beta \in (0, \alpha]$ such that $a(\beta) \leqslant b(\alpha)$. Let $\varepsilon \in (0, \lambda]$ be an arbitrary number and $x^*(t) = x(t; t_0, x_0)$ any solution of (3.1) such that $\|x_0\| < \beta$, $t_0 \in \mathbb{R}_+$. Then $b(\|x_0\|) \leqslant a(\|x_0\|) < a(\beta) < b(\alpha)$, i.e. $\|x_0\| < \alpha$ and therefore, the inequality

$$\|x^*(t)\| < \lambda \quad \text{for } t \geqslant t_0 \tag{5.9}$$

holds. Choose a constant $\gamma = \gamma(\varepsilon) \in (0, \varepsilon]$ such that $a(\gamma) < b(\varepsilon)$. Let $T > q\Gamma(q)a(\alpha)/c(\gamma)$, $T = T(\varepsilon) > 0$. We now prove that

$$\|x^*(t)\| < \varepsilon \quad \text{for } t \geqslant t_0 + T. \tag{5.10}$$

Assume

$$\|x^*(t)\| \geqslant \gamma \quad \text{for every } t \in [t_0, t_0 + T]. \tag{5.11}$$

Then from Lemma 3 (applied to the interval $[t_0, t_0+T]$ and $\Delta = \mathbb{R}^n$) we get

$$
\begin{aligned}
(5.12)\qquad & V(t_0+T, x^*(t_0+T)) \\
&\leqslant V(t_0, x_0) - \frac{1}{\Gamma(q)}\int_{t_0}^{t_0+T} (t_0+T-s)^{q-1} c(\|x^*(s)\|)\,\mathrm{d}s \\
&\leqslant a(\|x_0\|) - \frac{c(\gamma)}{\Gamma(q)}\int_{t_0}^{t_0+T} (t_0+T-s)^{q-1}\,\mathrm{d}s = a(\|x_0\|) - \frac{c(\gamma)}{\Gamma(q)}\frac{T}{q} \\
&< a(\alpha) - \frac{c(\gamma)}{\Gamma(q)}\frac{T}{q} < 0.
\end{aligned}
$$

The contradiction proves that there exists $t^* \in [t_0, t_0+T]$ such that $\|x^*(t^*)\| < \gamma$. By Lemma 3 for $t \geqslant t^*$ and $\Delta = \mathbb{R}^n$ the inequality

$$
V(t, x^*(t)) \leqslant V(t^*, x^*(t^*)) - \frac{1}{\Gamma(q)}\int_{t^*}^{t} (t-s)^{q-1} c(\|x^*(s)\|)\,\mathrm{d}s \leqslant V(t^*, x^*(t^*))
$$

holds. Then for any $t \geqslant t^*$ the inequalities

$$
b(\|x^*(t)\|) \leqslant V(t, x^*(t)) \leqslant V(t^*, x^*(t^*)) \leqslant a(\|x^*(t^*)\|) \leqslant a(\gamma) < b(\varepsilon)
$$

hold. Therefore (5.10) holds for all $t \geqslant t^*$ (hence for $t \geqslant t_0 + T$). $\square$

6. Applications

Example 7. Consider the system of fractional differential equations

$$
\begin{aligned}
(6.1)\qquad {}^cD^q x_1(t) &= -g_1(t)x_1 - g_2(t)x_2, \\
{}^cD^q x_2(t) &= -g_1(t)x_2 + g_2(t)x_1 \quad \text{for } t \geqslant t_0
\end{aligned}
$$

with initial conditions

$$
x_1(t_0) = x_1^0 \quad \text{and} \quad x_2(t_0) = x_2^0,
$$

where $x_1, x_2 \in \mathbb{R}$, $g_1(t) = 0.5/(t^q\Gamma(1-q)) + 1$ and $g_2 \in C(\mathbb{R}_+, \mathbb{R})$ is an arbitrary function.

Now (6.1) is equivalent to (3.1) with $x = (x_1, x_2) \in \mathbb{R}^2$, $f = (f_1, f_2)$, where $f_1(t, x_1, x_2) = -g_1(t)x_1 - g_2(t)x_2$ and $f_2(t, x_1, x_2) = -g_1(t)x_2 + g_2(t)x_1$.

Consider $V(t, x_1, x_2) = x_1^2 + x_2^2$ for $t \in \mathbb{R}_+$, $x = (x_1, x_2) \in \mathbb{R}^2$ and $x \in \overline{B}(\lambda)$, where $\lambda > 0$. Then condition 2 (ii) of Theorem 2 is satisfied for $a, b \in \mathcal{K}$ with $b(s) = \frac{1}{2}s$, $a(s) = 2s$.

