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NEUTRAL SET DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to establish an existence and uniqueness result for a class of the set functional differential equations of neutral type

$$\begin{cases} D_H X(t) = F(t, X_t, D_H X_t), \\ X|_{[-r,0]} = \Psi, \end{cases}$$

where $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ is a given function, $K_c(E)$ is the family of all nonempty compact and convex subsets of a separable Banach space E, $\mathcal{C}_0$ denotes the space of all continuous set-valued functions X from $[-r,0]$ into $K_c(E)$, $\mathfrak{L}_0^1$ is the space of all integrally bounded set-valued functions $X\colon [-r,0] \to K_c(E)$, $\Psi \in \mathcal{C}_0$ and D_H is the Hukuhara derivative. The continuous dependence of solutions on initial data and parameters is also studied.

Keywords: neutral type; existence; uniqueness; continous dependence

MSC 2010: 34K40, 34A12

1. Introduction

The study of set differential equations as an independent subject is relatively new. The first results in this area were obtained in [16], [5], [21]. Some recent results of interest can be found in [6], [14], [17], [19], [18], [20]. For more results, references and details we refer the reader to the book [13]. We also refer the reader to the first book [23] devoted exclusively to the subject of set differential equations on Banach spaces and their applications to differential inclusions with nonconvex right hand sides. The set differential equations with delay were studied in [1], [19], [22] and [24].

In this paper we are concerned with the set differential equation of neutral type

$$\begin{cases} D_H X(t) = F(t, X_t, D_H X_t), \\ X|_{[-r,0]} = \Psi, \end{cases}$$

where $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ is a given function, $K_c(E)$ is the family of all nonempty compact and convex subsets of a separable Banach space E, $\mathcal{C}_0$ denotes the space of all continuous set-valued functions X from $[-r,0]$ into $K_c(E)$, $\mathfrak{L}_0^1$ is the space of all integrally bounded set-valued functions $X\colon [-r,0] \to K_c(E)$, $\Psi \in \mathcal{C}_0$ and D_H is the Hukuhara derivative. The literature on related ordinary neutral differential equations is very extensive and we refer the reader to the book [8] for details. To our knowledge, there is no paper on set differential equations of neutral type. Some results for ordinary neutral differential equations in a finite dimensional Banach space were established in the papers [4], [12], and [11].

2. Preliminaries

In the following, E is a separable Banach space with the norm $\|\cdot\|$. We denote by $K_c(E)$ the family of all nonempty compact and convex subsets of E. By 0 we will denote the zero element of the space E. Also, θ will denote the null set-valued function $\theta\colon [a,b] \to K_c(E)$ defined by $\theta(t) = \{0\}$ for all $t \in [a,b]$. The Hausdorff-Pompeiu metric $\mathcal{H}$ on $K_c(E)$ is defined by

$$\mathcal{H}(A,B) = \max\Big\{ \sup_{x\in A} \inf_{y\in B} \|x-y\|,\ \sup_{y\in B} \inf_{x\in A} \|x-y\| \Big\}.$$

It is known in [2], [7] that $(K_c(E), \mathcal{H})$ is a complete and separable metric space. If $C([a,b], K_c(E))$ denotes the space of all continuous set-valued functions X from $[a,b]$ into $K_c(E)$, then it is well known that $C([a,b], K_c(E))$ is a complete and separable metric space with respect to the metric (see [10])

$$\mathcal{H}_{[a,b]}(X,Y) := \sup_{t\in[a,b]} \mathcal{H}(X(t), Y(t)).$$

From [23], we recall some notions in the theory of measurable set-valued functions. We denote by $\mu(\cdot)$ the Lebesgue measure on $[a,b]$. A set-valued function $X\colon [a,b] \to K_c(E)$ is called a *simple set-valued function* if it is constant on each of the sets $K_i \subset [a,b]$, $1 \leqslant i \leqslant n$, which produce a finite system of pairwise disjoint, Lebesgue measurable sets covering $[a,b]$. A set-valued function $X\colon [a,b] \to K_c(E)$ is called *strongly measurable* if it is almost everywhere (a.e.) in $[a,b]$ the pointwise limit of a sequence $X_n\colon [a,b] \to K_c(E)$, $n \geqslant 1$, of simple set-valued functions. We remark that the above definitions derive from standard notions for functions with values in a metric space. Then the notion of strong measurability is equivalent to the following *Luzin's property* (see [23]): For any $\varepsilon > 0$ there exists a closed set $K_\varepsilon \subset [a,b]$ with $\mu([a,b] \setminus K_\varepsilon) < \varepsilon$ and such that the restriction $X|_{K_\varepsilon}$ of X to K_ε is continuous. Note that the strong measurability of a set-valued function $X\colon [a,b] \to K_c(E)$ implies the measurability of X; that is, the set $X^{-1}(A) := \{t \in [a,b];\ X(t) \cap A \neq \emptyset\}$ is Lebesgue measurable for any closed set A. A set-valued function $X\colon [a,b] \to K_c(E)$ is called *integrally bounded* on $[a,b]$ if there exists a function $m(\cdot) \in L^1([a,b], \mathbb{R}_+)$ such that $\mathcal{H}(X(t), \{0\}) \leqslant m(t)$ a.e. on $[a,b]$. We denote by $\mathcal{M}([a,b], K_c(E))$ the set of all strongly measurable set-valued functions from $[a,b]$ to $K_c(E)$. Let us denote by $L^1([a,b], K_c(E))$ the space of all integrally bounded set-valued functions $X \in \mathcal{M}([a,b], K_c(E))$, where two multifunctions $X, Y \in L^1([a,b], K_c(E))$ are considered to be identical if $X(t) = Y(t)$ a.e. on $[a,b]$. Then $L^1([a,b], K_c(E))$ is a complete metric space with respect to the metric ([7], [10])

$$\mathcal{H}_{1,[a,b]}(X,Y) = \int_a^b \mathcal{H}(X(t), Y(t))\,\mathrm{d}t.$$

Also, we recall that a set-valued function $X\colon [a,b] \to K_c(E)$ is called *essentially bounded* on $[a,b]$ if there exists a constant $N > 0$ such that $\mathcal{H}(X(t), \{0\}) \leqslant N$ a.e. on $[a,b]$ (see [23]). We denote by $L^\infty([a,b], K_c(E))$ the space of all essentially bounded set-valued functions $X \in \mathcal{M}([a,b], K_c(E))$, where two multifunctions $X, Y \in L^1([a,b], K_c(E))$ are considered to be identical if $X(t) = Y(t)$ a.e. on $[a,b]$. Then $L^\infty([a,b], K_c(E))$ is a complete metric space with respect to the metric defined by (see [1])

$$\mathcal{H}_\infty(X,Y) := \inf\{N > 0;\ \mathcal{H}(X(t), Y(t)) < N \text{ a.e. on } [a,b]\}.$$

Next, for a given $N > 0$, let $L^{1,N}([a,b], K_c(E))$ be the space of all set-valued functions $X \in L^\infty([a,b], K_c(E))$ with $\mathcal{H}_\infty(X, \theta) \leqslant N$ and equipped with the metric $\mathcal{H}_{1,[a,b]}$.

Remark 2.1. From Theorem 1.4.5 and Theorem 2.2.5 in [15] (see also [23]) it follows that $L^1([a,b], K_c(E))$ can be regarded as the Banach space of vector-valued Bochner integrable functions, so that $L^1([a,b], K_c(E))$ is separable and the theory of Bochner integration can be applied to integrally bounded set-valued functions from $[a,b]$ into a given infinite dimensional Banach space.

Let $A, B \subset E$. The set $C \subset E$ satisfying $A = B + C$ is known as the *geometric difference* of the sets A and B and is denoted by $A - B$. We remark that

$A - A = \{0\}$ for any $A \subset E$. But, $A + (-1)A \neq \{0\}$ for any $A \subset E \setminus \{0\}$, where $(-1)A := \{-x;\ x \in A\}$.

We say that a set-valued mapping $X\colon [a,b] \to K_c(E)$ is *Hukuhara differentiable* (or *H-differentiable*) at a point $t_0 \in [a,b]$ if there exists $D_H X(t_0) \in K_c(E)$ such that the limits

$$\lim_{h\to 0^+} \frac{X(t_0+h) - X(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{X(t_0) - X(t_0-h)}{h}$$

exist with respect to the Hausdorff-Pompeiu metric and are equal to $D_H X(t_0)$. In this definition, we assume that both the differences $X(t_0+h) - X(t_0)$ and $X(t_0) - X(t_0-h)$ exist for sufficiently small $h > 0$ such that $t_0 + h$ and $t_0 - h$ both belong to $[a,b]$. A set $D_H X(t_0) \in K_c(E)$ is called the *H-derivative* of X at the point $t_0 \in [a,b]$. A set-valued mapping $X\colon [a,b] \to K_c(E)$ is called *H-differentiable* on $[a,b]$ if $D_H X(t)$ exists for each point $t \in [a,b]$. At the end points of $[a,b]$ we consider only the one-sided H-derivatives. The following three propositions are well known (see [16], [14], [17], [23]).

