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Marietjie Frick; Bernardo Llano; Rita Zuazua
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SELF-DICLIQUE CIRCULANT DIGRAPHS

Marietjie Frick, Pretoria, Bernardo Llano, Rita Zuazua, Mexico City

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Abstract. We study a particular digraph dynamical system, the so called digraph diclique operator. Dicliques have frequently appeared in the literature the last years in connection with the construction and analysis of different types of networks, for instance biochemical, neural, ecological, sociological and computer networks among others. Let $D=(V,A)$ be a reflexive digraph (or network). Consider X and Y (not necessarily disjoint) nonempty subsets of vertices (or nodes) of D. A disimplex $K(X,Y)$ of D is the subdigraph of D with vertex set $X\cup Y$ and arc set $\{(x,y)\colon x\in X,\ y\in Y\}$ (when $X\cap Y\neq\emptyset$, loops are not considered). A disimplex $K(X,Y)$ of D is called a diclique of D if $K(X,Y)$ is not a proper subdigraph of any other disimplex of D. The diclique digraph $\vec{k}(D)$ of a digraph D is the digraph whose vertex set is the set of all dicliques of D and $(K(X,Y),K(X',Y'))$ is an arc of $\vec{k}(D)$ if and only if $Y\cap X'\neq\emptyset$. We say that a digraph D is self-diclique if $\vec{k}(D)$ is isomorphic to D. In this paper, we provide a characterization of the self-diclique circulant digraphs and an infinite family of non-circulant self-diclique digraphs.

Keywords: circulant digraph; diclique; diclique operator; self-diclique digraph; graph dynamics

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1. Introduction

A digraph $D=(V,A)$ is called a *reflexive digraph* if the arc set $A\subseteq V\times V$ of D is a reflexive binary relation (that is, every vertex has a loop). Consider (not necessarily disjoint) nonempty subsets X and Y of vertices of $V(D)$. The *disimplex* $K(X,Y)$ of D is the subdigraph of D with vertex set $X\cup Y$ and arc set $\{(x,y)\colon x\in X,\ y\in Y\}$ (when $X\cap Y\neq\emptyset$, loops are not considered). A disimplex $K(X,Y)$ is called a *diclique* of D if $K(X,Y)$ is not a proper subdigraph of any other disimplex of the digraph D.

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The *diclique digraph* $\vec{k}(D)$ of a digraph D is defined by

$$V(\vec{k}(D)) = \{K(X,Y)\colon\ K(X,Y) \text{ is a diclique of } D\} \quad \text{and}$$
$$A(\vec{k}(D)) = \{(K(X,Y), K(X',Y'))\colon\ Y \cap X' \neq \emptyset\}.$$

In this paper, we study a particular digraph dynamical system, the so called *digraph diclique operator* (to be defined at the beginning of the next section). Dicliques have frequently appeared in the literature the last years in connection with the construction and analysis of different types of networks. They have been useful tools in general systems theory, where the representation of binary relations, the data structure to store them in the computer, the optimization of algorithms in particular disciplines, the determination of subsystems and finding the input and the output variables of a given subsystem are usual problems (see [5]). Dicliques also are applied in search techniques to find internally densely connected groups of nodes in directed networks with concrete applications to real-world networks as for example, Google's web pages and problems concerning the distribution of large amounts of information (as e-mails) as flows in a very complex network of transmission channels (see [6]). We also point out that "dicliques were strongly used in mathematical models to support economic based decision-making with applications to economical networks and Leontief models of energy markets, towards automated model construction and analysis" (see [4], the citation is taken from page 128 of this reference).

This setting can model problems from biochemical and neural networks in biochemistry and neuroscience, respectively, ecological complex systems, networks for social interactions in sociology or social psychology as well as from computer networks. A (di)graph is a natural representation of this kind of networks, where the vertices (or nodes) stand for molecules, neurons, species, persons, institutions or computers and the edges (or arcs) represent the interaction between the different nodes. It is an important research goal to study how networks change, evolve or behave after applying a given operator once or a series of iterations (see [8] for a comprehensive study on graph dynamics).

