Josef Mikeš; Karl Strambach
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SHELLS OF MONOTONE CURVES

Josef Mikeš, Olomouc, Karl Strambach, Erlangen

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Abstract. We determine in $\mathbb{R}^n$ the form of curves C corresponding to strictly monotone functions as well as the components of affine connections ∇ for which any image of C under a compact-free group Ω of affinities containing the translation group is a geodesic with respect to ∇. Special attention is paid to the case that Ω contains many dilatations or that C is a curve in $\mathbb{R}^3$. If C is a curve in $\mathbb{R}^3$ and Ω is the translation group then we calculate not only the components of the curvature and the Weyl tensor but we also decide when ∇ yields a flat or metrizable space and compute the corresponding metric tensor.

Keywords: geodesic; shell of a curve; affine connection; (pseudo-)Riemannian metric; projective equivalence

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1. Introduction

Already E. Beltrami showed that a differentiable curve is a local geodesic with respect to an affine connection ∇ precisely if it is a solution of an Abelian differential equation having as coefficients expressions in Christoffel symbols associated with ∇. But the investigation under which conditions systems S of differentiable curves consist of geodesics with respect to an affine connection ∇ started only in 2002 (cf. [6], [5]) with systems S of lines of 2-dimensional topological geometries (cf. [18]). An analogous treatment of systems of lines of shift spaces (cf. [2], [1]) turned out to be difficult, because the lines of a natural generalization of a Grünwald plane (cf. [14]) for $n \geqslant 3$ can never consist of geodesics with respect to an affine connection (cf. [13]). To obtain geodesics one has to reduce the set of curves which should be geodesics.

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Let C be a curve in $\mathbb{R}^n$ which is described by a sequence of differentiable strictly monotone and surjective functions. The aim of this paper is to study conditions for the following situation: all images of C under a given group Ω of affine transformations containing all translations of $\mathbb{R}^n$ are geodesics with respect to an affine connection ∇. If Ω coincides with the group T of affine translations, then we obtain concrete information on the components of the affine connection ∇ and about the form of the functions describing the curves C. If the group Ω is a semidirect product of the translation group T of $\mathbb{R}^n$ with a group consisting of many dilatations, then we determine explicitly the components of ∇ and the form of the curves C the images of which under Ω consist of geodesics with respect to ∇.

Special attention is paid to curves C in the 3-dimensional affine space and to the translation group $\Omega = T$ in $\mathbb{R}^3$. In this case we determine the form of the curves C, and we calculate the components of ∇ as well as the components of the curvature and the Weyl tensor. Moreover, we are able to decide when ∇ yields a flat or metrizable space and to calculate the corresponding metric tensor.

In the proofs the solutions of Riccati and Abelian differential equations play an important role. For this reason we explicitly describe the solutions of Abelian differential equations which are derivatives of strictly monotone and surjective functions on $\mathbb{R}$.

2. Strictly monotone curves

Let C be a curve homeomorphic to $\mathbb{R}$ which is a closed subset of $\mathbb{R}^n$, $n \geqslant 2$. We consider a curve of the form

$$C = (t, f_2(t), f_3(t), \ldots, f_n(t)), \quad t \in \mathbb{R},$$

such that the functions $f_i(t)\colon \mathbb{R} \to \mathbb{R}$, $i = 2, 3, \ldots, n$, are differentiable, surjective and strictly monotone. We will call such curves strictly monotone.

If T is the translation group of $\mathbb{R}^n$, then we call the set

$$C(T) = \{(t + u_1, f_2(t) + u_2, f_3(t) + u_3, \ldots, f_n(t) + u_n),\ t \in \mathbb{R}\},$$
$$\text{where } u_1, \ldots, u_n \in \mathbb{R},$$

the *shell of* C and the set

$$\widehat{C}(T) = \{(t + u_1, f_2(t) + u_2, f_3(t + v_3) + u_3, \ldots, f_n(t + v_n) + u_n),\ t \in \mathbb{R}\},$$
$$\text{where } u_1, \ldots, u_n, v_3, \ldots, v_n \in \mathbb{R},$$

the *extended shell of* C.

If Σ is a Lie group of affine transformations of $\mathbb{R}^n$ which contains the translation group, then the sets $C(T)^\Sigma$ and $\widehat{C}(T)^\Sigma$ of all images of $C(T)$ and $\widehat{C}(T)$, respectively, under the group Σ are called the Σ-*shell* and *extended* Σ-*shell of* C.

Let Δ be the group consisting of the mappings

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, s_3x_3, \ldots, s_nx_n), \quad s_i \in \mathbb{R} \setminus \{0\}, \tag{2.1}$$

and $\widehat{\Delta}$ the group of the mappings

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, s_2x_2, s_3x_3, \ldots, s_nx_n), \quad s_i \in \mathbb{R} \setminus \{0\}. \tag{2.2}$$

If Θ and $\widehat{\Theta}$ are, respectively, the semidirect products $T\Delta$ and $T\widehat{\Delta}$, where T is the translation group of $\mathbb{R}^n$, then the extended Θ-shell and the extended $\widehat{\Theta}$-shell of C are the sets

$$\begin{aligned} \widehat{C}(T)^\Theta = \{&(t+u_1, f_2(t)+u_2, s_3f_3(t+v_3)+u_3, \ldots, s_nf_n(t+v_n)+u_n), \\ &t \in \mathbb{R}\}, \text{where } u_1, \ldots, u_n, v_3, \ldots, v_n \in \mathbb{R},\ s_3, \ldots, s_n \in \mathbb{R} \setminus \{0\}, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \widehat{C}(T)^{\widehat{\Theta}} = \{&(t+u_1, s_2f_2(t)+u_2, s_3f_3(t+v_3)+u_3, \ldots, s_nf_n(t+v_n)+u_n),\ t \in \mathbb{R}\}, \\ &\text{where } u_1, \ldots, u_n, v_3, \ldots, v_n \in \mathbb{R},\ s_2, \ldots, s_n \in \mathbb{R} \setminus \{0\}. \end{aligned} \tag{2.4}$$

3. Riccati differential equations

For later use we consider the special Riccati differential equations with unknown function $y = y(t)$:

$$y' + a_1y^2 + a_2y + a_3 = 0, \tag{3.1}$$

where a_i are constants (cf. [10], A4.9, or [11], pages 33 and 41).

(3.1.1) If $a_i = 0$ for $i \in \{1, 2\}$, then $y = -a_3t + c$ for $c \in \mathbb{R}$.

(3.1.2) If $a_1 = 0$ and $a_2 \neq 0$, then $y = -a_3/a_2 + c\mathrm{e}^{-a_2t}$ for $c \in \mathbb{R}$.

(3.1.3) If $a_1 \neq 0$ and $a_2^2 = 4a_1a_3$, then we have

$$y = -\frac{a_2}{2a_1} + \frac{c_1}{c_1a_1t + c_2} \quad \text{with } c_1, c_2 \in \mathbb{R} \text{ and } (c_1, c_2) \neq (0, 0).$$

(3.1.4) If $a_1 \neq 0$ and $\lambda^2 = 4a_1a_3 - a_2^2 > 0$, then

$$y = -\frac{a_2}{2a_1} + \frac{\lambda}{2a_1} \operatorname{cotan} \frac{\lambda}{2}(t + c) \quad \text{with } c \in \mathbb{R}.$$

(3.1.5) If $a_1 \neq 0$ and $\lambda^2 = a_2^2 - 4a_1a_3 > 0$, then

$$y = -\frac{a_2}{2a_1} + \frac{\lambda}{2a_1} \frac{c_1 \mathrm{e}^{\lambda t/2} - c_2 \mathrm{e}^{-\lambda t/2}}{c_1 \mathrm{e}^{\lambda t/2} + c_2 \mathrm{e}^{-\lambda t/2}} \quad \text{with } c_1, c_2 \in \mathbb{R} \text{ and } (c_1, c_2) \neq (0,0).$$

Also in (3.1.3) and (3.1.5) the solution depends only on one parameter which is defined on the projective line.

Lemma 3.1. *If $f = f(t)$ is a differentiable function such that its derivative f' is a nonconstant solution of equation* (3.1), *then f is strictly monotone and surjective if and only if $f = -a_2^{-1}(a_3 t + c\mathrm{e}^{-a_2 t}) + d$ with $d \in \mathbb{R}$; it is a solution of $f'' + a_2 f' + a_3 = 0$ with $a_2 \in \mathbb{R} \setminus \{0\}$ and $ca_3 > 0$.*

P r o o f. If f is monotone and surjective, then the derivative f' cannot be a solution (3.1.3) or (3.1.4). If f' is a solution (3.1.5), then f has the form

$$f = \alpha t + \beta \ln(c_1 \mathrm{e}^{\lambda t/2} + c_2 \mathrm{e}^{-\lambda t/2}) + \delta$$

with fixed $\alpha, \beta, \delta \in \mathbb{R}$, $\beta \neq 0$. Since $\ln(c_1 \mathrm{e}^{\lambda t/2} + c_2 \mathrm{e}^{-\lambda t/2})$ must be defined for all $t \in \mathbb{R}$ it follows that $c_1, c_2 \geqslant 0$, $(c_1, c_2) \neq (0,0)$. But then there exists a value t_0 such that $f'(t_0) = 0$.

If f' is a solution (3.1.1), then $f' = -a_3 t + c$ with a constant c. It follows that $f = -\widetilde{a}_3 t^2 + ct + d$ with a constant d. Since f must be surjective we would have $\widetilde{a}_3 = 0$ and $f = ct + d$.