Proposition 2.2. *If $Y\colon [a,b] \to K_c(E)$ is continuous, then it is integrable on $[a,b]$. Moreover, in this case, the set-valued function $X\colon [a,b] \to K_c(E)$, defined by*

$$X(t) := X_0 + \int_a^t Y(s)\,\mathrm{d}s, \quad t \in [a,b],\ X_0 \in K_c(E), \tag{1}$$

is H-differentiable on $[a,b]$ and $D_H X(t) = Y(t)$ for $t \in [a,b]$.

Proposition 2.3. *Let $X\colon [a,b] \to K_c(E)$ be H-differentiable a.e. on $[a,b]$ and assume that $D_H X(t) \in L^1([a,b], K_c(E))$. Then for any $t \in [a,b]$ we have*

$$X(t) = X(\tau) + \int_\tau^t D_H X(s)\,\mathrm{d}s$$

for $\tau, t \in [a,b]$.

We recall that a mapping $X\colon [a,b] \to K_c(E)$ is said to be *absolutely continuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each family $\{(s_k, t_k);\ k = 1,2,\dots,n\}$ of disjoint open intervals in $[a,b]$ with $\sum_{k=1}^n (t_k - s_k) < \delta$, we have

$$\sum_{k=1}^n \mathcal{H}(X(t_k), X(s_k)) < \varepsilon.$$

We denote by $AC([a,b], K_c(E))$ the space of all absolutely continuous set-valued functions from $[a,b]$ into $K_c(E)$.

Proposition 2.4. *Let $X\colon\ [a,b] \to K_c(E)$ be an integrally bounded set-valued function. Then the set-valued function $X\colon\ [a,b] \to K_c(E)$ defined by (1) is absolutely continuous, $D_H X(t)$ exists a.e. on $[a,b]$, and $D_H X(t) = Y(t)$ a.e. on $[a,b]$.*

We denote by $A([a,b], K_c(E))$ the set of all set-valued functions $X \in AC([a,b], K_c(E))$ having the property that they are a.e. H-differentiable on $[a,b]$ and $D_H X \in L^1([a,b], K_c(E))$. It is easy to check that

$$\begin{aligned}\mathfrak{H}_{[a,b]}(X,Y) &:= \mathcal{H}_{[a,b]}(X,Y) + \mathcal{H}_{1,[a,b]}(D_H X, D_H Y)\\ &= \sup_{t\in[a,b]} \mathcal{H}(X(t),Y(t)) + \int_a^b \mathcal{H}(D_H X(t), D_H Y(t))\,\mathrm{d}t\end{aligned}$$

is a metric on $A([a,b], K_c(E))$.

Lemma 2.5. *$A([a,b], K_c(E))$ is a complete metric space with respect to the metric $\mathfrak{H}_{[a,b]}$.*

P r o o f. Let $\{X_n\}_{n\geqslant 1}$ be a Cauchy sequence in $A([a,b], K_c(E))$, i.e.,

$$\lim_{m,n\to\infty} \mathfrak{H}_{[a,b]}(X_m, X_n) = 0.$$

Then it follows that $\lim\limits_{m,n\to\infty} \mathcal{H}_{[a,b]}(X_m, X_n) = 0$. Since $C([a,b], K_c(E))$ is a complete metric space, there is a continuous set-valued function $X\colon\ [a,b] \to K_c(E)$ such that $\lim\limits_{n\to\infty} \mathcal{H}_{[a,b]}(X_n, X) = 0$. Further, since

$$\lim_{m,n\to\infty} \mathcal{H}_{1,[a,b]}(D_H X_m, D_H X_n) = 0,$$

$\{D_H X_n\}_{n\geqslant 1}$ is a Cauchy sequence in $L^1([a,b], K_c(E))$. Since $L^1([a,b], K_c(E))$ is a complete metric space, there is a continuous set-valued function $Y \in L^1([a,b], K_c(E))$ such that $\lim\limits_{n\to\infty} \mathcal{H}_{1,[a,b]}(D_H X_n, Y) = 0$. Moreover, for $t \in [a,b]$ we have

$$\begin{aligned}\mathcal{H}\Big(\int_a^t D_H X_n(s)\,\mathrm{d}s, \int_a^t Y(s)\,\mathrm{d}s\Big) &\leqslant \int_a^t \mathcal{H}(D_H X_n(s), Y(s))\,\mathrm{d}s\\ &\leqslant \mathcal{H}_{1,[a,b]}(D_H X_n, Y) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\mathcal{H}\Big(X(t), X(a) + \int_a^t Y(s)\,\mathrm{d}s\Big) &\leqslant \mathcal{H}(X(t), X_n(t)) + \mathcal{H}\Big(X_n(t), X(a) + \int_a^t Y(s)\,\mathrm{d}s\Big)\\ &= \mathcal{H}(X(t), X_n(t)) + \mathcal{H}\Big(X_n(a) + \int_a^t D_H X_n(s)\,\mathrm{d}s, X(a) + \int_a^t Y(s)\,\mathrm{d}s\Big)\\ &\leqslant \mathcal{H}(X(t), X_n(t)) + \mathcal{H}(X(a), X_n(a)) + \mathcal{H}\Big(\int_a^t D_H X_n(s)\,\mathrm{d}s, \int_a^t Y(s)\,\mathrm{d}s\Big) \to 0\end{aligned}$$

as $n \to \infty$, and so $X(t) = X(a) + \int_a^t Y(s)\,\mathrm{d}s$ for any $t \in [a, b]$. By Proposition 4 it follows that $X \in A([a, b], K_c(E))$. Obviously, $\lim\limits_{n\to\infty} \mathfrak{H}_{[a,b]}(X_n, X) = 0$, and the proof is complete. $\square$

Let $r > 0$ be given. In the following, for any $b > 0$ we will write $\mathcal{C}_{[b]}$, $\mathfrak{L}^1_{[b]}$, $\mathfrak{L}^{1,N}_{[b]}$ and $\mathcal{A}_{[b]}$ instead of $C([-r, b], K_c(E))$, $L^1([-r, b], K_c(E))$, $L^{1,N}([-r, b], K_c(E))$ and $A([-r, b], K_c(E))$, respectively. Then we write $\mathcal{H}_{[b]}$, $\mathcal{H}_{1,[b]}$ and $\mathfrak{H}_{[b]}$ instead of $\mathcal{H}_{[-r,b]}$, $\mathcal{H}_{1,[-r,b]}$ and $\mathfrak{H}_{[-r,b]}$, respectively. Also, for a given $t \in [0, b]$ we will write $\mathcal{C}_t$, $\mathfrak{L}^1_t$, $\mathfrak{L}^{1,N}_t$ and $\mathcal{A}_t$ instead of $C([t-r, t], K_c(E))$, $L^1([t-r, t], K_c(E))$, $L^{1,N}([t-r, t], K_c(E))$ and $A([t-r, t], K_c(E))$, respectively. Then we denote by $\mathcal{H}_t$, $\mathcal{H}_{1,t}$ and $\mathfrak{H}_t$ the metric on $\mathcal{C}_t$, $\mathfrak{L}^1_t$ and $\mathcal{A}_t$, respectively. Obviously, $X \in \mathcal{A}_{[b]}$ implies that $X \in \mathcal{A}_t$ for any $t \in [0, b]$.

If $t \in [0, b]$ and $X\colon [t-r, t] \to K_c(E)$ are given, then we define the set-valued function $X_t\colon [-r, 0] \to K_c(E)$ by $X_t(s) = X(t+s)$.

Lemma 2.6. *If $t \in [0, b]$ and $X \in \mathcal{A}_t$ are given, then $X_t \in \mathcal{A}_0$ and*

$$D_H X_t(s) = (D_H X)_t(s) \quad \text{for a.e. } s \in [-r, 0]. \tag{2}$$

P r o o f. First, we show that X_t is absolutely continuous on $[-r, 0]$. For this, let us remark that if $\{(s_k, t_k);\ k = 1, 2, \ldots, n\}$ is an arbitrary family of disjoint open intervals in $[-r, 0]$, then $\{(t+s_k, t+t_k);\ k = 1, 2, \ldots, n\}$ is a family of disjoint open intervals in $[t-r, t]$. Since X is absolutely continuous on $[t-r, t]$, hence for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} (t_k - s_k) = \sum_{k=1}^{n} [(t+t_k) - (t+s_k)] < \delta$$

implies

$$\sum_{k=1}^{n} \mathcal{H}(X_t(t_k), X_t(s_k)) = \sum_{k=1}^{n} \mathcal{H}(X(t+t_k), X(t+s_k)) < \varepsilon,$$

that is, X_t is absolutely continuous on $[-r, 0]$. Next, we show that X_t is a.e. H-differentiable on $[-r, 0]$ and (2) holds. Since X is a.e. H-differentiable on $[t-r, t]$ we have that

$$\lim_{h\to 0^+} \frac{X(t+s+h) - X(t+s)}{h} = D_H X(t+s),$$
$$\lim_{h\to 0^+} \frac{X(t+s) - X(t+s-h)}{h} = D_H X(t+s)$$

exist for a.e. $s \in [-r, 0]$. Let $s_0 \in [-r, 0]$ be such that the above limits exist. Then both the differences $X(t+s_0+h) - X(t+s_0)$ and $X(t+s_0) - X(t+s_0-h)$ exist for sufficiently small $h > 0$ such that $t + s_0 + h$ and $t + s_0 - h$ both belong to $[t - r, t]$. It follows that