In the first part of this work, we focus on the behavior pattern of the diclique operator when applied to the class of circulant digraphs. This class of digraphs is well studied from the theoretical point of view, but also they are important models for distributed loop computer networks, where, for example, their connectivity and diameter play a crucial role. For more details, we suggest the interesting survey [2] on this topic by Bermond, Comellas and Hsu and their extensive list of references concerning the construction and applications of distributed loop networks, as well as the optimization problems therein.

2. Preliminaries

The *digraph diclique operator* is a pair $(\mathcal{D}, \varphi)$, where $\mathcal{D}$ is a set of digraphs and $\varphi\colon \mathcal{D} \to \mathcal{D}$ is a mapping recursively defined by $\varphi(D) = \varphi^1(D) = \vec{k}(D)$ and

$$\varphi^n(D) = \varphi(\varphi^{n-1}(D))$$

for $n \geqslant 2$ and some $D \in \mathcal{D}$. These definitions were introduced by E. Prisner (see [8]) in terms of bisimplices, bicliques and the biclique operator. Since that terminology has been used in the literature for the study of maximal bipartite subgraphs of undirected graphs, we use the definitions stated above for the case of digraphs (see [7]).

We say that a digraph D is *self-diclique* if $\vec{k}(D)$ is isomorphic to D and denote this fact by $\vec{k}(D) \cong D$. In the research of digraph operators as well as graph operators, one of the main goals is the study of the iterated operator behavior (convergence, divergence and periodicity). In particular, a digraph D is said to be *$\vec{k}$-periodic* if $\vec{k}^n(D) \cong D$ for some $n \in \mathbb{N}$ (by convention $\vec{k}^1(D) = \vec{k}(D)$ and we recursively define $\vec{k}^n(D) = \vec{k}(\vec{k}^{n-1}(D))$ for every $n \geqslant 2$). Clearly, if $n = 1$ in the last definition, then D is self-diclique.

Let $\mathbb{Z}_n$ be the cyclic group of the residues modulo a positive integer n and $\emptyset \neq J \subseteq \mathbb{Z}_n \setminus \{0\}$. The *circulant digraph* $\vec{C}_n(J)$ is defined by $V(\vec{C}_n(J)) = \mathbb{Z}_n$ and

$$A(\vec{C}_n(J)) = \{(i,j)\colon\ i,j \in \mathbb{Z}_n \text{ and } j - i \in J\}.$$

In particular, $\vec{C}_n(1) = \vec{C}_n$, the directed cycle of order n, which is obviously self-diclique.

A circulant digraph D is vertex transitive, that is, its automorphism group acts transitively on the vertex set $V(D)$. Observe that $\vec{C}_{2m+1}(J)$ is a circulant (rotational) tournament if and only if $|\{i, -i\} \cap J| = 1$ for every $i \in \mathbb{Z}_{2m+1} \setminus \{0\}$.

A digraph D is *strongly connected* if for every pair of vertices $u, v \in V(D)$ there exists a directed path from u to v.

Prisner posed the following problem (see [8] Problem 39 on page 207).

Problem 2.1. Are there, besides the directed cycles, more $\vec{k}$-periodic digraphs in the family of all finite strongly connected digraphs?

The only $\vec{k}$-periodic digraphs that have so far been reported in the literature are self-diclique digraphs. Zelinka [9] was the first to find a self-diclique digraph that is not a directed cycle. He showed that $\vec{k}(\vec{O}_3) \cong \vec{O}_3$, where $\vec{O}_3$ is the Eulerian orientation of the octahedron given by $V(\vec{O}_3) = \mathbb{Z}_6$ and

$$A(\vec{O}_3)=\{(0,1),(0,2),(1,2),(1,3),(2,3),(2,4),(3,4),(3,5),(4,5),(4,0),(5,0),(5,1)\}.$$

Figueroa and Llano [3] observed that $\vec{O}_3$ is isomorphic to the circulant digraph $\vec{C}_6(1,2)$. Then they showed that $\vec{C}_n(1,2)$ is self-diclique for every $n \geqslant 5$. Note that $\vec{C}_5(1,2)$ is the only regular tournament on 5 vertices up to isomorphism.