If f' is a solution (3.1.2), then $f' = -a_3/a_2 + c\mathrm{e}^{-a_2 t}$ with $c \in \mathbb{R}$, and hence $f = -a_2^{-1}(a_3 t + c\mathrm{e}^{-a_2 t}) + d$, where $d \in \mathbb{R}$. If $c = 0$, then f' would be constant. For $c \neq 0$ the function f is strictly monotone if and only if $ca_3 \geqslant 0$, and surjective if and only if $a_3 \neq 0$. $\square$

4. Abelian differential equations

We consider Abelian differential equations

$$y' = \alpha + \beta y + \gamma y^2 + \varepsilon y^3 \quad \text{with } \varepsilon \neq 0, \tag{4.1}$$

where $\alpha, \beta, \gamma, \varepsilon \in \mathbb{R}$, and we are interested in real functions f with $f' = y$ which satisfy the conditions of Lemma 3.1.

With differential equation (4.1) we associate the cubic algebraic equation

$$\alpha + \beta y + \gamma y^2 + \varepsilon y^3 = 0. \tag{4.2}$$

Because $\varepsilon \neq 0$, the cubic equation (4.2) has a real solution $y = y_1$ and hence equation (4.1) has a solution $y(t) = y_1$ for all $t \in \mathbb{R}$. Since through any point of $\mathbb{R}^2$ there is precisely one solution of (4.1) we have

Lemma 4.1. *Any solution $y(t) = y_1 \neq 0$ of equation (4.1) corresponds to a linear function $f = y_1 t + c$ and f is strictly monotone and surjective.*

Theorem 4.1. *Let f be a real nonlinear function such that the derivative f' is a solution of the differential equation (4.1). Then f is strictly monotone and surjective if and only if the cubic equation (4.2) has three real roots y_i, $i = 1, 2, 3$, such that one of the following conditions holds:*
- (i) *$y_2 \neq y_1 = y_3 = 0$ and $f' = y$ is contained in the open interval determined by 0 and y_2;*
- (ii) *$y_1 y_2 > 0$ and the root y_3 of (4.2) is not contained in the open interval determined by y_1 and y_2, whereas $f' = y$ is contained in this interval.*

P r o o f. Let f' be a solution of the differential equation (4.1). Since f is monotone and surjective one has either $f' = y \geqslant 0$, $\varepsilon > 0$, or $f' = y \leqslant 0$, $\varepsilon < 0$.

We show that for f' there exists a bound. If this is not the case, then there is a $t_0 \in \mathbb{R}$ for all $t > t_0$ if $\varepsilon > 0$, and for all $t < t_0$ if $\varepsilon < 0$, such that the right hand side of equation (4.1) is for $\varepsilon > 0$ greater and for $\varepsilon < 0$ smaller than $\varepsilon_* y^3$, where sign $\varepsilon_* =$ sign ε.

Let $y_*(t)$ be a function which is the solution of the differential equation

$$y_*' = \varepsilon_* y_*^3 \tag{4.3}$$

with the initial condition $y_*(t_0) = y(t_0)$. Then one has $|y(t)| > |y_*(t)|$ for all $t > t_0$, or for all $t < t_0$. Integrating (4.3) we obtain

$$\frac{1}{y_*(t)^2} = \mathrm{const} - 2\varepsilon_* t.$$

If $\varepsilon > 0$, then for sufficiently large t, while if $\varepsilon < 0$, then for sufficiently small t, the value $\mathrm{const} - 2\varepsilon^* t$ is negative. This means that there exists a value t^* for which $y_*(t^*)$ is infinite. This yields that $y(t)$ has a vertical asymptote, which is a contradiction. From the fact that $y(t)$ is bounded it follows that equation (4.2) has only real roots.

Lemma 3.1 yields that we have to consider only strictly monotone solutions of (4.1) which are contained between two different roots of (4.2).

Let y_1, y_2, y_3 be different roots of (4.2) such that $y_1 < y_2$, $y_1 y_2 \neq 0$ and y_3 is not contained in the interval bounded by y_1 and y_2. Then between y_1 and y_2 there exists a strictly monotone solution y with asymptotes y_1 and y_2. Hence $y_1 y_2 > 0$ and for the solution $y(t)$ of the differential equation (4.1) one has

$$y_1 < y(t) < y_2 \quad \text{for all } t. \tag{4.4}$$

It follows from (4.4) by integration that for the solution of (4.1) through the point $(t_0, f(t_0))$ the inequalities

$$\begin{aligned} f(t_0) + y_1(t - t_0) < f(t) < f(t_0) + y_2(t - t_0) \quad &\text{for all } t > t_0 \text{ and} \\ f(t_0) + y_1(t - t_0) > f(t) > f(t_0) + y_2(t - t_0) \quad &\text{for all } t < t_0 \end{aligned}$$

hold. Hence the function $f(t)$ is surjective on $\mathbb{R}$ and we have part (2) of the claim.

Let $y_1 = 0$ be a simple root of equation (4.2) and let y_2 and y_3 be the other roots such that y_3 is not contained in the open interval bounded by 0 and y_2. Then equation (4.1) has the form

$$y' = y(y - y_2)\varepsilon(y - y_3). \tag{4.5}$$

The general solution of equation (4.5) has in the case $y_2 \neq y_3$ the form

$$|y|\,|y - y_2|^{\varrho_2}|y - y_3|^{\varrho_3} = \varrho_0 \mathrm{e}^{\varrho_1 t}, \quad \varrho_i = \text{const}, \tag{4.6}$$

whereas for $y_2 = y_3$ we obtain

$$|y|\,|y - y_2|^{\varrho_2} \exp\left(\frac{\varrho_3}{y - y_2}\right) = \varrho_0 \mathrm{e}^{\varrho_1 t}, \quad \varrho_i = \text{const}. \tag{4.7}$$

We are interested in a solution $y(t)$ for which $0 < |y(t)| < |y_2|$ holds. This solution $y(t)$ has zero as an asymptote. From (4.6) and (4.7) it follows that for $y(t) \to 0$ the solution $y(t)$ may be approximated by a function $\varrho_* \mathrm{e}^{\sigma t}$. Integration yields that $f(t)$ is approximated by $\varrho_{**} \mathrm{e}^{\sigma t} + \text{const}$. But then $f(t)$ would be bounded, which is a contradiction.

Finally we consider the case that 0 is a double root of the associated cubic equation (4.2). Then equation (4.1) has the form

$$y' = y^2 \varepsilon(y - y_2), \quad \text{with } y_2 \neq 0.$$

The function y is strictly increasing, or strictly decreasing, if and only if $\varepsilon(y - y_2) > 0$, or < 0, respectively. Then one has $y(t_0) < y(t) < y_2$ for $t > t_0$, or $y(t_0) > y(t) > y_2$ for $t < t_0$, respectively. Integrating these inequalities we obtain $f(t_0) + y(t_0)t <$

$f(t) < f(t_0) + y_2 t$ for $t > t_0$, and $f(t_0) + y(t_0)t < f(t) < f(t_0) + y_2 t$ for $t < t_0$, where $y = f'$. It follows that f is defined for all $t > t_0$ and for all $t < t_0$.

For all $t < t_0$ and for all $t > t_0$, there exist real numbers q_1 and q_2 such that

$$q_1 y^2 < y' = y^2 \varepsilon (y - y_2) < q_2 y^2$$

holds. Integrating this inequality we obtain for $t < t_0$ as well as for $t > t_0$, that

$$\frac{1}{c_1 - q_1 t} > y(t) > \frac{1}{c_2 - q_2 t},$$

and further integration yields

$$d_1 - \frac{1}{q_1} \ln |c_1 - q_1 t| > f(t) > d_2 - \frac{1}{q_2} \ln |c_2 - q_2 t|$$

with constants $d_i \in \mathbb{R}$. Since the subfunction $d_1 - q_1^{-1} \ln |c_1 - q_1 t|$ and the upper function $d_2 - q_2^{-1} \ln |c_2 - q_2 t|$ exist for all t in the intervals $(-\infty, t_0)$ and (t_0, ∞), respectively, the monotone and surjective function $f(t)$ exists for all t and is surjective on $\mathbb{R}$. This proves Theorem 4.1. $\square$

Corollary 4.1. *Let $f' = y$ and $y' = \alpha + \beta y + \gamma y^2 + \varepsilon y^3$, where $\alpha, \beta, \gamma, \varepsilon \in \mathbb{R}$ and $\varepsilon \neq 0$. Then this equation has solutions such that the function f on $\mathbb{R}$ is nonlinear strictly monotone and surjective if and only if one of the following conditions holds:*
(1) *$\alpha = \beta = 0$ and $\gamma \neq 0$,*
(2) *$\alpha = 0$, $\gamma^2 - 4\beta\varepsilon > 0$ and $\beta\varepsilon > 0$,*
(3) *$\alpha \neq 0$ and $D > 0$,*
(4) *$\alpha \neq 0$, $D = 0$, $(2\gamma^3 - 9\beta\gamma\varepsilon + 27\alpha\varepsilon^2)^2 = 4(\gamma^2 - 3\beta\varepsilon)^3 \neq 0$ and $(9\alpha\varepsilon^2 - 4\beta\gamma\varepsilon + \gamma^3) \times (\beta\gamma - 9\alpha\varepsilon)\varepsilon > 0$,*
where $D = 18\alpha\beta\gamma\varepsilon + \beta^2\gamma^2 - 4\beta^3\varepsilon - 4\alpha\gamma^3 - 27\alpha^2\varepsilon^2$.