$$\begin{aligned}
\lim_{h\to 0^+} \frac{X_t(s_0+h) - X_t(s_0)}{h} &= \lim_{h\to 0^+} \frac{X(t+s_0+h) - X(t+s_0)}{h} \\
&= D_H X(t+s_0) = (D_H X)_t(s_0), \\
\lim_{h\to 0^+} \frac{X_t(s_0) - X_t(s_0-h)}{h} &= \lim_{h\to 0^+} \frac{X(t+s_0) - X(t+s_0-h)}{h} \\
&= D_H X(t+s_0) = (D_H X)_t(s_0),
\end{aligned}$$

that is, X_t is H-differentiable at $s_0 \in [-r, 0]$. Hence (2) holds. Finally, we show that $D_H X_t \in \mathfrak{L}^1_0$. Since $X \in \mathcal{A}_t$, hence $D_H X \in \mathfrak{L}^1_t$, that is, $D_H X$ is strongly measurable and integrally bounded on $[t - r, t]$. Therefore, there exists a sequence $\{X^n\}_{n\geqslant 1}$ of simple set-valued functions from $[t - r, t]$ into $K_c(E)$ such that $\lim\limits_{n\to\infty} \mathcal{H}(X^n(\tau), D_H X(\tau)) = 0$ for a.e. $\tau \in [t - r, t]$ (see [23], page 2). Also, it is easy to check that $\{X^n_t\}_{n\geqslant 1}$ is a sequence of simple set-valued functions from $[-r, 0]$ into $K_c(E)$. It follows that

$$\lim_{n\to\infty} \mathcal{H}(X^n_t(s), D_H X_t(s)) = \lim_{n\to\infty} \mathcal{H}(X^n(t+s), D_H X(t+s)) = 0$$

for a.e. $s \in [-r, 0]$, that is, $D_H X_t$ is strongly measurable on $[-r, 0]$. Obviously, $D_H X_t$ is integrally bounded on $[-r, 0]$, and thus $D_H X_t \in \mathfrak{L}^1_0$. $\square$

Lemma 2.7. *If $t \in [0, b]$ is given, then $\mathfrak{H}_0(X_t, Y_t) = \mathfrak{H}_t(X, Y)$ for any $X, Y \in \mathcal{A}_t$. In particular, $\mathcal{H}_0(X_t, Y_t) = \mathcal{H}_t(X, Y)$ for any $X, Y \in \mathcal{C}_t$, and $\mathcal{H}_{1,0}(X_t, Y_t) = \mathcal{H}_{1,t}(X, Y)$ for any $X, Y \in \mathfrak{L}^1_t$.*

Proof. Indeed, we have

$$\begin{aligned}
\mathfrak{H}_0(X_t, Y_t) &= \sup_{s\in[-r,0]} \mathcal{H}(X_t(s), Y_t(s)) + \int_{-r}^{0} \mathcal{H}(D_H X_t(s), D_H Y_t(s))\,\mathrm{d}s \\
&= \sup_{s\in[-r,0]} \mathcal{H}(X(t+s), Y(t+s)) + \int_{-r}^{0} \mathcal{H}(D_H X(t+s), D_H Y(t+s))\,\mathrm{d}s \\
&= \sup_{\sigma\in[t-r,t]} \mathcal{H}(X(\sigma), Y(\sigma)) + \int_{t-r}^{t} \mathcal{H}(D_H X(\sigma), D_H Y(\sigma))\,\mathrm{d}\sigma,
\end{aligned}$$

that is, $\mathfrak{H}_0(X_t, Y_t) = \mathfrak{H}_t(X, Y)$. $\square$

3. Existence and uniqueness

In this section we consider the set differential equation of neutral type

$$\begin{cases} D_H X(t) = F(t, X_t, D_H X_t), \\ X|_{[-r,0]} = \Psi, \end{cases} \tag{3}$$

where $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ is a given function and $\Psi \in \mathcal{A}_0$. By a solution of the initial value problem (3) on some interval $[-r, T]$ we mean a set-valued function $X \in \mathcal{A}_{[T]}$ such that $X_0 = \Psi$ and $D_H X(t) = F(t, X_t, D_H X_t)$ for a.e. $t \in [0, T]$.

We say that $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ is a *Carathéodory set-valued function* if

(C_1) *for a.e.* $t \in [0,b]$, $F(t,\cdot,\cdot)$ *is continuous,*

(C_2) *for any* $(\Psi,\Phi) \in \mathcal{C}_0 \times \mathfrak{L}_0^1$, $F(\cdot,\Psi,\Phi)$ *is strongly measurable,*

(C_3) *for any bounded* $\mathcal{B} \subset \mathcal{C}_0 \times \mathfrak{L}_0^1$, *there exists an* $m(\cdot) \in L^1([0,b],\mathbb{R}_+)$ *such that*

$$\mathcal{H}(F(t,\Psi,\Phi),\{0\}) \leqslant m(t) \quad \textit{for a.e. } t \in [0,b] \textit{ and for any } (\Psi,\Phi) \in \mathcal{B}.$$

Lemma 3.1. *Let* $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ *be a Carathéodory set-valued function and let* $X \in \mathcal{A}_{[b]}$ *be given. Then*

(a) *the function* $U\colon [0,b] \to \mathcal{C}_0$ *defined by* $U(t) = X_t$ *is continuous on* $[0,b]$*;*

(b) *the function* $V\colon [0,b] \to \mathfrak{L}_0^1$ *defined by* $U(t) = D_H X_t$ *is strongly measurable and integrally bounded on* $[0,b]$*;*

(c) *the set-valued function* $t \mapsto F(t, X_t, D_H X_t)$ *is strongly measurable and integrally bounded on* $[0,b]$.

Proof. (a) Let $t_0 \in [0,b]$ be given and let $t_n \in [a,b]$, $n \geqslant 1$, be any sequence such that $t_n \to t_0$ as $n \to \infty$. Since X is uniformly continuous on $[-r,b]$, hence $\mathcal{H}_0(X_{t_n}, X_{t_0}) = \sup\limits_{-r \leqslant s \leqslant 0} \mathcal{H}(X(t_n+s), X(t_0+s)) \to 0$ as $n \to \infty$. It follows that $\mathcal{H}_0(U(t_n), U(t_0)) \to 0$ as $n \to \infty$, and so U is continuous on $[0,b]$.

(b) Since $D_H X \in \mathfrak{L}_{[b]}^1$, there exists a sequence $\{Y^n\}_{n \geqslant 1}$ of simple set-valued functions from $[-r,b]$ into $K_c(E)$ such that $\lim\limits_{n\to\infty} \mathcal{H}(Y^n(t), D_H X(t)) = 0$ for a.e. $t \in [-r,b]$. Obviously, $\{Y_t^n\}_{n \geqslant 1}$ is a sequence of simple set-valued functions from $[-r,0]$ into $\mathfrak{L}_0^1$ and

$$\mathcal{H}_{1,0}(Y_t^n, D_H X_t) = \int_{-r}^0 \mathcal{H}(Y^n(t+s), D_H X(t+s))\,\mathrm{d}s \to 0$$

for a.e. $t \in [0,b]$. It follows that V is strongly measurable on $[0,b]$. Moreover, since $D_H X \in \mathfrak{L}^1_{[b]}$ and

$$\begin{aligned}\int_0^b \mathcal{H}_{1,0}(D_H X_t, \theta)\,\mathrm{d}t &= \int_0^b \Big(\int_{-r}^0 \mathcal{H}(D_H X(t+s), \{0\})\,\mathrm{d}s\Big)\,\mathrm{d}t\\ &= \int_0^b \Big(\int_{t-r}^t \mathcal{H}(D_H X(\tau), \{0\})\,\mathrm{d}\tau\Big)\,\mathrm{d}t\\ &\leqslant \int_0^b \Big(\int_{-r}^b \mathcal{H}(D_H X(\tau), \{0\})\,\mathrm{d}\tau\Big)\,\mathrm{d}t\\ &= b\int_{-r}^b \mathcal{H}(D_H X(\tau), \{0\})\,\mathrm{d}\tau,\end{aligned}$$

V is integrally bounded on $[0,b]$.