The problem of characterizing self-diclique digraphs is still far from being solved. Problem 4 of [3] asks, in particular, for a characterization of self-diclique circulant and of self-diclique tournaments. In this paper, we provide

(i) a characterization of the self-diclique circulant digraphs (Theorem 3.3) and conclude that $\vec{C}_5(1,2)$ and $\vec{C}_3$ are the only self-diclique circulant tournaments (Corollary 3.4) and

(ii) an infinite family of non-circulant self-diclique digraphs (see Section 4).

We denote by $D\langle S\rangle$ the subdigraph of D induced by S. The sets $N^+(v)$ and $N^-(v)$ denote the out-neighborhood and the in-neighborhood of a vertex v of a digraph D, respectively. An arc between vertices u and v of D is symmetric if (u,v), $(v,u) \in A(D)$. We use [1] for the general terminology on digraphs.

3. Self-diclique circulant digraphs

Let $\emptyset \neq A \subseteq \mathbb{Z}_n$ and $i \in \mathbb{Z}_n$. We define

$$-A = \{-a\colon\ a \in A\} \quad \text{and} \quad i + A = \{i + a\colon\ a \in A\}$$

(the additive inverses and the sums are taken modulo n).

Proposition 3.1. *For every $n \geqslant 7$ and $3 \leqslant k \leqslant \lfloor (n-1)/2 \rfloor$, the circulant digraphs $\vec{C}_n(1,2,\ldots,k)$ are not self-diclique.*

Proof. Let $J = \{1,2,\ldots,k\}$. Observe that the circulant $\vec{C}_n(J)$ has no symmetric arcs, $N^+(0) = J$ and $N^-(0) = -J$. We have that $K(\{0,1\},J)$ and $K(-J,\{-1,0\})$ are dicliques of $\vec{C}_n(J)$ containing vertex 0. Since $\vec{C}_n(J)$ is vertex-transitive,

(i) the disimplices $K(\{i,i+1\},i+J)$ and $K(i-J,\{i-1,i\})$ are dicliques containing vertex i for every $i \in \mathbb{Z}_n$ and

(ii) $K(-J,\{-1,0\}) \neq K(\{-k,-k+1\},-k+J)$ since $|J| \geqslant 3$.

Therefore, $|V(\vec{k}(\vec{C}_n(J)))| \geqslant 2n$ and thus $\vec{k}(\vec{C}_n(J)) \ncong \vec{C}_n(J)$. $\square$

Note, for example, that the tournament $\vec{C}_7(1,2,3)$ has two distinct dicliques

$$F_i = K(i + \{0,1\}, i + \{1,2,3\}) \quad \text{and} \quad G_i = K(i + \{0,1,2\}, i + \{2,3\})$$

for every $i \in \mathbb{Z}_7$. The digraph $\vec{C}_n(1,2,3,4)$ with $n \geqslant 9$ has three distinct dicliques

$$K(i+\{0,1\}, i+\{1,2,3,4\}),$$
$$K(i+\{0,1,2\}, i+\{2,3,4\}) \quad \text{and} \quad K(i+\{0,1,2,3\}, i+\{3,4\})$$

for every $i \in \mathbb{Z}_n$ and hence $|V(\vec{k}(\vec{C}_n(1,2,3,4)))| \geqslant 3n$.