Proof. First we assume that 0 is a root of the cubic equation (4.2) corresponding to equation (4.1). Clearly, zero cannot be a threefold root of (4.2). Then equation (4.1) has a solution which is a derivative of a strictly monotone and surjective function if and only if either 0 is a root of multiplicity 2 or equation (4.2) has besides 0 two different roots y_1, y_2 with $y_i \neq 0$, $i = 1, 2$ and $y_1 y_2 > 0$ (Theorem 4.1). Zero is a root of multiplicity 2 if and only if $\alpha = \beta = 0$ but $\gamma \neq 0$, which is the claim 1 of the assertion.

If 0 has multiplicity 1, then $\alpha = 0$ but $\beta \neq 0$ and $y' = \varepsilon y(y - y_1)(y - y_2)$. The fact that the roots y_1 and y_2 are different is equivalent to $D = \beta^2(\gamma^2 - 4\beta\varepsilon) > 0$, where D is the discriminant of (4.2). We have to decide under which circumstances $y_1 y_2 > 0$.

Since the coefficient β of the monome y coincides with $\varepsilon y_1 y_2$ we have $y_1 y_2 = \beta/\varepsilon$, and $y_1 y_2 > 0$ is equivalent to $\beta\varepsilon > 0$. This covers part 2 of the assertion.

Now we treat the case that the roots of equation (4.2) are all real, but none of them is equal to 0. Hence $\alpha \neq 0$. If y_1, y_2, y_3 are distinct, then (4.1) has a solution which is the derivative of a strictly monotone and surjective function on $\mathbb{R}$ if and only if $D > 0$, and this yields part (3) of the assertion.

Let y_1 and y_2 be different real roots of equation (4.2) with $y_i \neq 0$, $i = 1, 2$, such that y_2 has multiplicity 2. It follows from [8], pages 154–156, that $D = 0$, $(2\gamma^3 - 9\beta\gamma\varepsilon + 27\alpha\varepsilon^2)^2 = 4(\gamma^2 - 3\beta\varepsilon)^3 \neq 0$,

$$y_1 = \frac{9\alpha\varepsilon^2 - 4\beta\gamma\varepsilon + \gamma^3}{\varepsilon(3\beta\varepsilon - \gamma^2)} \quad \text{and} \quad y_2 = \frac{\beta\gamma - 9\alpha\varepsilon}{2(3\beta\varepsilon - \gamma^2)}.$$

According to Theorem 4.1 the differential equation (4.1) has a solution which is the derivative of a strictly monotone and surjective function if and only if $y_1 y_2 > 0$, i.e., $(9\alpha\varepsilon^2 - 4\beta\gamma\varepsilon + \gamma^3)(\beta\gamma - 9\alpha\varepsilon)\varepsilon > 0$. This completes the proof. $\square$

5. Affine connections

Since we apply results of differential geometry only to the n-dimensional space $\mathbb{R}^n$ there exist global coordinates and the components Γ^h_{ij}, $h, i, j \in \{1, 2, \ldots, n\}$, of any affine connection ∇ can be written in a unique way in these coordinates.

An affine connection ∇ is called symmetric if $\nabla_X Y = \nabla_Y X - [X, Y]$, where $[X, Y]$ is the Lie bracket, i.e., if for its components Γ^h_{ij} one has $\Gamma^h_{ij} = \Gamma^h_{ji}$ for all $h, i, j \in \{1, 2, \ldots, n\}$.

By a *geodesic* of ∇ we mean a piecewise C^2-curve $\gamma\colon I \to \mathbb{R}^n$ satisfying $\nabla_{\dot\gamma}\dot\gamma = \varrho\dot\gamma$, where $\varrho\colon I \to \mathbb{R}$ is a continuous function and $I \subset \mathbb{R}$ is an open interval (cf. [4], page 3, [16], page 122).

Using the components of ∇ the differential equation for geodesics has the form (cf. [16], page 144)

$$\ddot\gamma^h + \sum_{i,j=1}^{n} \Gamma^h_{ij}\dot\gamma^i\dot\gamma^j = \varrho(t)\dot\gamma^h, \quad h \in \{1, 2, \ldots, n\}. \tag{5.1}$$

This implies that the geodesics depend only on the symmetric part of the connection ∇. Hence we will always assume that ∇ is symmetric.

Let $\mathfrak{g}$ be a Lie algebra of a group G of diffeomorphisms and let $\nabla = \{\Gamma^h_{ij}\}$ be an affine connection. The Lie derivation $\mathfrak{L}_\xi\nabla$ along a nonzero element $\xi \in \mathfrak{g}$ is given in terms of the components of ∇ by

$$\mathfrak{L}_\xi \Gamma^h_{ij} \equiv \frac{\partial^2 \xi^h}{\partial x_i \partial x_j} + \sum_{\alpha=1}^{n} \Big(\xi^\alpha \frac{\partial \Gamma^h_{ij}}{\partial x_\alpha} - \frac{\partial \xi^h}{\partial x_\alpha} \Gamma^\alpha_{ij} + \frac{\partial \xi^\alpha}{\partial x_i} \Gamma^h_{\alpha j} + \frac{\partial \xi^\alpha}{\partial x_j} \Gamma^h_{\alpha i} \Big),$$

where $h, i, \ldots = 1, 2, \ldots, n$.

The group G preserves geodesics with respect of ∇ if and only if

$$\mathfrak{L}_\xi \Gamma^h_{ij} = \delta^h_i \psi_j + \delta^h_j \psi_i,$$

where δ^h_i is the Kronecker symbol and the ψ_i are differentiable functions [12], [15], page 143, [20].

The group G consists of affine mappings with respect to ∇ precisely if $\mathfrak{L}_\xi \Gamma^h_{ij} = 0$ or, equivalently, if and only if ψ_i vanishes. Moreover, if $\mathbb{R}^n$ is a (pseudo-)Riemannian space with respect to the metric tensor g, then the Lie group G is a group of isometries precisely if $\mathfrak{L}_\xi g = 0$ (cf. [20], page 43, [15], page 100).

A diffeomorphism φ is called a *geodesic mapping* if φ maps any geodesic onto a geodesic. In [14] we proved the following proposition:

Proposition 5.1. *Let S be a system of geodesics with respect to an affine connection ∇. If the translation group T of $\mathbb{R}^n$ consists of geodesic maps for S, then the affine connection ∇ may be chosen in such a way that the components Γ^h_{ij} are constant. Moreover, the components $\Gamma^\sigma_{\sigma\sigma}$, $\sigma = 1, \ldots, n$, are zero.*

We shall call the connections satisfying the conditions of Proposition 5.1 *natural connections of S*. With respect to a natural connection ∇° the translation group of $\mathbb{R}^n$ consists of affine transformations of S. Namely, for $\xi = (\delta^h_\sigma)^n_{h=1}$ one has $\mathfrak{L}_\xi \Gamma^h_{ij} \equiv (\partial/\partial x_\sigma) \Gamma^h_{ij} = 0$.

Proposition 5.2. *Let Δ, or $\widehat{\Delta}$ be, respectively, the group of mappings* (2.1), *or* (2.2). *If S is a system of geodesics such that Δ, or $\widehat{\Delta}$ consists of geodesic maps with respect to a natural affine connection ∇° of S, then only the following components Γ^h_{ij} can be different from zero:*

$$\Gamma^h_{1h} = \Gamma^h_{h1},\ h = 2, \ldots, n; \quad \Gamma^h_{2h} = \Gamma^h_{h2},\ h = 1, 3, \ldots, n; \quad \Gamma^1_{22}; \quad \Gamma^2_{11},$$

for the group Δ, and $\Gamma^h_{1h} = \Gamma^h_{h1}$, $h = 2, \ldots, n$, for the group $\widehat{\Delta}$.

Moreover, Δ as well as $\widehat{\Delta}$ is a group of affine mappings with respect to ∇°.

P r o o f. If $\Delta_\tau = \{(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{\tau-1}, s_\tau x_\tau, x_{\tau+1}, \ldots, x_n); s_\tau \in \mathbb{R}\}$, then a generator of its Lie algebra is given by $\xi_\tau = x_\tau \partial/\partial x_\tau = (x_\tau \delta^h_\tau)^n_{h=1}$.

For the Lie derivative $\mathfrak{L}_{\xi_\tau}\Gamma^h_{ij}$ along ξ_τ one has

$$\mathfrak{L}_{\xi_\tau}\Gamma^h_{ij} \equiv \delta^\tau_i \Gamma^h_{\tau j} + \delta^\tau_j \Gamma^h_{\tau i} - \delta^h_\tau \Gamma^\tau_{ij} = \delta^h_i \psi^{(\tau)}_j + \delta^h_j \psi^{(\tau)}_i, \tag{5.2}$$

where $h, i, j = 1, \dots, n$, and $\psi^{(\tau)}_i$ are differentiable functions.

From $h = i = j = \tau$ it follows that $\Gamma^\tau_{\tau\tau} = 2\psi^{(\tau)}_\tau$, and hence $\psi^{(\tau)}_\tau = 0$ (cf. Proposition 5.1). Using the left hand side of (5.2) for $h = i = j \neq \tau$ one has $\psi^{(\tau)}_i = 0$ for all $i \neq \tau$. Therefore $\psi^{(\tau)}_i = 0$ for all $i = 1, \dots, n$. Hence $\mathfrak{L}_{\xi_\tau}\Gamma^h_{ij} = 0$ and Δ_τ consists of affine transformations with respect to ∇°.