(c) Let $\varepsilon > 0$ be given. Since $\mathcal{C}_0 \times \mathfrak{L}^1_0$ and $K_c(E)$ are complete separable metric spaces and $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}^1_0 \to K_c(E)$ is a Carathéodory set-valued function, there exists a closed set $K^1_\varepsilon \subset [0,b]$ with $\mu([0,b] \setminus K^1_\varepsilon) < \varepsilon$ and such that the restriction $F|_{K^1_\varepsilon \times \mathcal{C}_0 \times \mathfrak{L}^1_0}$ of F to $K^1_\varepsilon \times \mathcal{C}_0 \times \mathfrak{L}^1_0$ is continuous (see [9]). Since $t \mapsto V(t) = D_H X_t\colon [0,b] \to \mathfrak{L}^1_0$ is strongly measurable, by Luzin's property there exists a closed set $K^2_\varepsilon \subset [0,b]$ with $\mu([0,b] \setminus K^2_\varepsilon) < \varepsilon$ and such that the restriction $V|_{K^2_\varepsilon}$ of V to K^2_ε is continuous. Let $K_\varepsilon := K^1_\varepsilon \cap K^2_\varepsilon$. Then K_ε is a closed subset of $[0,b]$ with $\mu([0,b] \setminus K_\varepsilon) < \varepsilon$ and such that the restriction $F(\cdot, U(\cdot), V(\cdot))|_{K_\varepsilon}$ of $F(\cdot, U(\cdot), V(\cdot))$ to K_ε is continuous. From Luzin's property it follows that $t \mapsto F(t, U(t), V(t))$ is strongly measurable on $[0,b]$. Obviously, $t \mapsto F(t, U(t), V(t))$ is integrally bounded on $[0,b]$. This completes the proof. $\square$

Remark 3.2. If $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}^1_0 \to K_c(E)$ is a Carathéodory set-valued function, then using Propositions 2.3 and 2.4 it is easy to show that a set-valued function $X \in \mathcal{A}_{[T]}$ is a solution of (3) on an interval $[-r, T]$ if and only if

$$X(t) = \begin{cases} \Psi(t) & \text{if } -r \leqslant t \leqslant 0,\\ \Psi(0) + \int_0^t F(s, X_s, D_H X_s)\,\mathrm{d}s & \text{if } \quad 0 \leqslant t \leqslant T. \end{cases} \tag{4}$$

For a given set-valued function $\Psi \in \mathcal{A}_0$, let $\Psi^0\colon [-r,b] \to K_c(E)$ be the set-valued function defined by

$$\Psi^0(t) := \begin{cases} \Psi(t) & \text{if } t \in [-r,0],\\ \Psi(0) & \text{if } t \in [0,b]. \end{cases} \tag{5}$$

Then it is easy to see that $\Psi^0 \in \mathcal{A}_{[b]}$. For given $\varrho > 0$ and $\Psi \in \mathcal{A}_0$, let

$$\mathcal{B}_\varrho(\Psi^0) := \{X \in \mathcal{A}_{[b]}\,;\ \mathfrak{H}_{[b]}(X, \Psi^0) \leqslant \varrho\}.$$

Theorem 3.3. *Suppose that* $F\colon [0,b]\times\mathcal{C}_0\times\mathfrak{L}_0^1\to K_c(E)$ *satisfies the conditions* (C_1) *and* (C_3) *and the following locally Lipschitz type condition: for any bounded set* $\mathcal{B}\subset\mathcal{A}_{[b]}$ *there exists* $L>0$ *such that*

$$\mathcal{H}(F(t,X_t,D_HX_t),F(t,Y_t,D_HY_t))\leqslant L\mathfrak{H}_t(X,Y) \tag{6}$$

for any $t\in[a,b]$ *and* $X,Y\in\mathcal{B}$. *Then, for every* $\Psi\in\mathcal{A}_0$, *there exists a unique solution* $X\colon[-r,T]\to K_c(E)$ *for the initial value problem* (3) *on some interval* $[-r,T]$ *with* $T\in(0,b]$.

Proof. Let $\varrho=\mathfrak{H}_{[0]}(\Psi,\theta)$ and let $L>0$ be such that (6) holds for for any $t\in[a,b]$ and $X,Y\in\mathcal{B}_\varrho(\Psi^0)$. From (C_3) it follows that there is an $m(\cdot)\in L^1([0,b],\mathbb{R}_+)$ such that $\mathcal{H}(F(t,X_t,D_HX_t),\{0\})\leqslant m(t)$ for a.e. $t\in[0,b]$ and any $X\in\mathcal{B}_\varrho(\Psi^0)$. We choose $T\in(0,b]$ such that $\int_0^T m(t)\,\mathrm{d}t<\varrho/2$. Then $\Psi^0\in\mathcal{A}_{[T]}$ and $\mathfrak{H}_{[T]}(\Psi^0,\theta)\leqslant\varrho$. Further, consider the set $\mathcal{N}_\varrho$ defined by

$$\mathcal{N}_\varrho:=\{X\in\mathcal{A}_{[T]};\ X_0=\Psi \text{ and } \mathfrak{H}_{[T]}(X,\Psi^0)\leqslant\varrho\}.$$

We remark that if $X\in\mathcal{N}_\varrho$, then $\mathfrak{H}_t(X,\theta)\leqslant\varrho$ for any $t\in[0,T]$. Further, let us consider the following successive approximation of absolutely continuous set-valued functions:

$$X^0(t)=\Psi^0(t),\quad t\in[0,T]$$

and

$$X^n(t)=\begin{cases}\Psi(t) & \text{if } -r\leqslant t\leqslant 0,\\ \Psi(0)+\int_0^t F(s,X_s^{n-1},D_HX_s^{n-1})\,\mathrm{d}s & \text{if } \quad 0\leqslant t\leqslant T,\end{cases}$$

for $n\geqslant 1$. We show that $X^n\in\mathcal{N}_\varrho$ for any $n\geqslant 1$. Obviously, $X^0\in\mathcal{N}_\varrho$. Let us assume that $X^1,X^2,\ldots,X^n\in\mathcal{N}_\varrho$. Then

$$\begin{aligned}\mathcal{H}(X^{n+1}(t),\Psi^0(t))&\leqslant\int_0^t\mathcal{H}(F(s,X_s^n,D_HX_s^n,\theta)\,\mathrm{d}s\\ &\leqslant\int_0^T m(t)\,\mathrm{d}t<\frac{\varrho}{2},\end{aligned}$$

and so

$$\mathcal{H}_{[T]}(X^{n+1},\Psi^0)=\sup_{-r\leqslant t\leqslant T}\mathcal{H}(X^{n+1}(t),\Psi^0(t))<\frac{\varrho}{2}.$$

Also

$$\begin{aligned}\int_{-r}^T\mathcal{H}(D_HX^{n+1}(t),D_H\Psi^0(t))\,\mathrm{d}t&=\int_0^T\mathcal{H}(F(s,X_s^n,D_HX_s^n),\theta)\,\mathrm{d}t\\ &\leqslant\int_0^T m(t)\,\mathrm{d}t<\frac{\varrho}{2}.\end{aligned}$$

It follows that

$$\mathfrak{H}_{[T]}(X^{n+1},\Psi^0)=\sup_{-r\leqslant t\leqslant T}\mathcal{H}(X^{n+1}(t),\Psi^0(t))\\
+\int_{-r}^{T}\mathcal{H}(D_HX^{n+1}(t),D_H\Psi^0(t))\,\mathrm{d}t<\varrho,$$

and thus $X^{n+1}\in\mathcal{N}_\varrho$. By mathematical induction it follows that $X^n\in\mathcal{N}_\varrho$ for any $n\geqslant 1$. Next, by (6) and Lemma 2.7, we have

$$\mathcal{H}(X^{n+1}(t),X^n(t))\leqslant\int_0^t\mathcal{H}(F(s,X_s^n,D_HX_s^n),F(s,X_s^{n-1},D_HX_s^{n-1}))\,\mathrm{d}s\\
\leqslant L\int_0^t\mathfrak{H}_0(X_s^n,X_s^{n-1})\,\mathrm{d}s=L\int_0^t\mathfrak{H}_s(X^n,X^{n-1})\,\mathrm{d}s$$

and

$$\int_{t-r}^t\mathcal{H}(D_HX^{n+1}(s),D_HX^n(s))\,\mathrm{d}s=\int_0^t\mathcal{H}(F(s,X_s^n,D_HX_s^n),F(s,X_s^{n-1},D_HX_s^{n-1}))\,\mathrm{d}s\\
\leqslant L\int_0^t\mathfrak{H}_0(X_s^n,X_s^{n-1})\,\mathrm{d}s=L\int_0^t\mathfrak{H}_s(X^n,X^{n-1})\,\mathrm{d}s.$$

Therefore,

$$\mathfrak{H}_t(X^{n+1},X^n)=\sup_{t-r\leqslant\tau\leqslant t}\mathcal{H}(X^{n+1}(\tau),X^n(\tau))\\
+\int_{t-r}^t\mathcal{H}(D_HX^{n+1}(\tau),D_HX^n(\tau))\,\mathrm{d}\tau\\
\leqslant 2L\int_0^t\mathfrak{H}_s(X^n,X^{n-1})\,\mathrm{d}s.$$