For any $x_1, x_2 \in \mathbb{R}$: $x = (x_1, x_2) \in \overline{B}(\lambda)$ from (3.6) and (3.8) we obtain

$$
\begin{aligned}
(6.2)\quad & {}_{+}^{c}D^q_{(6.1)}V(t, x_1, x_2) \\
= & -((x_1^0)^2 + (x_2^0)^2)\frac{1}{t^q\Gamma(1-q)} + \limsup_{h\to 0^+}\frac{1}{h^q}\Big\{(x_1^2 + x_2^2)\sum_{r=0}^{[(t-t_0)/h]}(-1)^r qCr \\
& - 2h^q(x_1(-g_1(t)x_1 - g_2(t)x_2) + x_2(-g_1(t)x_2 + g_2(t)x_1))\sum_{r=1}^{[(t-t_0)/h]}(-1)^r qCr \\
& + h^{2q}((-g_1(t)x_1 - g_2(t)x_2)^2 + (-g_1(t)x_2 + g_2(t)x_1)^2)\sum_{r=1}^{[(t-t_0)/h]}(-1)^r qCr\Big\}.
\end{aligned}
$$

Applying (3.7) and (3.8) to (6.2) yields

$$
\begin{aligned}
{}_{+}^{c}D^q_{(6.1)}V(t, x_1, x_2) = & \frac{x_1^2 + x_2^2 - (x_1^0)^2 - (x_2^0)^2}{t^q\Gamma(1-q)} \\
& - 2(x_1(-g_1(t)x_1 - g_2(t)x_2) + x_2(-g_1(t)x_2 + g_2(t)x_1)) \\
\leqslant & (x_1^2 + x_2^2)\Big(\frac{1}{t^q\Gamma(1-q)} - 2g_1(t)\Big) = -2(x_1^2 + x_2^2) = -2V(t, x_1, x_2).
\end{aligned}
$$

The comparison FrDE is (3.3) and by Example 2 the zero solution of (3.3) is uniformly stable. According to Theorem 2, the zero solution of (6.1) is uniformly stable. Also, since condition 1(i) of Theorem 3 is satisfied, the zero solution of (6.1) is uniformly asymptotically stable.

If we use (3.11) then

$$
\begin{aligned}
(6.3)\quad D^qV(t, x_1, x_2) = & \limsup_{h\to 0}\frac{1}{h^q}\{x_1^2 + x_2^2 - (x_1 - h^q(-g_1(t)x_1 - g_2(t)x_2))^2 \\
& + (x_2 - h^q(-g_1(t)x_2 + g_2(t)x_1))^2\} \\
= & \limsup_{h\to 0}\frac{1}{h^q}\{-2h^q[x_1(-g_1(t)x_1 - g_2(t)x_2) + x_2(-g_1(t)x_2 + g_2(t)x_1)] \\
& + h^{2q}[(-g_1(t)x_1 - g_2(t)x_2)^2 + (-g_1(t)x_2 + g_2(t)x_1)^2]\} \\
= & x_1(-g_1(t)x_1 - g_2(t)x_2) + x_2(-g_1(t)x_2 + g_2(t)x_1) = -(x_1^2 + x_2^2)g_1(t) \\
= & -\Big(\frac{0.5}{t^q\Gamma(1-q)} + 1\Big)V(t, x_1, x_2),
\end{aligned}
$$

and we obtain the comparison Caputo fractional differential equation

$$
{}^{c}D^q u(t) = -\Big(\frac{0.5}{t^q\Gamma(1-q)} + 1\Big)u
$$

which is more difficult to solve than the scalar FrDE (3.3).

R e m a r k 5. Using an inequality for any continuous function (for the proof see [1]) and Theorem 1 from the paper [6], one can prove the stability of the zero solution of (6.1) in a different way which is easier for this particular example.

Now we illustrate the usefulness of (3.6) in the case when the quadratic Lyapunov function gives us no result.

E x a m p l e 8. Consider the initial value problem for the scalar fractional differential equation

(6.4)
$$ {}^{c}D^{q}x(t) = \frac{-0.25q - 0.75/t^{q}\Gamma(1-q) - 2^{q-2}\cos(2t + q\pi/2))}{1+\cos^2(t)}x, \quad t \geqslant t_0,\ x(t_0) = x_0, $$

where $0 < q < 1$. Let

$$ f(t,x) = \frac{-0.25q - 0.75/t^{q}\Gamma(1-q) - 2^{q-2}\cos(2t + q\pi/2)}{1+\cos^2(t)}x. $$

Consider the quadratic Lyapunov function $V(t,x) = x^2$. Since the sign of the function $-0.25q - 0.75/(t^{q}\Gamma(1-q)) - 2^{q-2}\cos(2t+q\pi/2)$ changes, so does the fractional derivative of V (see Fig. 2 for some values of q). Therefore, the quadratic function is not applicable to the fractional equation (6.4).

