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MULTILEVEL CORRECTION ADAPTIVE FINITE ELEMENT METHOD FOR SEMILINEAR ELLIPTIC EQUATION

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Cordially dedicated to Prof. Ivo Babuška on the occasion of his 90th birthday

Abstract. A type of adaptive finite element method is presented for semilinear elliptic problems based on multilevel correction scheme. The main idea of the method is to transform the semilinear elliptic equation into a sequence of linearized boundary value problems on the adaptive partitions and some semilinear elliptic problems on very low dimensional finite element spaces. Hence, solving the semilinear elliptic problem can reach almost the same efficiency as the adaptive method for the associated boundary value problem. The convergence and optimal complexity of the new scheme can be derived theoretically and demonstrated numerically.

Keywords: semilinear elliptic problem; multilevel correction; adaptive finite element method

MSC 2010: 65N30, 35J61, 65B99, 62F35

1. Introduction

The purpose of this paper is to propose a multilevel correction type of adaptive finite element (AFEM) method for semilinear elliptic equations. Furthermore, we also give the corresponding convergence and optimality analysis in a general setting for the nonlinear term. The concept of adaptive finite element method was proposed by Babuška and his collaborators in [2], [3], [4], [5], which led to much work about the a posteriori error estimates, mesh refinement, convergence and optimal complexity and so on. For linear partial differential equations, especially, for the Possion

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equation and its variants, the theory is well-developed. For instance, Dörfler [9] introduced Dörfler's marking and proved strict energy error reduction for the Laplace problem provided the initial mesh is fine enough. Morin, Nochetto, and Siebert [15], [16] proved that there is no strict energy error reduction in general by introducing the concept of data oscillation and interior node property. Mekchay and Nochetto [14] proved a similar result for general second order elliptic operators by introducing the new concept of total error, that is, the sum of energy error and oscillations. For more results, please refer to the papers [2], [3], [4], [5], [7], [17] and the papers cited therein.

Besides for the linear boundary value problem, AFEM is also an efficient method for nonlinear elliptic equations (see, e.g., [10], [11]) and eigenvalue problems (see, e.g., [12], [19]). In this paper, a new adaptive scheme is designed for the semilinear elliptic problem based on the multilevel correction method (see [13]). With the new proposed method, solving semilinear elliptic problem will not be much more difficult than the solution of the corresponding boundary value problem. And we adopt the techniques in [10], [13] to prove the convergence and optimal complexity for this AFEM.

An outline of this paper goes as follows. In Section 2, we introduce some basic notation, the finite element method for the semilinear elliptic equation and some settings in this paper. In Section 3, we construct the multilevel correction adaptive algorithm and the corresponding convergence and complexity analysis are given in Section 4 and Section 5, respectively. In Section 6, some numerical results are presented to verify the theoretical results. Finally, some concluding remarks are given in the last section.

2. Discretization by finite element method

In this paper, the letter C (with or without subscripts) is used to denote a constant which may be different at different places. For convenience, the symbols $x_1 \lesssim y_1$, $x_2 \gtrsim y_2$ and $x_3 \approx y_3$ mean that $x_1 \leqslant C_1 y_1$, $x_2 \geqslant c_2 y_2$ and $c_3 x_3 \leqslant y_3 \leqslant C_3 x_3$. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ denote a bounded domain with Lipschitz boundary $\partial\Omega$. We use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms $\|\cdot\|_{s,p,\Omega}$ and seminorms $|\cdot|_{s,p,\Omega}$ (see, e.g, [1]). For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H^1_0(\Omega) = \{v \in H^1(\Omega)\colon\ v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of traces. For simplicity, we use $\|\cdot\|_{s,\Omega}$ to denote $\|\cdot\|_{s,2,\Omega}$ and V to denote $H^1_0(\Omega)$ in the rest of the paper.

Here, we consider the following type of semilinear elliptic equation:

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) + b(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\mathcal{A} = (a_{i,j})_{d\times d}$ is a symmetric positive definite matrix with $a_{i,j} \in W^{1,\infty}$ $(i,j = 1,2,\ldots,d)$, $b(x,u)$ is a nonlinear function corresponding to the second variable.

In this paper, we assume the nonlinear term has the property such that (2.1) has a unique solution $u \in H_0^1(\Omega)$. The weak form of the semilinear problem (2.1) can be described as: Find $u \in V$ such that

$$a(u,v) + (b(\cdot,u),v) = (f,v) \quad \forall\, v \in V, \tag{2.2}$$

where

$$a(u,v) = (\mathcal{A}\nabla u, \nabla v). \tag{2.3}$$

Obviously, $a(u,v)$ is bounded and coercive on V, i.e.,

$$a(u,v) \leqslant C_a\|u\|_{1,\Omega}\|v\|_{1,\Omega} \quad \text{and} \quad c_a\|u\|_{1,\Omega}^2 \leqslant a(u,u) \quad \forall\, u,v \in V. \tag{2.4}$$

So $\|w\|_{a,\Omega} = \sqrt{a(w,w)}$ satisfies $\|w\|_{a,\Omega} \approx \|w\|_{1,\Omega}$.

Now, we introduce the finite element method for semilinear elliptic problem (2.2). First we generate a shape regular decomposition of the computing domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ into triangles for $d = 2$, or tetrahedrons or hexahedrons for $d = 3$ (cf. [6], [8]). The mesh diameter h describes the maximum diameter of all cells $T \in \mathcal{T}_h$. Based on the mesh $\mathcal{T}_h$, we construct the finite element space $V_h \subset V$.