Let us consider the sequence of real functions $\{g_n\}_{n\geqslant 1}$ given by $g_n(t)=\mathfrak{H}_t(X^n, X^{n-1})$, $n\geqslant 1$, $t\in[0,T]$. Then $g_{n+1}(t)\leqslant 2L\int_0^t g_n(s)\,\mathrm{d}s$ for $n\geqslant 1$ and $t\in[0,T]$. Since $g_1(t)=\mathfrak{H}_t(X^1,X^0)\leqslant\varrho$, the last inequality implies that $g_n(t)\leqslant\varrho(2Lt)^n/n!$, $n\geqslant 1$, $t\in[0,T]$, and thus

$$\lim_{n\to\infty}\mathfrak{H}_t(X^n,X^{n-1})=0\quad\text{for } t\in[0,T].$$

Further, since

$$
\begin{aligned}
&\mathfrak{H}_{[T]}(X^n, X^{n-1}) \\
&\quad = \sup_{-r\leqslant\tau\leqslant T} \mathcal{H}(X^n(\tau), X^{n-1}(\tau)) + \int_{-r}^{T} \mathcal{H}(D_H X^n(\tau), D_H X^{n-1}(\tau))\,\mathrm{d}\tau \\
&\quad = \sup_{-r\leqslant\tau\leqslant T} \mathcal{H}(\widehat{X}^n(\tau), \widehat{X}^{n-1}(\tau)) + \int_{-r}^{T} \mathcal{H}(D_H \widehat{X}^n(\tau), D_H \widehat{X}^{n-1}(\tau))\,\mathrm{d}\tau \\
&\quad = \sup_{T-r\leqslant\sigma\leqslant 2T} \mathcal{H}(\widehat{X}^n(\sigma), \widehat{X}^{n-1}(\sigma)) + \int_{T-r}^{2T} \mathcal{H}(D_H \widehat{X}^n(\sigma), D_H \widehat{X}^{n-1}(\sigma))\,\mathrm{d}\sigma \\
&\quad = \sup_{T-r\leqslant\sigma\leqslant T} \mathcal{H}(X^n(\sigma), X^{n-1}(\sigma)) + \int_{T-r}^{T} \mathcal{H}(D_H X^n(\sigma), D_H X^{n-1}(\sigma))\,\mathrm{d}\sigma,
\end{aligned}
$$

it follows that $\mathfrak{H}_{[T]}(X^n, X^{n-1}) = \mathfrak{H}_T(X^n, X^{n-1}) \leqslant \varrho(2LT)^n/n!$, $n \geqslant 1$, and $\mathfrak{H}_{[T]}(X^n, X^{n-1}) = \mathfrak{H}_T(X^n, X^{n-1}) \to 0$ as $n \to \infty$. Further, for any $m > n$ we have

$$
\begin{aligned}
\mathfrak{H}_{[T]}(X^m, X^n) &\leqslant \mathfrak{H}_{[T]}(X^m, X^{m-1}) + \mathfrak{H}_{[T]}(X^{m-1}, X^{m-2}) + \ldots + \mathfrak{H}_{[T]}(X^{n+1}, X^n) \\
&\leqslant \varrho \sum_{k=n}^{m-1} \frac{(2LT)^{k+1}}{(k+1)!}.
\end{aligned}
$$

Since this last sum is part of the series for e^{2LT}, it follows that we can make $\mathfrak{H}_{[T]}(X^m, X^n)$ less than any $\varepsilon > 0$ by taking n sufficiently large. Therefore, $\{X^n\}_{n\geqslant 1}$ is a Cauchy sequence in $\mathcal{A}_{[T]}$. Due to Lemma 2.5 there exists a set-valued function $X \in \mathcal{A}_{[T]}$ such that $X|_{[-r,0]} = \Psi$ and $\lim_{n\to\infty} \mathfrak{H}_{[T]}(X^n, X) = 0$. Moreover, since $\mathfrak{H}_{[T]}(X^n, \Psi^0) < \varrho$, $n \geqslant 1$, it is easy to see that $\mathfrak{H}_{[T]}(X, \Psi^0) \leqslant \mathfrak{H}_{[T]}(X^n, X) + \mathfrak{H}_{[T]}(X^n, \Psi^0)$ implies $\mathfrak{H}_{[T]}(X, \Psi^0) \leqslant \varrho$, that is, $X \in \mathcal{N}_\varrho$. Next, from (6) we have that

$$
\begin{aligned}
&\mathcal{H}\Big(\int_0^t F(\tau, X_\tau^n, D_H X_\tau^n)\,\mathrm{d}\tau, \int_0^t F(\tau, X_\tau, D_H X_\tau)\,\mathrm{d}\tau\Big) \\
&\qquad \leqslant \int_0^t \mathcal{H}(F(\tau, X_\tau^n, D_H X_\tau^n), F(\tau, X_\tau, D_H X_\tau))\,\mathrm{d}\tau \leqslant L\int_0^t \mathfrak{H}_0(X_\tau^n, X_\tau)\,\mathrm{d}\tau \\
&\qquad \leqslant L\int_0^T \mathfrak{H}_\tau(X^n, X)\,\mathrm{d}\tau \leqslant L\int_0^T \mathfrak{H}_{[T]}(X^n, X)\,\mathrm{d}\tau \to 0
\end{aligned}
$$

as $n \to \infty$. Then, we obtain that

$$
\begin{aligned}
&\lim_{n\to\infty} \mathcal{H}\Big(X^n(t), \Psi(0) + \int_0^t F(\tau, X_\tau, D_H X_\tau)\,\mathrm{d}\tau\Big) \\
&\qquad \leqslant \lim_{n\to\infty} \mathcal{H}\Big(\int_0^t F(\tau, X_\tau^n, D_H X_\tau^n)\,\mathrm{d}\tau, \int_0^t F(\tau, X_\tau, D_H X_\tau)\,\mathrm{d}\tau\Big) = 0.
\end{aligned}
$$

It follows that

$$X(t) = \begin{cases} \Psi(t) & \text{if } -r \leqslant t \leqslant 0, \\ \Psi(0) + \int_0^t F(\tau, X_\tau, D_H X_\tau)\,\mathrm{d}\tau & \text{if } \quad 0 \leqslant t \leqslant T, \end{cases}$$

which represents a solution of (3) on $[0,T]$. We shall show now that (3) has exactly one solution $X \in \mathcal{A}_{[T]}$. Suppose that X, $Y \in \mathcal{A}_{[T]}$ are two solutions of (3). Then we have that

$$\begin{aligned}\mathcal{H}(X(t), Y(t)) &\leqslant \int_0^t \mathcal{H}(F(\tau, X_\tau, (D_H X)_\tau), F(\tau, Y_\tau, (D_H Y)_\tau))\,\mathrm{d}\tau \\ &\leqslant L \int_0^t \mathfrak{H}_0(X_\tau, Y_\tau)\,\mathrm{d}\tau = L \int_0^t \mathfrak{H}_\tau(X, Y)\,\mathrm{d}\tau\end{aligned}$$

and

$$\begin{aligned}\int_{t-r}^t \mathcal{H}(D_H X(\tau), D_H Y(\tau))\,\mathrm{d}\tau &= \int_0^t \mathcal{H}(D_H X(\tau), D_H Y(\tau))\,\mathrm{d}\tau \\ &\leqslant \int_0^t \mathcal{H}(F(\tau, X_\tau, D_H X_\tau), F(\tau, Y_\tau, D_H Y_\tau))\,\mathrm{d}\tau \\ &\leqslant L \int_0^t \mathfrak{H}_\tau(X, Y)\,\mathrm{d}\tau.\end{aligned}$$

It follows that

$$\mathfrak{H}_t(X, Y) \leqslant 2L \int_0^t \mathfrak{H}_\tau(X, Y)\,\mathrm{d}\tau, \quad t \in [0, T],$$

and Gronwall's lemma implies that $\mathfrak{H}_t(X,Y) = 0$ for $t \in [0,T]$. By Lemma 2.7, we obtain $\mathfrak{H}_0(X_t, Y_t) = 0$ for $t \in [0,T]$, that is, $X_t = Y_t$ for $t \in [0,T]$. Therefore, $X(t) = X_t(0) = Y_t(0) = Y(t)$ for $t \in [0,T]$, and hence (3) has a unique solution. This completes the proof. □

Remark 3.4. It is easy to see that the result of Theorem 3.3 remanins also true if the Lipschitz type condition (6) is satisfied on any bounded set $\mathcal{B} \subset \mathcal{A}_{[b]}^N$, where $\mathcal{A}_{[b]}^N$ is the set of all set-valued functions $X \in \mathcal{A}_{[b]}$ with $D_H X \in \mathfrak{L}_{[b]}^{1,N}$.

Theorem 3.5. *Suppose that $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ satisfies all the conditions of Theorem* 3.3. *Then the largest interval of existence of the solution of* (3) *is* $[0,b]$.