Now, an analysis of equations (5.2) yields that

$$\Gamma^h_{\tau\tau} = \Gamma^\tau_{ij} = \Gamma^h_{i\tau} = 0, \quad \text{for } h, i, j = 1, \dots, n, \ \tau = 3, \dots, n,$$

for the group Δ, and

$$\Gamma^h_{\tau\tau} = \Gamma^\tau_{ij} = \Gamma^h_{i\tau} = 0, \quad \text{for } h, i, j = 1, \dots, n, \ \tau = 2, \dots, n,$$

for the group $\widehat{\Delta}$. Considering the complementary set of the components Γ^h_{ij} we obtain the assertion. □

If a connection ∇ has components Γ^h_{ij}, $h, i, j, \in \{1, \dots, n\}$, the components R^h_{ijk}, $h, i, j, k \in \{1, \dots, n\}$ of the curvature tensor R of ∇ are given by (cf. [4], page 8, [17], pages 29–31, [19], page 27).

$$R^h_{ijk} = \frac{\partial}{\partial x^j}\Gamma^h_{ik} - \frac{\partial}{\partial x^k}\Gamma^h_{ij} + \sum_{\alpha=1}^{n}(\Gamma^\alpha_{ik}\Gamma^h_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^h_{\alpha k}). \tag{5.3}$$

The curvature tensor R of ∇ is often called the Riemannian tensor of ∇.

The Ricci tensor belonging to R has components $R_{ij} = \sum\limits_{\alpha=1}^{n} R^\alpha_{i\alpha j}$.

We remark that the Ricci tensor sometimes is defined with the opposite sign, see [4], page 8, [19], page 39.

In particular, ∇ is the *Levi-Civita connection* of a (pseudo-)Riemannian space with the metric $g = (g_{ij})$ if $\nabla g = 0$, i.e.,

$$\frac{\partial}{\partial x_k} g_{ij} = \sum_{\alpha=1}^{n}(g_{i\alpha}\Gamma^\alpha_{jk} + g_{j\alpha}\Gamma^\alpha_{ik}), \tag{5.4}$$

where the components Γ^h_{ij} (called the Christoffel symbols) are given by

$$\Gamma^h_{ij} = \frac{1}{2}\sum_{\alpha=1}^{n} g^{h\alpha}\Big(\frac{\partial}{\partial x^i} g_{j\alpha} + \frac{\partial}{\partial x^j} g_{i\alpha} - \frac{\partial}{\partial x^\alpha} g_{ij}\Big);$$

here $(g^{h\alpha})$ denotes the inverse matrix of (g_{ij}). Then for g there exists a unique symmetric affine connection ∇ such $\nabla g = 0$.

The integrability conditions of (5.4) have the following form [4], page 79:

$$\sum_{\alpha=1}^{n}(g_{h\alpha}R^{\alpha}_{ijk} + g_{i\alpha}R^{\alpha}_{hjk}) = 0. \tag{5.5}$$

The Riemannian tensor (cf. [15], page 61) vanishes if and only if the space is locally (pseudo-)Euclidean (also called a *flat space*), i.e., if there exists a local coordinate system $(x_1, x_2, \ldots, x_n)$ such that the metric is given by

$$\mathrm{d}s^2 = \sum_{\alpha=1}^{n} \varepsilon_\alpha \, \mathrm{d}x_\alpha^2, \quad \varepsilon_\alpha = \pm 1.$$

The components W^h_{ijk} of the Weyl tensor of a projective curvature have the following form ([4], page 88, [12], [15], page 133, [19], page 79):

$$\begin{aligned} W^h_{ijk} = R^h_{ijk} &- \frac{1}{n-1}\delta^h_i(R_{jk} - R_{kj}) \\ &- \frac{1}{n^2-1}[\delta^h_k(nR_{ij} + R_{ji}) - \delta^h_j(nR_{ik} + R_{ki})]. \end{aligned} \tag{5.6}$$

For $n \geqslant 3$ the Weyl tensor vanishes if and only if there exists a geodesic mapping onto a Euclidean space such that the geodesics with respect to ∇ are locally geodesically mapped onto the Euclidean lines ([4], page 89, [12], [15], page 138, [19], page 81). These spaces are called *(locally) projective Euclidean.* The notion of being projective Euclidean determines an equivalence relation within the class of geodesic mappings.

6. Dilatation shells of monotone curves

In this section we classify strictly monotone curves such that their images under groups of affinities containing the translation group as well as many dilatations are geodesics with respect to an affine natural connection.

Theorem 6.1. *Let $C(T)^\Theta$ be the Θ-shell of a strictly monotone and surjective curve $C \subset \mathbb{R}^n$. Then $C(T)^\Theta$ consists of geodesics with respect to a natural connection ∇° if and only if the components Γ^h_{ij} of ∇° and the functions $f_i(t)$ describing C may be chosen in one of the following three ways:*

(1) *$f_i(t) = \alpha_i t$ with $\alpha_i \neq 0$, for $i = 2, \ldots, n$, and all components Γ^h_{ij} vanish.*

(2) *$f_2 = -\Gamma^2_{11}(2\Gamma^2_{12})^{-1}t + c\exp(-2\Gamma^2_{12}t)$ with a constant c, $\Gamma^2_{11}\Gamma^2_{12}c > 0$,*

$$f_h = c_h \int \mathrm{e}^{\overline{\omega}_h t + (\widehat{\omega}_h/\beta)\mathrm{e}^{\beta t}}\,\mathrm{d}t + c^0_h, \tag{6.1}$$

where c_h and c^0_h are constants, $3 \leqslant h \leqslant n$, $\overline{\omega}_h\Gamma^2_{12} > 0$,

$$\overline{\omega}_h = -2\Gamma^h_{1h} - \frac{2\alpha}{\beta}(\Gamma^1_{12} - \Gamma^h_{2h}) \quad \text{and} \quad \widehat{\omega}_h = 2c(\Gamma^1_{12} - \Gamma^h_{2h}) < 0. \tag{6.2}$$

(3) *f_2' is contained in the open interval determined by y_2^- and y_2^+, where*

$$\lim_{t\to-\infty} f_2' = y_2^- \quad \text{and} \quad \lim_{t\to\infty} f_2' = y_2^+, \tag{6.3}$$

and y_2^- as well as y_2^+ are different roots of the cubic equation

$$\Gamma^1_{22}x^3 + 2\Gamma^1_{12}x^2 - 2\Gamma^2_{12}x - \Gamma^2_{11} = 0$$

such that either y_2^- or y_2^+ is zero and f_2' is contained in the open interval $(-2\Gamma^1_{12}/\Gamma^1_{22}, 0)$, or $y_2^- y_2^+ > 0$ holds.

For $h = 3, \ldots, n$ one has

$$f_h = c_h \int \mathrm{e}^{\int(\overline{\omega}_h + \widehat{\omega}_h f_2' + \varepsilon(f_2')^2)\,\mathrm{d}t}\,\mathrm{d}t + c^0_h, \tag{6.4}$$

where c_h and c^0_h are constants, $\overline{\omega}_h = -2\Gamma^h_{1h}$, $\widehat{\omega}_h = 2(\Gamma^1_{12} - \Gamma^h_{2h})$, and

$$\text{(a)}\quad \overline{\omega}_h + \widehat{\omega}_h y^- + \varepsilon(y^-)^2 \leqslant 0 \qquad \text{and} \qquad \text{(b)}\quad \overline{\omega}_h + \widehat{\omega}_h y^+ + \varepsilon(y^+)^2 \geqslant 0. \tag{6.5}$$

P r o o f. Assume that $C(T)^\Theta$ consists of geodesics with respect to a natural connection ∇°. It follows from Proposition 5.2 that the set $C(T)^\Theta$ consists of geodesics only if the components of ∇° which can be different from zero are

$$\Gamma^h_{1h} = \Gamma^h_{h1},\ h = 2, \ldots, n; \quad \Gamma^h_{2h} = \Gamma^h_{h2},\ h = 1, 3, \ldots, n; \quad \Gamma^1_{22}; \quad \Gamma^2_{11}.$$

Let

$$l = (t + u_1, f_2(t) + u_2, s_3 f_3(t) + u_3, \ldots, s_n f_n(t) + u_n), \quad t \in \mathbb{R},$$

be a curve of $C(T)^\Theta$. Then

$$\dot{l} = (1, f_2'(t), s_3 f_3'(t), \ldots, s_n f_n'(t)),$$
$$\ddot{l} = (0, f_2''(t), s_3 f_3''(t), \ldots, s_n f_n''(t)).$$

According to (5.1) the curve l is a geodesic if and only if

$$\ddot{l}^h + \sum_{\alpha,\beta=1}^{n} \Gamma^h_{\alpha\beta} \dot{l}^\alpha \dot{l}^\beta = \varrho(t)\, \dot{l}^h, \quad h = 1, 2, \dots, n. \tag{6.6}$$

We obtain:
for $h = 1$:

$$\varrho(t) = 2\Gamma^1_{12} f_2'(t) + \Gamma^1_{22}(f_2'(t))^2, \tag{6.7}$$

for $h = 2$:

$$f_2''(t) + \Gamma^2_{11} + 2\Gamma^2_{12} f_2'(t) = (2\Gamma^1_{12} f_2'(t) + \Gamma^1_{22}(f_2'(t))^2) f_2'(t), \tag{6.8}$$

and for $h > 2$:

$$c f_h''(t) = (-2\Gamma^h_{1h} + 2(\Gamma^1_{12} - \Gamma^h_{2h}) f_2'(t) + \Gamma^1_{22}(f_2'(t))^2) f_h'(t). \tag{6.9}$$

If f_2 is a linear function, then $f_2' = c = \text{const} \neq 0$. Hence $f_h''(t) = \omega_h f_h'(t)$ with $\omega = \text{const}$, and we have case 1 of the claim because the only surjective solution of the differential equation $f_h''(t) = \omega_h f_h'(t)$ is linear.