We remark that the dicliques of $\vec{C}_n(J) = \vec{C}_n(1,2)$ are exactly $K(i-1+J, i+J)$; $i = 0,1,\ldots,n-1$, so in this case, $|V(\vec{k}(\vec{C}_n(J)))| = n$ (see [3], Theorem 1 for the details).

It is a well-known result that a circulant digraph $\vec{C}_n(j_1, j_2, \ldots, j_l)$ is connected if and only if $\gcd(n, j_1, j_2, \ldots, j_l) = 1$ (see [1], page 81). If it is connected, it is clearly strongly connected.

Proposition 3.2. *The strongly connected circulant digraphs $\vec{C}_n(j_1, j_2, \ldots, j_l)$, where $j_l > l$, $l \geqslant 2$ and $j_r \neq -j_s$ for every $r, s \in \{1, 2, \ldots, l\}$, are not self-diclique.*

Proof. Let $J = \{j_1, j_2, \ldots, j_l\}$. Observe that $N^+(0) = J$ and $N^-(0) = -J$. We have that $K(\{0\}, J)$ and $K(-J, \{0\})$ are dicliques of $\vec{C}_n(J)$ containing the vertex 0. Since $\vec{C}_n(J)$ is vertex-transitive,

(i) the disimplices $K(\{i\}, i+J)$ and $K(i-J, \{i\})$ are dicliques containing vertex i for every $i \in \mathbb{Z}_n$ and

(ii) $K(-J, \{0\}) \ncong K(\{-j_k\}, -j_k+J)$ for every $j_k \in J$ with $1 \leqslant k \leqslant l$ (also recall that $|J| = l \geqslant 2$).

Therefore, $|V(\vec{k}(\vec{C}_n(J)))| \geqslant 2n$ and hence $\vec{k}(\vec{C}_n(J)) \ncong \vec{C}_n(J)$. □

For example, $\vec{C}_7(2,3)$ has dicliques $K(\{i\}, i+\{2,3\})$ and $K(i+\{4,5\}, \{i\})$ for every $i \in \mathbb{Z}_7$.

Theorem 3.3. *The digraphs $\vec{C}_n$ $(n \geqslant 3)$ and $\vec{C}_n(1,2)$ $(n \geqslant 5)$ are the only self-diclique circulant digraphs without symmetric arcs.*

Proof. By Theorem 1 of [3], $\vec{C}_n(1,2)$ is self-diclique for every $n \geqslant 5$. Propositions 3.1 and 3.2 show that any other strongly connected circulant digraph without symmetric arcs is not self-diclique. □

Corollary 3.4. *$\vec{C}_3$ and $\vec{C}_5(1,2)$ are the only self-diclique circulant tournaments.*

4. An infinite family of non-circulant self-diclique digraphs

Let $m \geqslant 3$. We define the digraph D_m by $V(D_m) = \mathbb{Z}_{2m}$ and

$$A(D_m) = \{(i, i+j) \colon\ i = 0, 2, 4, \ldots, 2m-2; j = 1, 2, 3\} \\ \cup \{(i, i+j) \colon\ i = 1, 3, 5, \ldots, 2m-1; j = 1, 2\}.$$

Observe that D_3 is a semiregular tournament of order 6.

Remark 4.1. The function $\phi\colon\ \mathbb{Z}_{2m} \to \mathbb{Z}_{2m}$ defined by $\phi\colon\ j \to j + 2i \bmod 2m$ is an automorphism of D_m for every $j \in \mathbb{Z}_{2m}$ and $i = 0, 1, \ldots, m-1$.

It is clear that D_m is a strongly connected non-circulant digraph without symmetric arcs. By the definition of D_m, we have that $D_m\langle 2k + \{0, 1, 2, 3\}\rangle \cong TT_4$ for every $k = 0, 1, \ldots, m-1$, where TT_4 denotes the transitive tournament on 4 vertices.