P r o o f. Let $X\colon [-r,\beta) \to K_c(E)$ be the solution of (3) existing on $[-r,\beta)$, $0 < \beta < b$. Also, we suppose, by contradiction, that the value of β cannot be increased. Let us consider $0 \leqslant s < t < \beta$. Then we have

$$\mathcal{H}(X(t), X(s)) \leqslant \int_s^t \mathcal{H}(F(\tau, X_\tau, D_H X_\tau), \theta)\,\mathrm{d}\tau \leqslant \int_s^t m(\tau)\,\mathrm{d}\tau.$$

Since $m(\cdot) \in L^1([0,\beta],\mathbb{R}_+)$ we have $\int_s^t m(s)\,\mathrm{d}s \to 0$ as $s,t \to \beta^-$, which implies that $\lim\limits_{t\to\beta^-} X(t)$ exists. Hence, if we take $X(\beta) = \lim\limits_{t\to\beta^-} X(t)$, then the function X can be extended by continuity to $[0,\beta]$. Further, consider the initial value problem

$$\begin{cases} D_H Y(t) = G(t, Y_t, D_H Y_t), & 0 \leqslant t < b-\beta, \\ Y|_{[-(\sigma+\beta),0]} = \Phi \end{cases}$$

where $G(t, Y_t, D_H Y_t) = F(t+\beta, Y_{t+\beta}, D_H Y_{t+\beta})$ for $0 \leqslant t < b-\beta$ and Φ is defined by $\Phi(s) = X(s+\beta)$ for $s \in [-(\sigma+\beta),0]$. By Theorem 3.3, there exists a solution $Y\colon [-(\sigma+\beta),\widetilde{\beta}) \to E$ of the initial value problem (3), where $\widetilde{\beta} \in (0, b-\beta]$. It follows that $\widetilde{X}\colon [-r,\beta+\widetilde{\beta}] \to E$, given by

$$\widetilde{X}(t) = \begin{cases} X(t), & \text{for } t \in [-r,\beta], \\ Y(t-\beta), & \text{for } t \in [\beta,\beta+\widetilde{\beta}], \end{cases}$$

is a solution of the initial value problem (3). Therefore, the solution X can be continued beyond β, contradicting the assumption that β cannot be increased. This contradiction completes the proof. □

4. Continuous dependence

For a given $\Psi \in \mathcal{A}_0$ and a set-valued function $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ which satisfies the conditions of Theorem 3.3 let us denote by $X(t,\Psi,F,\beta)$ the unique solution on $[-r,\beta]$ of the initial value problem (3).

Theorem 4.1. *Suppose that the set-valued functions $F,G\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ satisfy the conditions of Theorem 3.3 and that there exists a $\lambda \geqslant 0$ such that*

$$\mathcal{H}_0(F(t,\Psi,\Phi), G(t,\Psi,\Phi)) \leqslant \lambda$$

for all $(t,\Psi,\Phi) \in [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1$. Then for any $\Psi,\Phi \in \mathcal{A}_0$ we have

$$\mathfrak{H}_t(X,Y) \leqslant 2\mathfrak{H}_0(\Psi,\Phi)\mathrm{e}^{Lt} + \frac{2\lambda}{L}(\mathrm{e}^{Lt}-1) \quad \text{for } 0 \leqslant t \leqslant \beta,$$

where $X(\cdot) = X(\cdot,\Psi,F,\beta_1)$, $Y(\cdot) = X(\cdot,\Psi,G,\beta_2)$ and $\beta = \min(\beta_1,\beta_2)$.

P r o o f. For any $t \in [0,\beta]$ we have

$$\begin{aligned}
\mathcal{H}(X(t),Y(t)) &\leqslant \mathcal{H}(\Psi(0),\Phi(0)) + \int_0^t \mathcal{H}(F(s,X_s,D_H X_s),G(s,Y_s,D_H Y_s))\,\mathrm{d}s \\
&\leqslant \mathcal{H}_0(\Psi,\Phi) + \int_0^t \mathcal{H}(F(s,X_s,D_H X_s),F(s,Y_s,D_H Y_s))\,\mathrm{d}s \\
&\quad + \int_0^t \mathcal{H}(F(s,Y_s,D_H Y_s),G(s,Y_s,D_H Y_s))\,\mathrm{d}s \\
&\leqslant \mathcal{H}_0(\Psi,\Phi) + L\int_0^t \mathfrak{H}_0(X_s,Y_s)\,\mathrm{d}s + \int_0^t \lambda\,\mathrm{d}s \\
&= \mathcal{H}_0(\Psi,\Phi) + \lambda t + L\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s.
\end{aligned}$$

It follows that

$$\sup_{s\in[0,t]} \mathcal{H}(X(s),Y(s)) \leqslant \mathcal{H}_0(\Psi,\Phi) + \lambda t + L\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s.$$

Since

$$\begin{aligned}
\sup_{s\in[t-r,0]} \mathcal{H}(X(s),Y(s)) &= \sup_{s\in[t-r,0]} \mathcal{H}(\Psi(s),\Phi(s)) \\
&\leqslant \sup_{s\in[-r,0]} \mathcal{H}(\Psi(s),\Phi(s)) = \mathcal{H}_0(\Psi,\Phi),
\end{aligned}$$

we obtain that

$$\sup_{s\in[t-r,t]} \mathcal{H}(X(s),Y(s)) \leqslant 2\mathcal{H}_0(\Psi,\Phi) + \lambda t + L\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s,$$

that is,

$$\mathcal{H}_t(X,Y) \leqslant 2\mathcal{H}_0(\Psi,\Phi) + \lambda t + L\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s. \tag{7}$$

Further,

$$\begin{aligned}
\int_0^t \mathcal{H}(D_H X_s, D_H Y_s)\,\mathrm{d}s &= \int_0^t \mathcal{H}(F(s,X_s,D_H X_s),G(s,Y_s,D_H Y_s))\,\mathrm{d}s \\
&\leqslant \int_0^t \mathcal{H}(F(s,X_s,D_H X_s),F(s,Y_s,D_H Y_s))\,\mathrm{d}s \\
&\quad + \int_0^t \mathcal{H}(F(s,Y_s,D_H Y_s),G(s,Y_s,D_H Y_s))\,\mathrm{d}s \\
&\leqslant L\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s + \int_0^t \lambda\,\mathrm{d}s
\end{aligned}$$

and

$$\int_{t-r}^{0} \mathcal{H}(D_H X(s), D_H Y(s))\,\mathrm{d}s \leqslant \int_{-r}^{0} \mathcal{H}(D_H \Psi(s), D_H \Phi(s))\,\mathrm{d}s = \mathcal{H}_{1,0}(D_H \Psi, D_H \Phi).$$

Thus, we obtain

$$\begin{aligned}(8)\qquad \int_{t-r}^{t} \mathcal{H}(D_H X_s, D_H Y_s)\,\mathrm{d}s &\leqslant \int_{t-r}^{0} \mathcal{H}(D_H X_s, D_H Y_s)\,\mathrm{d}s + \int_{0}^{t} \mathcal{H}(D_H X_s, D_H Y_s)\,\mathrm{d}s \\ &\leqslant \mathcal{H}_{1,0}(D_H \Psi, D_H \Phi) + \lambda t + L\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s.\end{aligned}$$

By using (7) and (8), we obtain

$$\mathfrak{H}_t(X,Y) \leqslant 2\mathfrak{H}_0(\Psi,\Phi) + 2\lambda t + 2L\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s.$$

Applying Gronwall's Lemma [3] yields

$$\mathfrak{H}_t(X,Y) \leqslant 2\mathfrak{H}_0(\Psi,\Phi)\mathrm{e}^{2Lt} + \frac{\lambda}{L}(\mathrm{e}^{2Lt} - 1)$$

for $t \in [0,\beta]$ and this completes the proof. □

Corollary 4.2. *Let $\Psi,\Phi \in \mathcal{A}_0$. If $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ satisfies the conditions of Theorem* 3.3*, then*

$$\mathfrak{H}_t(X,Y) \leqslant 2\mathfrak{H}_0(\Psi,\Phi)\mathrm{e}^{2Lt} \quad \textit{for } 0 \leqslant t \leqslant \beta,$$

where $X(\cdot) = X(\cdot,\Psi,F,\beta_1)$, $Y(\cdot) = X(\cdot,\Phi,F,\beta_2)$ and $\beta = \min(\beta_1,\beta_2)$.

Now consider the functional differential equation with a parameter

$$(9)\qquad \begin{cases} D_H X(t) = F(t, X_t, D_H X_t, \Lambda), \\ X|_{[-r,0]} = \Psi, \end{cases}$$

where $F\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \times K_c(E_1) \to K_c(E)$ and E_1 is a real Banach space. In the following we suppose:

(H1) For each $\Lambda \in K_c(E_1)$ the set-valued function $F(\cdot,\cdot,\cdot,\Lambda)\colon [0,b] \times \mathcal{C}_0 \times \mathfrak{L}_0^1 \to K_c(E)$ satisfies the conditions (C_1) and (C_3).

(H2) For any bounded set $\mathcal{B} \times \mathcal{B}_1 \subset \mathcal{A}_{[b]} \times K_c(E_1)$ there exists $L > 0$ such that

$$\mathcal{H}(F(t, X_t, D_H X_t, \Lambda), F(t, Y_t, D_H Y_t, \Omega) \leqslant L[\mathfrak{H}_t(X,Y) + \mathcal{H}(\Lambda,\Omega)]$$

for any $t \in [a,b]$, $X, Y \in \mathcal{B}$ and $\Lambda, \Omega \in \mathcal{B}_1$.

For each $\Lambda \in \mathcal{B}_1$, the existence of a unique solution of (9) is ensured by Theorem 3.3.