Let f_2 be a nonlinear function and

$$\alpha = -\Gamma^2_{11}, \quad \beta = -2\Gamma^2_{12}, \quad \gamma = 2\Gamma^1_{12}, \quad \varepsilon = \Gamma^1_{22}.$$

a) If $\varepsilon = 0$ then f_2 is strictly monotone and surjective if and only if

$$f_2 = \frac{1}{\beta}(-\alpha t + c\mathrm{e}^{\beta t}) + d \quad \text{with } d \in \mathbb{R},\ \beta \neq 0 \text{ and } \alpha c > 0.$$

Substituting for f_2' in (6.9) we obtain

$$f_h''(t) = (\overline{\omega}_h + \widehat{\omega}_h \mathrm{e}^{\beta t}) f_h'(t), \tag{6.10}$$

where $\overline{\omega}$ and $\widehat{\omega}$ are constants having form (6.2). The integration of (6.10) yields (6.1).

If $\widehat{\omega}_h = 0$, then f_h is not surjective. Detailed analysis of (6.1) yields that f_h, $h = 3, \dots, n$, is surjective if and only if $\overline{\omega}_h < 0$, $\widehat{\omega}_h \beta < 0$ since in all other cases there exists a horizontal asymptote.

b) Let be $\varepsilon \neq 0$.

If $\alpha = \beta = 0$ then f_2 is strictly monotone and surjective if and only if f_2' is contained in the open interval $(0, -\gamma/\varepsilon)$, see Theorem 3.1, part 1.

Let $\alpha \neq 0$ or $\beta \neq 0$, then f_2 is strictly monotone and surjective if and only if the cubic equation $\alpha + \beta y + \gamma y^2 + \varepsilon y^3 = 0$ has 3 real roots such that f_2' is contained in the open interval determined by two different positive or negative roots, but this interval does not contain the third root, see Theorem 3.1, part 2. Integration of (6.9) gives (6.4). It remains to prove that the relations (6.5) are satisfied. Clearly $f_h(t)$, $h = 3, \ldots, n$, is monotone and surjective.

Let y_2^- and y_2^+ be as in (6.3). If (6.5 a) or (6.5 b) holds, then, respectively,

$$\lim_{t \to -\infty, \text{ or } \infty} \overline{\omega}_h + \widehat{\omega}_h f_2' + \varepsilon (f_2')^2 \leqslant 0 \quad \text{or} \quad \geqslant 0.$$

A detailed analysis yields $\lim\limits_{t \to -\infty, \text{ or } \infty} f_h(t) = -c_h \infty$ or $c_h \infty$, from which it follows that $f_h(t)$ is surjective.

If neither (6.5 a) nor (6.5 b) holds, then $f_h(t)$ is not surjective. □

Corollary 6.1. *Let $C(T)^{\widehat{\Theta}}$ and $\widehat{C}(T)^{\widehat{\Theta}}$ be the sets of curves* (2.3) *or* (2.4), *respectively, where C is a strictly monotone and surjective curve. Then $C(T)^{\widehat{\Theta}}$ or $\widehat{C}(T)^{\widehat{\Theta}}$, consists of geodesics with respect to a natural connection ∇° if and only if $f_i = \alpha_i t$ with a constant $\alpha_i \neq 0$ for $i = 2, \ldots, n$, and all $\Gamma_{ij}^h = 0$.*

Proof. Let ∇° be an affine connection such that the $\widehat{\Theta}$-shell $C(T)^{\widehat{\Theta}}$ consists of geodesics with respect to ∇°. Then Proposition 5.2 implies that only

$$\Gamma_{1h}^h = \Gamma_{h1}^h, \quad h = 2, \ldots, n,$$

may be different from zero.

Then equations (5.8) and (5.9) reduce to

$$f_2''(t) = -2\Gamma_{12}^2 f_2' \quad \text{and} \quad f_h''(t) = -2\Gamma_{1h}^h f_h', \quad h > 2.$$

But then $f_i(t)$, $i = 2, \ldots, n$, are surjective if and only if all f_i are linear.

The set $C(T)^{\widehat{\Theta}}$ is a subset of the set $\widehat{C}(T)^{\widehat{\Theta}}$. Hence if $\widehat{C}(T)^{\widehat{\Theta}}$ consists of geodesics, then also $C(T)^{\widehat{\Theta}}$ consists of geodesics and the corollary is proved. □

Theorem 6.2. *Let $\widehat{C}(T)^{\Theta}$ be the set of curves* (2.3). *Then $\widehat{C}(T)^{\Theta}$ consists of geodesics with respect to a natural connection ∇° if and only if for the components Γ_{ij}^h of ∇° and for the functions $f_i(t)$ describing C one of the following assertions holds:*

(1) *$f_i(t) = \alpha_i t$ for $i = 2, \ldots, n$, and all components Γ_{ij}^h vanish.*
(2) *$f_2 = -\Gamma_{11}^2 (2\Gamma_{12}^2)^{-1} t + \beta \exp(-2\Gamma_{12}^2 t)$ with a constant $\beta \neq 0$, $f_h = c_h t$ with constants $c_h \neq 0$, $3 \leqslant h \leqslant n$, and the only nonzero components of Γ_{ij}^h are precisely Γ_{11}^2, $\Gamma_{12}^2 = \Gamma_{21}^2$. The connection ∇° is not flat and yields a space that is not locally projective Euclidean.*

P r o o f. Assume that $\widehat{C}(T)^{\Theta}$ consists of geodesics with respect to a natural connection ∇°. It follows from Proposition 5.2 that the set $\widehat{C}(T)^{\Theta}$ consists of geodesics only if the components of ∇° which can be different from zero are

$$\Gamma^h_{1h} = \Gamma^h_{h1},\ h = 2, \dots, n; \quad \Gamma^h_{2h} = \Gamma^h_{h2},\ h = 1, 3, \dots, n; \quad \Gamma^1_{22}; \quad \Gamma^2_{11}.$$

Let $l = (t + u_1, f_2(t) + u_2, s_3 f_3(t + v_3) + u_3, \dots, s_n f_n(t + v_n) + u_n)$, $t \in \mathbb{R}$, be a curve of $\widehat{C}(T)^{\Theta}$. Then

$$\begin{aligned} \dot{l} &= (1, f_2'(t), s_3 f_3'(t + v_3), \dots, s_n f_n'(t + v_n)), \\ \ddot{l} &= (0, f_2''(t), s_3 f_3''(t + v_3), \dots, s_n f_n''(t + v_n)). \end{aligned}$$

According to (5.1) the curve l is a geodesic if and only if (6.6) holds. For $h = 1$ and $h = 2$ we obtain (6.7) and (6.8). For $h > 2$ one has

$$c f_h''(t_h) = (-2\Gamma^h_{1h} + 2(\Gamma^1_{12} - \Gamma^h_{2h}) f_2'(t) + \Gamma^1_{22}(f_2'(t))^2) f_h'(t_h),$$

where $t_h = t + v_h$. Since t and t_h are independent variables we have

$$c f_h''(t_h) = \omega_h f_h'(t_h), \tag{6.11}$$

where

$$\omega_h = -2\Gamma^h_{1h} + 2(\Gamma^1_{12} - \Gamma^h_{2h}) f_2'(t) + \Gamma^1_{22}(f_2'(t))^2 \tag{6.12}$$

is a constant. If $f_h'(t) \neq \text{const}$, $h = 3, \dots, n$, then no solution of the differential equation (6.11) is surjective. Hence $f_h(t) = c_h t$, with $c_h \neq 0$ and $\omega_h = 0$ for $h = 3, \dots, n$.

If $f_2'(t) = c_2 \neq 0$ is constant then we are in case 1.

If $f_2'(t)$ is not constant, then because of Lemma 2.1 we can choose the function $f_2 = ct + \beta \exp(\alpha t)$ with nonzero constants α, β, c. Substituting this in (6.8) and in (6.12) we obtain

$$\Gamma^1_{22} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = 0, \quad \Gamma^2_{11} = \alpha c, \quad \Gamma^2_{12} = \Gamma^2_{21} = -\alpha/2$$

and

$$\Gamma^h_{1h} = \Gamma^h_{2h} = 0, \quad h = 3, \dots, n.$$

The connection ∇° is not flat, since the component $R^2_{112} = (\Gamma^2_{12})^2$ of the curvature tensor is not zero. Moreover, from $W^2_{112} = (\Gamma^2_{12})^2 \neq 0$ it follows that ∇° is not projectively flat. Taking this into account the theorem is completely proved. $\square$

If ∇ is a linear connection which is in the same projective class as ∇° and if P is a non-projective Euclidean space with respect to ∇° belonging to the second case of Theorem 5.2, then we can conclude the following:

Remark 6.1. Any space P is *subprojective*, i.e., the geodesic lines with respect to ∇ in some coordinate system of $\mathbb{R}^n$ are described by $(n-1)$ equations such that not all but $(n-2)$ of them are linear, cf. Kagan [9].