Theorem 4.2. *The digraph D_m is self-diclique for every $m \geqslant 3$.*

Proof. Define the subsets $X_0 = \{0, 1\}$, $Y_0 = \{1, 2, 3\}$, $W_0 = \{0, 1, 2\}$ and $Z_0 = \{2, 3\}$ of vertices of D_m. Using the definition of D_m, the disimplices $K_0 = K(X_0, Y_0)$ and $L_0 = K(W_0, Z_0)$ are dicliques of D_m. By Remark 4.1, the disimplices

$$K_{2i} = K(X_{2i}, Y_{2i}) \quad \text{and} \quad L_{2i} = K(W_{2i}, Z_{2i})$$

are also dicliques of D_m, where

$$X_{2i} = \{2i, 2i+1\} = 2i + X_0, \quad Y_{2i} = \{2i+1, 2i+2, 2i+3\} = 2i + Y_0, \\ W_{2i} = \{2i, 2i+1, 2i+2\} = 2i + W_0 \quad \text{and} \quad Z_{2i} = \{2i+2, 2i+3\} = 2i + Z_0,$$

$i = 0, 1, \ldots, m-1$ and the addition is taken modulo $2m$. Also, by the definition of D_m, the set $\mathcal{K} = \{K_{2i}, L_{2i} \colon\ i = 0, 1, 2, \ldots, m-1\}$ contains every diclique of the digraph. Now, we consider the diclique digraph $\vec{k}(D_m)$ whose vertex set is $\mathcal{K}$. Notice that

$$Y_{2i} \cap W_{2i} \neq \emptyset, \quad Y_{2i} \cap X_{2(i+1)} = Y_{2i} \cap X_{2i+2} \neq \emptyset, \\ Y_{2i} \cap W_{2(i+1)} = Y_{2i} \cap W_{2i+2} \neq \emptyset \quad \text{and} \quad Z_{2i} \cap W_{2i+2} \neq \emptyset$$

for every $i = 0, 1, \ldots, m-1$, and $Z_{2m-2} \cap X_0 \neq \emptyset$.

Therefore, the arc set of $\vec{k}(D_m)$ is equal to

$$A(\vec{k}(D_m)) = \{(K_{2i}, L_{2i}) \colon\ i = 0, 1, \ldots, m-1\} \cup \{(L_{2m-2}, K_0)\} \\ \cup \{(K_{2i}, K_{2i+2}), (K_{2i}, L_{2i+2}), (L_{2i}, L_{2i+2}) \colon\ i = 0, 1, \ldots, m-1\}$$

(the sum of the subindices is taken modulo $2m$).

Define the digraph homomorphism $\varphi\colon\ V(D_m) \to V(\vec{k}(D_m))$ by $\varphi(2i) = K_{2i}$ and $\varphi(2i+1) = L_{2i}$ for every $i = 0, 1, 2, \ldots, m-1$. Using the definition of D_m, it is straightforward to check that φ is a digraph isomorphism. $\square$

O p e n q u e s t i o n. Are there, besides the digraphs $\vec{C}_n$ $(n \geqslant 3)$, $\vec{C}_n(1,2)$ $(n \geqslant 5)$ and D_m $(m \geqslant 3)$ any other strong self-diclique digraphs without symmetric arcs?

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Authors' addresses: *Marietjie Frick*, University of South Africa, P.O. Box 392, UNISA, 0003, Pretoria, South Africa, e-mail: `marietjie.frick@gmail.com`; *Bernardo Llano*, Departamento de Matemáticas, Universidad Autónoma Metropolitana-Iztapalapa, San Rafael Atlixco 186, Colonia Vicentina, 09340, Mexico City, Mexico, e-mail: `llano@xanum.uam.mx`; *Rita Zuazua*, Facultad de Ciencias, Universidad Nacional Autónoma de México, Ciudad Universitaria, Av. Universidad 3000, Circuito Exterior S/N, Delegación Coyoacán, C.P. 04510, Mexico City, Mexico, e-mail: `ritazuazua@ciencias.unam.mx`.