Theorem 4.3. *Suppose that the set-valued function* $F\colon [0,b]\times\mathcal{C}_0\times\mathfrak{L}_0^1\times K_c(E_1) \to K_c(E)$ *satisfies the conditions* (H1) *and* (H2). *For* $\Psi, \Phi \in \mathcal{A}_0$ *we denote by* $X(\cdot) = X(\cdot,\Psi,\Lambda)$ *and* $Y(\cdot) = Y(\cdot,\Phi,\Omega)$ *the solution of* (9) *corresponding to parameters* Λ *and* Ω*, respectively, on* $[0,\beta]$, $\beta \leqslant b$. *Then we have*

$$\mathfrak{H}_t(X,Y) \leqslant [2\mathfrak{H}_0(\Psi,\Phi) + 2K\beta\mathcal{H}(\Lambda,\Omega)]\mathrm{e}^{2Kt} \quad \textit{for } 0 \leqslant t \leqslant \beta.$$

Proof. From Remark 3.2 we have that

$$X(t) = \begin{cases} \Psi(t) & \text{if } -r \leqslant t \leqslant 0,\\ \Psi(0) + \int_0^t F(\tau, X_\tau, D_H X_\tau, \Lambda)\,\mathrm{d}\tau & \text{if } \quad 0 \leqslant t \leqslant \beta, \end{cases}$$

and

$$Y(t) = \begin{cases} \Phi(t) & \text{if } -r \leqslant t \leqslant 0,\\ \Phi(0) + \int_0^t F(\tau, Y_\tau, D_H Y_\tau, \Omega)\,\mathrm{d}\tau & \text{if } \quad 0 \leqslant t \leqslant \beta. \end{cases}$$

Let $t \in [0,\beta]$. Proceeding exactly in the same way as in Theorem 4.1, we obtain

$$\mathcal{H}_t(X,Y) \leqslant 2\mathfrak{H}_0(\Psi,\Phi) + K\beta\mathcal{H}(\Lambda,\Omega) + K\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s \tag{10}$$

and

$$\begin{aligned} \int_{t-r}^t &\mathcal{H}(D_H X_s, D_H Y_s)\,\mathrm{d}s \\ &\leqslant \mathcal{H}_{1,0}(D_H\Psi, D_H\Phi) + K\beta\mathcal{H}(\Lambda,\Omega) + K\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s. \end{aligned} \tag{11}$$

Using (10) and (11), we obtain

$$\mathfrak{H}_t(X,Y) \leqslant 2\mathfrak{H}_0(\Psi,\Phi) + 2K\beta\mathcal{H}(\Lambda,\Omega) + 2K\int_0^t \mathfrak{H}_s(X,Y)\,\mathrm{d}s.$$

Applying Gronwall's Lemma [3] yields

$$\mathfrak{H}_t(X,Y) \leqslant [2\mathfrak{H}_0(\Psi,\Phi) + 2K\beta\mathcal{H}(\Lambda,\Omega)]\mathrm{e}^{2Kt}$$

for $t \in [0,\beta]$ and this completes the proof. $\square$

5. Examples

1. For $b, r > 0$ we consider the set differential equation of neutral type

$$(12)\qquad \begin{cases} D_H X(t) = \displaystyle\int_{-r}^{0} \frac{t}{1+(t+s)^2} X(t+s)\,\mathrm{d}s \\ \qquad\qquad + \displaystyle\int_{-r}^{0} \frac{t+s}{1+s^2+t^2} D_H X(t+s)\,\mathrm{d}s, \quad t \in [0,b], \\ X|_{[-r,0]} = \Psi, \end{cases}$$

where $\Psi \in \mathcal{A}_0$ is given. For any $X \in \mathcal{A}_{[b]}$, let

$$F(t, X_t, D_H X_t) = \int_{-r}^{0} \frac{t}{1+(t+s)^2} X_t(s)\,\mathrm{d}s + \int_{-r}^{0} \frac{1+t}{1+s^2+t^2} D_H X_t(s)\,\mathrm{d}s,\ t \in [0,b].$$

Obviously, $t \mapsto F(t, X_t, D_H X_t)$ is strongly measurable for each fixed $X \in \mathcal{A}_{[b]}$. Next, for any $X, Y \in \mathcal{A}_{[b]}$ and $t \in [0,b]$, we have that

$$\begin{aligned} &\mathcal{H}(F(t, X_t, D_H X_t), F(t, Y_t, D_H Y_t)) \\ &\quad \leqslant \int_{-r}^{0} \frac{t}{1+(t+s)^2} \mathcal{H}(X_t(s), Y_t(s))\,\mathrm{d}s + \int_{-r}^{0} \frac{1+t}{1+s^2+t^2} \mathcal{H}(D_H X_t(s), D_H X_t(s))\,\mathrm{d}s \\ &\quad \leqslant br\mathcal{H}_0(X_t, Y_t) \quad + (1+b) \int_{-r}^{0} \mathcal{H}(D_H X_t(s), D_H X_t(s))\,\mathrm{d}s \\ &\quad \leqslant L\biggl[\mathcal{H}_t(X,Y) + \int_{-r}^{0} \mathcal{H}_t(D_H X(s), D_H X(s))\,\mathrm{d}s\biggr] = L\mathfrak{H}_t(X,Y), \end{aligned}$$

where $L := \min\{br, 1+b\}$. Hence the Lipschitz type condition (6) is satisfied for any $t \in [0,b]$ and $X, Y \in \mathcal{A}_{[b]}$. Since $F(t, \theta, \theta) = \{0\}$ for any $t \in [0,b]$, for a given $\varrho > 0$ and any $X \in B_\varrho(\Psi^0)$ we have

$$\begin{aligned} \mathcal{H}(F(t, X_t, D_H X_t), \{0\}) \leqslant L\mathfrak{H}_t(X, \theta) &\leqslant L[\mathfrak{H}_t(X, \Psi^0) + \mathfrak{H}_t(\Psi^0, \theta)] \\ &\leqslant L\varrho + L\mathfrak{H}_t(\Psi^0, \theta). \end{aligned}$$

Obviously, the function $m(t) := L\varrho + L\mathfrak{H}_t(\Psi^0, \theta)$is Lebesgue integrable on $[0,b]$ and $\mathcal{H}(F(t, X_t, D_H X_t), \{0\}) \leqslant m(t)$ for a.e. $t \in [0,b]$. It follows that all the conditions from Theorem 3.3 are satisfied.

2. In the following, we consider the set differential equation of neutral type

$$(13)\qquad \begin{cases} D_H X(t) = g_{X,A}(t) D_H X(t-1) + \displaystyle\int_{-1}^{0} s^2 D_H X(t+s)\,\mathrm{d}s, \quad t \in [0,2], \\ X|_{[-1,0]} = \theta, \end{cases}$$

where $g_{X,A}(t) = \mathcal{H}(X(t-1), A) = \mathcal{H}(X_t(-1), A)$. Here $A \in K_c(E)$ is a symmetric set with $\mathcal{H}(A, \{0\}) = 1$. We recall that $A \subset E$ is a symmetric set if $A = (-1)A := \{-x;\ x \in A\}$. For any $X \in \mathcal{A}_{[2]}$, let

$$F(t, X_t, D_H X_t) = g_{X,A}(t) D_H X_t(-1) + \int_{-1}^{0} s^2 D_H X_t(s)\,\mathrm{d}s, \quad t \in [0,2].$$

Obviously, $t \mapsto F(t, X_t, D_H X_t)$ is strongly measurable for each fixed $X \in \mathcal{A}_{[2]}$. We will show that there exists no constant $L > 0$ such that

$$\mathcal{H}(F(t, X_t, D_H X_t), F(t, Y_t, D_H Y_t)) \leqslant L \mathfrak{H}_t(X, Y) \tag{14}$$

for any $t \in [0,2]$ and any $X, Y \in \mathcal{A}_{[2]}$. If the Lipschitz type condition (14) is satisfied, taking $Y(t) = \theta$, $t \in [-1,2]$, there must exist $L > 0$ such that $\mathcal{H}(F(t, X_t, D_H X_t), \{0\}) \leqslant L \mathfrak{H}_t(X, \theta)$ for any $t \in [0,2]$ and any $X \in \mathcal{A}_{[2]}$. The last inequality can be written as

$$\mathcal{H}(g_{X,A}(t) D_H X_t(-1) + \int_{-1}^{0} s^2 \mathcal{H}(D_H X_t(s), \theta)\,\mathrm{d}s \leqslant L \mathfrak{H}_t(X, \theta) \tag{15}$$

for any $t \in [0,2]$ and $X \in \mathcal{A}_{[2]}$. Next, for a given $n > 3L$, let $X^n \colon [-1,2] \to K_c(E)$ be defined by

$$X^n(t) = \begin{cases} \{0\} & \text{if} \quad -1 \leqslant t \leqslant 1 - \dfrac{1}{n}, \\ \Big(-t + 1 - \dfrac{1}{n}\Big)A & \text{if} \quad 1 - \dfrac{1}{n} \leqslant t \leqslant 1 - \dfrac{1}{2n}, \\ (t-1)A & \text{if} \quad 1 - \dfrac{1}{2n} \leqslant t \leqslant 1 + \dfrac{1}{2n}, \\ \Big(-t + 1 + \dfrac{1}{n}\Big)A & \text{if} \quad 1 + \dfrac{1}{2n} \leqslant t \leqslant 1 + \dfrac{1}{n}, \\ \{0\} & \text{if} \quad 1 + \dfrac{1}{n} \leqslant t \leqslant 2. \end{cases}$$