Remark 6.2. Since the only components of ∇° and of the curvature R different from zero are Γ^2_{11}, Γ^2_{12} $(=\Gamma^2_{21})$, and R^2_{112} $(=R^2_{121})$, one obtains $\nabla^\circ R=0$, where the covariant derivative of R is given by

$$\nabla^\circ_l R^h_{ijk} \equiv \partial_l R^h_{ijk} + \Gamma^h_{\alpha l} R^\alpha_{ijk} - \Gamma^\alpha_{il} R^h_{\alpha jk} - \Gamma^\alpha_{jl} R^h_{i\alpha k} - \Gamma^\alpha_{kl} R^h_{ij\alpha}.$$

Hence any space P is *symmetric* in the sense of E. Cartan, cf. [3], [19].

Remark 6.3. No space P is *metrizable* with respect to a linear connection ∇ in the same projective class as ∇° (cf. Sinyukov [19]); P is not even a Weyl space, i.e., it does not satisfy $\nabla g=\omega g$ for a metric g (cf. [7], [15], Theorem 6.15, page 174).

7. Extended shells of monotone curves

Choosing a suitable coordinate system we may assume that for a fixed r with $2\leqslant r\leqslant n$ the functions $f_i(t)=\alpha_i t+\beta_i$, $\beta_i\in\mathbb{R}$, $\alpha_i\neq 0$, $i<r$, are linear, whereas f_i, $i\geqslant r$, are not linear.

Now we consider systems of curves which are invariant under the translation group; these can be seen as partial shift spaces.

Theorem 7.1. *If the curves of the extended shell $\widehat{C}(T)$ of a strictly monotone curve $C\subset\mathbb{R}^n$ are geodesics with respect to a natural connection ∇°, then for the functions $f_i(t)$, $i=2,\dots,n$, describing C the following identities hold:*

$$\begin{aligned}\alpha_h + \beta_h f'_h(t_h) + \gamma_h (f'_h(t_h))^2 + \varepsilon_h (f'_h(t_h))^3 \\ + \Gamma^h_{11} + 2\sum_{\sigma=2}^{n}(\Gamma^h_{1\sigma} - \Gamma^1_{1\sigma} f'_h(t_h)) f'_\sigma(t_\sigma) \\ + \sum_{\sigma,\varrho=2}^{n} (\Gamma^h_{\sigma\tau} - \Gamma^1_{\sigma\tau} f'_h(t_h)) f'_\sigma(t_\sigma) f'_\tau(t_\tau) \equiv 0,\end{aligned} \tag{7.1}$$

where $1<h\leqslant n$, α_h, β_h, γ_h, ε_h are constant coefficients and t_h are independent variables.

Moreover, the functions $f_h'(t_h)$ satisfy the following Abelian differential equations:

$$f_h''(t_h) = \alpha_h + \beta_h f_h'(t_h) + \gamma_h (f_h'(t_h))^2 + \varepsilon_h (f_h'(t_h))^3. \tag{7.2}$$

If $\varepsilon_h \neq 0$, then $(\beta_h, \gamma_h) \neq (0,0)$, and one of the conditions of Corollary 3.1 *is satisfied. If $\varepsilon_h = 0$ and f_h' is not constant, then $\gamma_h = 0$, but $\alpha_h \neq 0$.*

Proof. For a curve

$$x = (t + u_1, f_2(t) + u_2, f_3(t + v_3) + u_3, \ldots, f_n(t + v_n) + u_n), \quad t \in \mathbb{R},$$

of $\widehat{C}(T)$ one has

$$\begin{aligned}
\dot{x} &= (1, f_2{}'(t), f_3{}'(t + v_3), \ldots, f_n{}'(t + v_n)),\\
\ddot{x} &= (0, f_2{}''(t), f_3{}''(t + v_3), \ldots, f_n{}''(t + v_n)).
\end{aligned}$$

This curve is a geodesic if and only if relation (5.1) holds. We put in this relation $t_2 \equiv t$ and $t_\lambda \equiv t + v_\lambda$ for $\lambda > 2$.

For $h = 1$ one has

$$\varrho(t_2) = 2 \sum_{\sigma=2}^{n} \Gamma^1_{1\sigma} f_\sigma{}'(t_\sigma) + \sum_{\sigma,\tau=2}^{n} \Gamma^1_{\sigma\tau} f_\sigma{}'(t_\sigma) f_\tau{}'(t_\tau),$$

and for $h > 1$ we get

$$f_h{}''(t_h) + \Gamma^h_{11} + 2 \sum_{\sigma=2}^{n} \Gamma^h_{1\sigma} f_\sigma{}'(t_\sigma) + \sum_{\sigma,\tau=2}^{n} \Gamma^h_{\sigma\tau} f_\sigma{}'(t_\sigma) f_\tau{}'(t_\tau) = \varrho(t) f_h{}'(t_h). \tag{7.3}$$

Substituting $\varrho(t_2)$ into (7.3) we obtain for $h > 1$:

$$\begin{aligned}
f_h{}''(t_h) + \Gamma^h_{11} + 2 \sum_{\sigma=2}^{n} &(\Gamma^h_{1\sigma} - \Gamma^1_{1\sigma} f_h{}'(t_h)) f_\sigma{}'(t_\sigma)\\
&+ \sum_{\sigma,\varrho=k+1}^{n} (\Gamma^h_{\sigma\tau} - \Gamma^1_{\sigma\tau} f_h{}'(t_h)) f_\sigma{}'(t_\sigma) f_\tau{}'(t_\tau) \equiv 0.
\end{aligned} \tag{7.4}$$

Since we can fix in (7.4) all variables t_r $(h \neq r)$ we obtain for all functions $f_h'(t_h)$ Abelian differential equations (7.2), where $\alpha_h, \beta_h, \gamma_h, \varepsilon_h$ are constant coefficients. Moreover, if $\varepsilon_h \neq 0$ then the constants $\alpha_h, \beta_h, \gamma_h$ satisfy one of the conditions of Corollary 3.1, which in particular yields $(\beta_h, \gamma_h) \neq (0,0)$. If $\varepsilon_h = 0$, then Lemma 2.1 gives $\gamma_h = 0$ and $\alpha \neq 0$.

Putting (7.2) into (7.4) we obtain the identities (7.1). □

Proposition 7.1. *Let $\widehat{C}(T)$ be the extended shell of a strictly monotone curve C in $\mathbb{R}^n$ such that any curve of $\widehat{C}(T)$ is a geodesic with respect to a natural connection $\nabla^\circ = \{\Gamma^h_{ij}\}$. A nonconstant function $f'_h(t_h)$ $(h > 1)$, corresponding to C is a solution of an Abelian, but of no Riccati differential equation if and only if $\Gamma^1_{hh} \neq 0$. Moreover, in* (6.1) *we have $\varepsilon_h = \Gamma^1_{hh}$.*

Proof. Since $h > 1$ we use formula (7.1). We fix there all parameters t_σ with $\sigma > 1$ and obtain a polynomial in f'_h. The coefficient of $(f'_h)^3$ equals $\varepsilon_h - \Gamma^1_{hh}$. As f'_h is not constant we have $\Gamma^1_{hh} = \varepsilon_h$. Hence Γ^1_{hh} is different from zero if and only if $\varepsilon_h \neq 0$. The function f'_h is a solution of an Abelian differential equation, but of no Riccati equation if and only if $\varepsilon_h \neq 0$. □

8. Monotone curves in the 3-dimensional space

In this section we classify extended shells $\widehat{C}(T)$ of strictly monotone curves C such that any curve of $\widehat{C}(T)$ is a geodesic with respect to a natural connection ∇°.

Theorem 8.1. *Let $\widehat{C}(T)$ be the extended shell of a strictly monotone curve $C \subset \mathbb{R}^3$ such that neither of the derivatives f'_2, f'_3 of the functions describing C is constant. Then any curve of $\widehat{C}(T)$ is a geodesic with respect to ∇° if and only if the functions f_i, $i = 2, 3$, may be chosen as*

$$f_i = -\frac{1}{\beta_i}(\alpha_i t + c_i \mathrm{e}^{-\beta_i t}), \tag{8.1}$$

where all α_i, β_i, c_i are constants different from zero and the conditions

$$\alpha_2 = -\Gamma^2_{11}, \quad \beta_2 = -2\Gamma^2_{12} = -2\Gamma^2_{21}, \quad \alpha_3 = -\Gamma^3_{11}, \quad \beta_3 = -2\Gamma^3_{13} = -2\Gamma^3_{31} \tag{8.2}$$

are satisfied, whereas all the other components of ∇° vanish.

Proof. Any curve of $\widehat{C}(T)$ is a geodesic with respect to ∇° if and only if formulas (7.1) hold for $h \in \{2, 3\}$ and $\Gamma^1_{11} = \Gamma^2_{22} = \Gamma^3_{33} = 0$. Then an analysis of this situation yields (8.2) as well as

$$\gamma_2 = 2\Gamma^1_{12} = 2\Gamma^1_{21}, \quad \varepsilon_2 = 0, \quad \gamma_3 = 2\Gamma^1_{13} = 2\Gamma^1_{31}, \quad \varepsilon_3 = 0,$$

and all the other components of ∇° are zero. Moreover, from Theorem 5.2 it follows that also $\gamma_2 = \gamma_3 = 0$.