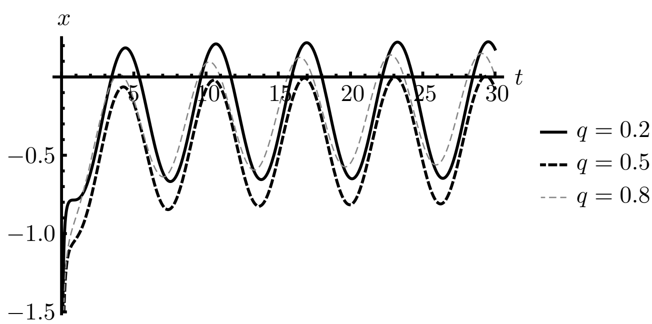

Figure 2. Example 8: $q = 0.5, 0.7$, and 0.8.

Now, consider the Lyapunov function $V(t,x) = (1+\cos^2(t))x^2$. The inequalities $\|x\| \leqslant V(t,x) \leqslant 2\|x\|$ are satisfied for any $x \in \mathbb{R}$. Let $x, x_0 \in \mathbb{R}$ with $|x| \leqslant \lambda$, $|x_0| \leqslant \lambda$, where $\lambda > 0$ is a given number. Applying formula (3.6) to this Lyapunov function, we obtain

(6.5)
$$ {}^{c}_{+}D^{q}_{(6.4)}V(t,x) = \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{(1+\cos^2(t))x^2 - (1+\cos^2(t_0))x_0^2 $$
$$ - \sum_{r=1}^{[(t-t_0)/h]} (-1)^{r+1} qCr[(1+\cos^2(t-rh))(x - h^q f(t,x))^2 - (1+\cos^2(t_0))x_0^2]\Big\}. $$

Using (3.7), (3.8), $\cos^2(t) = 0.5 + 0.5\cos(2t)$, and $D_0^q(\cos(2t)) = 2^q\cos(2t + q\pi/2)$, we obtain

$$\begin{aligned}
{}_{+}^{c}D^q_{(6.4)}V(t,x) &= -(1+\cos^2(t_0))x_0^2\frac{(t-t_0)^{-q}}{\Gamma(1-q)}\\
&\quad + x^2\limsup_{h\to 0^+}\frac{1}{h^q}\Big\{\sum_{r=0}^{[(t-t_0)/h]}(-1)^r qCr(1+\cos^2(t-rh))\\
&\quad - 2h^q f(t,x)\sum_{r=1}^{[(t-t_0)/h]}(-1)^r qCr(1+\cos^2(t-rh))\\
&\quad + h^{2q}(f(t,x))^2\sum_{r=1}^{[(t-t_0)/h]}(-1)^r qCr(1+\cos^2(t-rh))\Big\}\\
&\leqslant x^2\Big(\frac{1}{t^q\Gamma(1-q)} + D_0^q\cos^2(t)\Big)\\
&\quad - 2xf(t,x)\limsup_{h\to 0^+}\sum_{r=1}^{[(t-t_0)/h]}(-1)^r qCr(1+\cos^2(t-rh))\\
&\quad + f(t,x)\limsup_{h\to 0^+}h^q\sum_{r=1}^{[(t-t_0)/h]}(-1)^r qCr(1+\cos^2(t-rh))\\
&= x^2\Big(\frac{1.5}{t^q\Gamma(1-q)} + 2^{q-1}\cos\Big(2t+\frac{q\pi}{2}\Big)\Big)\\
&\quad + 2x^2\frac{-0.25q - 0.75/t^q\Gamma(1-q) - 2^{q-2}\cos(2t+q\pi/2)}{1+\cos^2(t)}(1+\cos^2(t)) \leqslant 0.
\end{aligned}$$

According to Corollary 4, the zero solution is uniformly stable.

7. Conclusions

In this paper we use Lyapunov functions to study the stability of the zero solution of a nonlinear nonautonomous fractional differential equation. We introduce the derivative of the Lyapunov function based on the Caputo fractional Dini derivative of a function. Comparison results using this new definition and scalar fractional differential equations are presented and sufficient conditions for stability, uniform stability and asymptotic uniform stability are obtained.

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