Then $X^n \in \mathcal{A}_{[2]}$ and $X^n(t) = \theta$ for any $t \in [-1,0]$. Further, since $\mathcal{H}(D_H X^n(\tau), \theta) = 1$ for $1 \leqslant \tau \leqslant 1 + 1/n$ and $\mathcal{H}(D_H X^n(\tau), \theta) = 0$ for $1 + 1/n \leqslant \tau \leqslant 2$, we have for $t = 2$ that

$$\begin{aligned} \mathfrak{H}_2(X^n, \theta) &= \sup_{\tau \in [1,2]} \mathcal{H}(X^n(\tau), \{0\}) + \int_1^2 \mathcal{H}(D_H X^n(\tau), \theta)\,\mathrm{d}\tau \\ &= \frac{1}{2n} + \int_1^{1+1/n} \mathcal{H}(D_H X^n(\tau), \theta)\,\mathrm{d}\tau = \frac{1}{2n} + \frac{1}{n} = \frac{3}{2n}. \end{aligned}$$

Now, for $t = 2$, from (15) we have

$$\mathcal{H}(g_{X^n,A}(1)D_H X^n(1) + \int_{-1}^{0} s^2 \mathcal{H}(D_H X_2^n(s), \theta)\,\mathrm{d}s \leqslant L\mathfrak{H}_2(X^n, \theta).$$

Since $\mathcal{H}(D_H X_2^n(s), \theta) = 1$ for $-1 \leqslant s \leqslant -1 + 1/n$ and $\mathcal{H}(D_H X_2^n(s), \theta) = 0$ for $-1 + 1/n \leqslant s \leqslant 0$, we conclude that

$$\int_{-1}^{0} D_H X_2^n(s)\,\mathrm{d}s = \int_{-1}^{-1+1/n} s^2 D_H X_2^n(s)\,\mathrm{d}s = \frac{1}{3}\Big[\Big(-1 + \frac{1}{n}\Big)^3 + 1\Big]A.$$

Since $X^n(1) = \{0\}$, $g_{X^n,A}(1) = 1$, $D_H X^n(1) = A$, $\mathfrak{H}_2(X^n, \theta) = 3/(2n)$ and $n > 3L$, we obtain from the last inequality that

$$1 + \frac{1}{3}\Big[\Big(-1 + \frac{1}{n}\Big)^3 + 1]\Big] \leqslant \frac{3L}{2n} < \frac{1}{2},$$

which is impossible. Therefore, the Lipschitz type condition (14) does not hold for any $X, Y \in \mathcal{A}_{[2]}$. Now, for a given $\varrho > 0$, let $\mathcal{B}_\varrho(\Psi^0) := \{X \in \mathcal{A}_{[2]}^N;\ \mathfrak{H}_{[2]}(X, \Psi^0) \leqslant \varrho\}$ (see Remark 3.4 for the definition of the space $\mathcal{A}_{[2]}^N$). In our case, $\Psi^0(t) = \theta$ for any $t \in [-1, 2]$. Next, we recall the following useful property of the Hausdorff-Pompeiu metric (see [13]): with $\beta := \max\{\lambda, \mu\}$, $\lambda, \mu > 0$, we have

$$\mathcal{H}(\lambda A, \mu B) \leqslant \beta\mathcal{H}(A, B) + |\lambda - \mu|[\mathcal{H}(A, \{0\}) + \mathcal{H}(B, \{0\})]$$

for all $A, B \in K_c(E)$. Further, for any $t \in [0, 2]$ and $X, Y \in \mathcal{A}_{[2]}^N$ we have

$$\begin{aligned}\mathcal{H}(F(t, X_t, D_H X_t), F(t, Y_t, D_H Y_t)) &\leqslant \mathcal{H}(g_{X,A}(t)D_H X_t(-1), g_{Y,A}(t)D_H Y_t(-1)) \\ &+ \int_{-1}^{0} s^2 \mathcal{H}(D_H X_t(s), D_H Y_t(s))\,\mathrm{d}s \leqslant \beta(t)\mathcal{H}(D_H X_t(-1), D_H Y_t(-1)) \\ &+ |g_{X,A}(t) - g_{Y,A}(t)|\,[\mathcal{H}(D_H X_t(-1), \{0\}) + \mathcal{H}(D_H Y_t(-1), \{0\})] \\ &+ \int_{-1}^{0} \mathcal{H}(D_H X_t(s), D_H Y_t(s))\,\mathrm{d}s,\end{aligned}$$

where $\beta(t) := \max\{g_{X,A}(t), g_{Y,A}(t)\} = \max\{\mathcal{H}(X_t(-1), A), \mathcal{H}(Y_t(-1), A)\}$. Since for any $t \in [0, 2]$ we have that

$$\begin{aligned}\beta(t) &= \max\{\mathcal{H}(X_t(-1), A), \mathcal{H}(Y_t(-1), A)\} \leqslant 2(1 + \varrho), \\ |g_X(t) - g_Y(t)| &= |\mathcal{H}(X_t(-1), A) - \mathcal{H}(Y_t(-1), A)| \leqslant \mathcal{H}(X_t(-1), Y_t(-1)) \\ &\leqslant \sup_{s \in [-1,0]} \mathcal{H}(X_t(s), Y_t(s)) = \mathcal{H}_0(X_t, Y_t) = \mathcal{H}_t(X, Y),\end{aligned}$$

and

$$\mathcal{H}(X_t(-1),\{0\}) \leqslant N,\ \ \mathcal{H}(Y_t(-1),\{0\}) \leqslant N,$$

it follows that

$$\begin{aligned}\mathcal{H}(F(t,X_t,D_HX_t),F(t,Y_t,D_HY_t)) &\leqslant L\Big[\mathcal{H}_t(X,Y)+\int_{-1}^{0}\mathcal{H}(D_HX_t(s),D_HY_t(s))\,\mathrm{d}s\Big]\\ &= L\mathfrak{H}_t(X,Y), \quad t\in[0,2],\end{aligned}$$

where $L=1+2N+2(1+\varrho)$. Hence the Lipschitz type condition (14) is satisfied for any $t\in[0,2]$ and $X,Y\in\mathcal{B}_\varrho(\Psi^0)$. Also, it is easy to check that $\mathcal{H}(F(t,X_t,D_HX_t),\{0\})\leqslant L(\varrho+rN)$ for a.e. $t\in[0,2]$. Hence all the conditions from Theorem 3.3 are satisfied.

Next, we suppose that $A\in K_c(E)$ is an arbitrary parameter and let

$$F(t,X_t,D_HX_t,A)=g_{X,A}(t)X_t(-1)+\int_{-1}^{0}s^2D_HX_t(s)\,\mathrm{d}s, \quad t\in[0,2].$$

Then for any $t\in[0,2]$, $X,Y\in\mathcal{B}_\varrho(\Psi^0)$ and $A,B\in K_c(E)$ we have

$$\begin{aligned}\mathcal{H}(F(t,X_t,D_HX_t,A),F(t,Y_t,D_HY_t,B)) &\leqslant \mathcal{H}(g_{X,A}(t)X_t(-1),g_{Y,B}(t)Y_t(-1))\\ &+\int_{-1}^{0}s^2\mathcal{H}(D_HX_t(s),D_HY_t(s))\,\mathrm{d}s\leqslant \beta(t)\mathcal{H}(X_t(-1),Y_t(-1))\\ &+|g_{X,A}(t)-g_{Y,B}(t)|\,[\mathcal{H}(X_t(-1),\{0\})+\mathcal{H}(Y_t(-1),\{0\})]\\ &+\int_{-1}^{0}\mathcal{H}(D_HX_t(s),D_HY_t(s))\,\mathrm{d}s.\end{aligned}$$

As above, since

$$\begin{aligned}|g_{X,A}(t)-g_{Y,B}(t)| &= |\mathcal{H}(X_t(-1),A)-\mathcal{H}(Y_t(-1),B)|\\ &\leqslant \mathcal{H}(A,B)+\mathcal{H}(X_t(-1),Y_t(-1))\leqslant \mathcal{H}(A,B)+\mathcal{H}_t(X,Y),\end{aligned}$$

we obtain

$$\begin{aligned}&\mathcal{H}(F(t,X_t,D_HX_t,A),F(t,Y_t,D_HY_t,B))\\ &\qquad\leqslant K\Big[\mathcal{H}(A,B)+\mathcal{H}_t(X,Y)+\int_{-1}^{0}\mathcal{H}(D_HX_t(s),D_HY_t(s))\,\mathrm{d}s\Big]\\ &\qquad= K[\mathcal{H}(A,B)+\mathfrak{H}_t(X,Y)], \quad t\in[0,2],\end{aligned}$$

where $K=\max\{1,2(1+\varrho),2N\}$. Hence the Lipschitz type condition (H_2) is satisfied for any $t\in[0,2]$, $X,Y\in\mathcal{B}_\varrho(\Psi^0)$ and $A,B\in K_c(E)$.

Finally, we remark that if $A \in K_c(E)$ is an arbitrary parameter then, following a reasoning similar to the previous one, it is easy to check that for the set differential equation of neutral type

$$\begin{cases} D_H X(t) = g_{X,A}(t)[A + X(t-1)] + \int_{-1}^{0} s^2 D_H X(t+s)\,\mathrm{d}s, \quad t \in [0,2], \\ X|_{[-1,0]} = \theta, \end{cases} \tag{16}$$

the Lipschitz condition (H_2) is satisfied only on bounded sets $\mathcal{B} \times \mathcal{B}_1 \subset \mathcal{A}^N_{[2]} \times K_c(E)$.

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