Since neither f'_2 nor f'_3 are constant, by Lemma 2.1 it follows that these functions may be chosen as in (8.1). □

If neither of f_2 and f_3 is constant, then precisely the following components R^h_{ijk} of the curvature tensor of the natural connection ∇° are different from zero: $R^2_{112} = -R^2_{121} = (\Gamma^2_{12})^2$ and $R^3_{113} = -R^3_{131} = (\Gamma^3_{13})^2$. The only nonzero component of the Ricci tensor is $R_{11} = -(\Gamma^2_{12})^2 - (\Gamma^3_{13})^2$, i.e., the Ricci tensor is symmetric and ∇° is equiaffine.

The nonzero components of the Weyl projective tensor are precisely the following ones:

$$W^2_{112} = -W^2_{121} = -W^3_{113} = W^3_{131} = \frac{1}{2}((\Gamma^2_{12})^2 - (\Gamma^3_{13})^2).$$

Therefore the natural connection ∇° yields a locally projective Euclidean space if and only if $\alpha_2 = \Gamma^2_{12} = \pm\Gamma^3_{13} = \pm\alpha_3$.

If the connection ∇° were metrizable, then the metric occurring in formula (5.4) would not be singular. But putting in (5.5) indices $h = 1, 2, 3$, $i = j = 1$, $k = 2$ we obtain $g_{h2} = 0$ for $h = 1, 2, 3$. This contradiction shows that ∇° never yields a metrizable space P. Moreover, the space P is symmetric, because $\nabla^\circ R = 0$ (see Remark 5.2). This yields that P (which is not projective Euclidean) is not projective metrizable (cf. Sinyukov [19], Theorem 7, page 86) and is not a Weyl space, cf. [7], [15], Theorem 6.15, page 174.

The affine connections $\overline{\nabla}$ and ∇° belong to the same projective equivalence class if and only if for the components Γ^h_{ij} of ∇° and $\overline{\Gamma}^h_{ij}$ of $\overline{\nabla}$ one has

$$\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i, \tag{8.3}$$

where the ψ_i are differentiable functions [15], page 131, [19], page 73. Hence taking $\psi_i = \Gamma^2_{12}\delta^1_i$ the affine connection $\overline{\nabla}$ with components $\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \Gamma^2_{12}\delta^1_i\delta^h_j + \Gamma^2_{12}\delta^1_j\delta^h_i$ belongs to the same projective equivalence class as the natural connection ∇°, and we obtain in the case $|\Gamma^2_{12}| = |\Gamma^3_{13}|$ that the only nonzero components of $\overline{\Gamma}^h_{ij}$ are

$$\overline{\Gamma}^1_{11} = \overline{\Gamma}^2_{12} = \overline{\Gamma}^2_{21} = 2\Gamma^2_{12}, \quad \text{if } \Gamma^2_{12} = -\Gamma^3_{13}, \tag{8.4}$$

or

$$\overline{\Gamma}^1_{11} = \overline{\Gamma}^2_{12} = \overline{\Gamma}^2_{21} = \overline{\Gamma}^3_{13} = \overline{\Gamma}^3_{31} = 2\Gamma^2_{12}, \quad \text{if } \Gamma^2_{12} = \Gamma^3_{13}. \tag{8.5}$$

Since $\overline{\Gamma}^2_{12} = \alpha \neq 0$ ($\beta_2 = 1/2\alpha$, $\beta_3 = \pm 1/2\alpha$) one verifies that the components of the metric tensor

$$\overline{g}_{ij} = \begin{pmatrix} (1 + (\alpha x_2)^2)\mathrm{e}^{2\alpha x_1} & \alpha x_2 \mathrm{e}^{2\alpha x_1} & 0 \\ \alpha x_2 \mathrm{e}^{2\alpha x_1} & \mathrm{e}^{2\alpha x_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{8.6}$$

belonging to the relations (8.4), and the components of

$$\overline{g}_{ij} = \begin{pmatrix} (1+(\alpha x_2)^2)+(\alpha x_3)^2)\mathrm{e}^{2\alpha x_1} & \alpha x_2 \mathrm{e}^{2\alpha x_1} & \alpha x_3 \mathrm{e}^{2\alpha x_1} \\ \alpha x_2 \mathrm{e}^{2\alpha x_1} & \mathrm{e}^{2\alpha x_1} & 0 \\ \alpha x_3 \mathrm{e}^{2\alpha x_1} & 0 & \mathrm{e}^{2\alpha x_1} \end{pmatrix} \tag{8.7}$$

belonging to the relations (8.5), satisfy the partial differential equations (5.4).

The determinants of these tensors are $\mathrm{e}^{4\alpha x_1}$ and $\mathrm{e}^{6\alpha x_1}$, respectively, for every (x_1, x_2, x_3). From the form of the matrices $\overline{g}_{ij}$ it follows that the corresponding metrics are positive definite. Hence we have

Proposition 8.1. *Let $\widehat{C}(T)$ be the extended shell of a strictly monotone curve $C \subset \mathbb{R}^3$ such that neither of the derivatives f_2', f_3' of the functions describing C is constant and any curve of $\widehat{C}(T)$ is a geodesic with respect to a natural connection ∇°. Although ∇° yields a space P which is neither projective metrizable nor a Weyl space, P is symmetric with respect to ∇°.*

But there exists a connection $\overline{\nabla} = \{\overline{\Gamma}_{ij}^h\}$ belonging to the same projective class as ∇° such that P with respect to $\overline{\nabla}$ is locally projective Euclidean. Moreover, there exist metric tensors (8.6) *and* (8.7) *yielding Riemannian spaces.*

Now we consider strictly monotone curves C such that one of the derivatives f_i' of the corresponding functions is constant.

Theorem 8.2. *Let $\widehat{C}(T)$ be the extended shell of a strictly monotone curve $C \subset \mathbb{R}^3$ such that the derivative f_r' is not constant, whereas f_s' is constant ($r, s \in \{2, 3\}$, $r \neq s$).*

Then any curve of $\widehat{C}(T)$ is a geodesic with respect to ∇° if and only if the corresponding functions f_r, f_s may be chosen as follows: $f_s = ct$ with a constant $c \neq 0$ and f_r is a solution of equation (7.2) *with coefficients*

$$\begin{aligned}
\alpha_r &= -\Gamma_{11}^r - 2c\Gamma_{1s}^r - c^2\Gamma_{ss}^r, \\
\beta_r &= -2\Gamma_{1r}^r + 2c\Gamma_{1s}^1 - 2c\Gamma_{rs}^r + c^2\Gamma_{ss}^1, \\
\gamma_r &= 2\Gamma_{1r}^1 + 2c\Gamma_{rs}^1, \\
\varepsilon_r &= \Gamma_{rr}^1,
\end{aligned}$$

which satisfy

$$\begin{gathered}
\Gamma_{11}^s + 2c(\Gamma_{1s}^s - c\Gamma_{1s}^1) - c^3\Gamma_{ss}^1 = 0, \\
2(\Gamma_{1r}^s - c\Gamma_{1r}^1) + 2c(\Gamma_{rs}^s - c\Gamma_{rs}^1) = 0, \\
\Gamma_{rr}^s - c\Gamma_{rr}^1 = 0.
\end{gathered} \tag{8.8}$$

If $\Gamma^1_{rr} \neq 0$, then f_r' is a solution of an Abelian differential equation which is not a Riccati equation. If $\Gamma^1_{rr} = 0$, then $\gamma_r = \varepsilon_r = 0$ and

$$f_r = -\frac{1}{\beta_r}\,(\alpha_r t + c_r \mathrm{e}^{-\beta_r t}), \quad \alpha_r \neq 0,\ \beta_r \neq 0,\ c_r \neq 0. \tag{8.9}$$

Proof. Any curve of $\widehat{C}(T)$ is a geodesic with respect to ∇° if and only if formulas (7.1) hold for $h \in \{r, s\}$. Since f_r' is not constant and f_s' is a nonzero constant c, an analysis of this situation yields (8.8) and (8.9). Using Theorem 5.2 we obtain the remaining claims. □

At the end of the paper we study properties of the extended shells $\widehat{C}(T)$ of increasing curves $C \subset \mathbb{R}^3$ such that any curve of $\widehat{C}(T)$ is a geodesic with respect to a natural connection ∇° having only few components different from zero.

Theorem 8.3. *Let $\widehat{C}(T)$ be the extended shell of a strictly monotone curve $C \subset \mathbb{R}^3$ such that any curve of $\widehat{C}(T)$ is a geodesic with respect to a natural connection $\nabla^\circ = \{\Gamma^h_{ij}\}$ having not all components, but at most*

$$\Gamma^2_{11}, \quad \Gamma^2_{12}\ (= \Gamma^2_{21}), \quad \Gamma^3_{11}, \quad \Gamma^3_{13}\ (= \Gamma^3_{31})$$

different from zero. Then:

(1) *If $\Gamma^2_{12} = \Gamma^3_{13} = 0$, then ∇° yields a flat space. Such a space is metrizable, i.e., there exists Euclidean metrics for it.*

(2) *The connection ∇° yields a projective Euclidean space if and only if $|\Gamma^2_{12}| = |\Gamma^3_{13}|$.*

(3) *If ∇° does not yield a flat space, i.e., if $(\Gamma^2_{12})^2 + (\Gamma^3_{13})^2 \neq 0$, then for the functions $f_i(t)$ describing C one of the following cases occurs:*

1) *Neither of the functions f_i is constant, the curves C have the form as in Theorem 7.1 and all Γ^2_{11}, Γ^2_{12}, Γ^3_{11}, Γ^3_{13} are different from zero.*

2) *For $r, s \in \{2, 3\}$, $r \neq s$, the function f_s may be chosen as $f_s = ct$ with a constant $c \neq 0$ and the function f_r has the form (8.9) with $\alpha_r = -\Gamma^r_{11} \neq 0$, $\beta_r = -2\Gamma^r_{1r} \neq 0$, $\Gamma^s_{11} = -2c\Gamma^s_{1s}$ and all the other components are zero,*

(4) *If ∇° is a non-flat natural connection, then ∇° is not metrizable. But there exists in the projective equivalence class of ∇° an affine connection $\overline{\nabla} = \{\overline{\Gamma}^h_{ij}\}$ which is metrizable such that $|\Gamma^2_{12}| = |\Gamma^3_{13}| \neq 0$; the only nonzero components of $\overline{\Gamma}^h_{ij}$ can be*

$$\overline{\Gamma}^1_{11} = \overline{\Gamma}^2_{12} = 2\Gamma^2_{12}, \quad \overline{\Gamma}^2_{11} = \Gamma^2_{11}, \quad \overline{\Gamma}^3_{11} = \Gamma^3_{11} \quad \textit{if } \Gamma^2_{12} = -\Gamma^3_{13}, \tag{8.10}$$

or

$$(8.11) \qquad \overline{\Gamma}^1_{11} = \overline{\Gamma}^2_{12} = \overline{\Gamma}^3_{13} = 2\Gamma^2_{12}, \quad \overline{\Gamma}^2_{11} = \Gamma^2_{11}, \quad \overline{\Gamma}^3_{11} = \Gamma^3_{11} \quad \textit{if } \Gamma^2_{12} = \Gamma^3_{13}$$

and the corresponding metric tensors $\overline{g}_{ij}$ are positive definite.

Proof. Using formula (5.3) we obtain that all components R^h_{ijk} of the curvature tensor are zero except

$$R^2_{112} = (\Gamma^2_{12})^2 \quad \text{and} \quad R^3_{113} = (\Gamma^3_{13})^2.$$

The assertion (1) follows from the fact that the curvature tensor vanishes.

The only possibly nonzero component of the Ricci tensor is $R_{11} = (\Gamma^2_{12})^2 + (\Gamma^3_{13})^2$. Applying formula (5.6) we obtain in our case that all components W^h_{ijk} are zero except $W^2_{112} = -W^3_{113} = \frac{1}{2}((\Gamma^2_{12})^2 - (\Gamma^3_{13})^2)$. Hence the Weyl tensor vanishes if and only if $|\Gamma^2_{12}| = |\Gamma^3_{13}|$ and the claim (2) is proved.

Since in case (3) the connection ∇° is not flat we have $\Gamma^2_{12} \neq 0$ or $\Gamma^3_{13} \neq 0$. If neither of the derivatives f'_i is constant, it follows from Theorem 7.1 that we are in case (3) part 1. If one of the derivatives f'_i is constant, then Theorem 7.2 yields that we have case (3), part 2.

If the connection ∇° is metrizable, then the metric g_{ij} occurring in formula (5.4) is not singular. Putting in (5.5) indices $i = 1$, $k = 1$, $l = 3$ and $j = 1, 2, 3$ we obtain $g_{1j} = 0$ for $j = 1, 2, 3$, which is a contradiction. Hence the first claim of (4) holds.

Taking in (8.3) the functions $\psi_i = \Gamma^2_{12}\delta^1_i$ the affine connection $\overline{\nabla}$ with components $\overline{\Gamma}^h_{ij} = \Gamma^h_{ij} + \Gamma^2_{12}\delta^1_i\delta^h_j + \Gamma^2_{12}\delta^1_j\delta^h_i$ belongs to the same projective equivalence class as the natural connection ∇° and the main part of the claim (4) holds.

Putting $\overline{\Gamma}^2_{12} = \alpha \neq 0$, $\overline{\Gamma}^2_{11} = \beta$ and $\overline{\Gamma}^3_{11} = \gamma$ one verifies that the components of the metric tensor

$$\overline{g}_{ij} = \begin{pmatrix} (1 + (\beta x_1 + \alpha x_2)^2)\mathrm{e}^{2\alpha x_1} + \gamma^2/\alpha^2 & (\beta x_1 + \alpha x_2)\mathrm{e}^{2\alpha x_1} & -\gamma/\alpha \\ (\beta x_1 + \alpha x_2)\mathrm{e}^{2\alpha x_1} & \mathrm{e}^{2\alpha x_1} & 0 \\ -\gamma/\alpha & 0 & 1 \end{pmatrix},$$

belonging to the relations (8.10), and the components of

$$\overline{g}_{ij} = \begin{pmatrix} (1+(\beta x_1+\alpha x_2)^2+(\gamma x_1+\alpha x_3)^2)\mathrm{e}^{2\alpha x_1} & (\beta x_1+\alpha x_2)\mathrm{e}^{2\alpha x_1} & (\gamma x_1+\alpha x_3)\mathrm{e}^{2\alpha x_1} \\ (\beta x_1 + \alpha x_2)\mathrm{e}^{2\alpha x_1} & \mathrm{e}^{2\alpha x_1} & 0 \\ (\gamma x_1 + \alpha x_3)\mathrm{e}^{2\alpha x_1} & 0 & \mathrm{e}^{2\alpha x_1} \end{pmatrix}$$

belonging to the relations (8.11), satisfy the partial differential equations (4.4).

The determinants of these tensors are $\mathrm{e}^{4\alpha x_1}$ and $\mathrm{e}^{6\alpha x_1}$, respectively, for every (x_1, x_2, x_3). From the form of the matrices $\overline{g}_{ij}$ it follows that the corresponding metrics are positive definite. $\square$

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References

[1] *D. Betten*: Topological shift spaces. Adv. Geom. *5* (2005), 107–118.
[2] *D. Betten*: Some classes of topological 3-spaces. Result. Math. *12* (1987), 37–61. (In German.)
[3] *E. Cartan*: Les espaces riemanniens symétriques. Verh. Internat. Math.-Kongr. *1* (1932), 152–161. (In French.)
[4] *L. P. Eisenhart*: Non-Riemannian geometry. American Mathematical Society Colloquium Publications 8. American Mathematical Society, Providence, 1990.
[5] *G. Gerlich*: Topological affine planes with affine connections. Adv. Geom. *5* (2005), 265–278.
[6] *G. Gerlich*: Stable projective planes with Riemannian metrics. Arch. Math. *79* (2002), 317–320.
[7] *I. Hinterleitner, J. Mikeš*: Geodesic mappings onto Weyl spaces. Proc. 8th Int. Conf. on Appl. Math. (APLIMAT 2009), Bratislava, pp. 423–430.
[8] *R. S. Irving*: Integers, Polynomials, and Rings. A Course in Algebra. Undergraduate Texts in Mathematics, Springer, New York, 2004.
[9] *V. F. Kagan*: Subprojective Spaces. Bibliothek der Russischen Wissenschaften Mathematik, Mechanik, Physik, Astronomie, Staatsverlag für Physikalisch-Mathematische Literatur, Moskva, 1961. (In Russian.)
[10] *E. Kamke*: Differentialgleichungen. Lösungsmethoden und Lösungen. 1. Gewöhnliche Differentialgleichungen. Akademische Verlagsgesellschaft, Leipzig, 1942. (In German.)
[11] *E. Kamke*: Differentialgleichungen Reeller Funktionen. Akademische Verlagsgesellschaft, Leipzig, 1930. (In German.)
[12] *J. Mikeš*: Geodesic mappings of affine-connected and Riemannian spaces. J. Math. Sci., New York *78* (1996), 311–333.
[13] *J. Mikeš, K. Strambach*: Grünwald shift spaces. Publ. Math. *83* (2013), 85–96.
[14] *J. Mikeš, K. Strambach*: Differentiable structures on elementary geometries. Result. Math. *53* (2009), 153–172.
[15] *J. Mikeš, A. Vanžurová, I. Hinterleitner*: Geodesic Mappings and Some Generalizations. Palacký University, Faculty of Science, Olomouc, 2009.
[16] *A. P. Norden*: Spaces with Affine Connection. Nauka, Moskva, 1976. (In Russian.)
[17] *A. Z. Petrov*: New Methods in the General Theory of Relativity. Hauptredaktion für Physikalisch-Mathematische Literatur, Nauka, Moskva, 1966. (In Russian.)
[18] *H. Salzmann, D. Betten, T. Grundhöfer, H. Hähl, R. Löwen, M. Stroppel*: Compact Projective Planes. With an Introduction to Octonion Geometry. De Gruyter Expositions in Mathematics 21, De Gruyter, Berlin, 1995.
[19] *N. S. Sinyukov*: Geodesic Mappings of Riemannian Spaces. Nauka, Moskva, 1979. (In Russian.)
[20] *K. Yano, S. Bochner*: Curvature and Betti Numbers. Annals of Mathematics Studies 32, Princeton University Press 9, Princeton, 1953.

Authors' addresses: Josef Mikeš, Univerzita Palackého v Olomouci, 17. listopadu 12, 771 46 Olomouc, Czech Republic, e-mail: `josef.mikes@upol.cz`; Karl Strambach, Department Mathematik, Universität Erlangen-Nürnberg, Cauerstr. 11, D-91058 Erlangen, Germany, e-mail: `strambach@mat.fau.de`.