The standard finite element scheme for semilinear equation (2.2) is: Find $\bar{u}_h \in V_h$ such that

$$a(\bar{u}_h, v_h) + (b(\cdot,\bar{u}_h), v_h) = (f, v_h) \quad \forall\, v_h \in V_h. \tag{2.5}$$

In this paper, we denote

$$\delta_h(u) = \inf_{\chi\in V_h} \|u - \chi\|_{a,\Omega}. \tag{2.6}$$

In order to design and analyze the multilevel correction type of AFEM, we introduce the following assumptions.

Assumption A. The nonlinear term $b(x,\cdot)$ has the estimates

$$(b(\cdot,v) - b(\cdot,w), \psi) \leqslant C_b\|v-w\|_{0,\Omega}\|\psi\|_{a,\Omega} \quad \forall\, v,w,\psi \in V \tag{2.7}$$

and

$$\|b(\cdot,v) - b(\cdot,w)\|_{0,\Omega} \leqslant C_b\|v-w\|_{a,\Omega} \quad \forall\, v,w,\psi \in V. \tag{2.8}$$

For generality, we only state the following assumptions about the error estimate for the discrete equation (2.5):

Assumption B. The discrete equation (2.5) has a unique solution $\bar{u}_h$ and the following error estimates hold:

$$\|u - \bar{u}_h\|_{a,\Omega} \lesssim \delta_h(u), \tag{2.9}$$

$$\|u - \bar{u}_h\|_{0,\Omega} \lesssim \eta_a(V_h)\|u - \bar{u}_h\|_{a,\Omega}, \tag{2.10}$$

where $\eta_a(V_h)$ depends on the finite dimensional space V_h and has the following property:

$$\lim_{h\to 0} \eta_a(V_h) = 0, \quad \eta_a(\widetilde{V}_h) \leqslant \eta_a(V_h) \quad \text{if } V_h \subset \widetilde{V}_h \subset V. \tag{2.11}$$

3. Adaptive finite element method based on multilevel correction

In this section, we propose the multilevel correction AFEM for (2.2). Given an initial triangulation, the AFEM runs along the following loop:

$$\text{Solve} \to \text{Estimate} \to \text{Mark} \to \text{Refine}$$

In order to analyze the AFEM for semilinear elliptic equation, we first recall some basic conclusions of the AFEM for linear elliptic equation (see [7]).

3.1. AFEM for linear elliptic equation. In this subsection, we recall AFEM for an elliptic boundary value problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} Lu := -\nabla\cdot(\mathcal{A}\nabla u) = f & \text{in } \Omega, \\ \qquad\qquad\qquad\quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The weak form of (3.1) can be described as: Find $u \in V$ such that

$$a(u, v) = (f, v) \quad \forall\, v \in V. \tag{3.2}$$

And the corresponding finite element approximation of (3.2) is: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h. \tag{3.3}$$

Now we review the residual type a posteriori error estimator for the solution of (3.3). Let $\mathbb{T} := \{\mathcal{T}_k\}_{k\in\mathbb{N}_+}$ denote the sequence of all conforming meshes by refining the initial mesh $\mathcal{T}_1$ and let V_{h_k} denote the corresponding finite element space defined on the mesh $\mathcal{T}_k \in \mathbb{T}$. In this paper, we use $\mathcal{E}_k$ to denote the set of interior faces (edges or sides) of $\mathcal{T}_k$. For any $v \in V_{h_k}$, we define the element residual $\widetilde{\mathcal{R}}_T(v)$ and jump residual $\widetilde{\mathcal{J}}_e(v)$ by

$$\widetilde{\mathcal{R}}_T(v) := f - Lv = f + \nabla \cdot (\mathcal{A}\nabla v) \quad \text{in } T \in \mathcal{T}_k, \tag{3.4}$$

$$\widetilde{\mathcal{J}}_e(v) := -\mathcal{A}\nabla v^+ \cdot \nu^+ - \mathcal{A}\nabla v^- \cdot \nu^- := [\mathcal{A}\nabla v]_e \cdot \nu_e \quad \text{on } e \in \mathcal{E}_k, \tag{3.5}$$

where e is the common side of elements T^+ and T^- with the unit outward normals ν^+ and ν^-, respectively, and $\nu_e = \nu^-$. Let ω_e be the union of elements which share the side e, and ω_T the union of elements sharing a side with T. For $T \in \mathcal{T}_k$, we define the local error estimator $\tilde{\eta}_k^2(v,T)$ by

$$\tilde{\eta}_k^2(v,T) := h_T^2 \|\widetilde{\mathcal{R}}_T(v)\|_{0,T}^2 + \sum_{e \in \mathcal{E}_k, e \subset \partial T} h_e \|\widetilde{\mathcal{J}}_e(v)\|_{0,e}^2 \tag{3.6}$$

and the oscillation $\widetilde{\operatorname{osc}}_k^2(v,T)$ by

$$\widetilde{\operatorname{osc}}_k^2(v,T) := h_T^2 \|\widetilde{\mathcal{R}}_T(v) - \mathbb{P}_T \widetilde{\mathcal{R}}_T(v)\|_{0,T}^2 + \sum_{e \in \mathcal{E}_k, e \subset \partial T} h_e \|\widetilde{\mathcal{J}}_e(v) - \mathbb{P}_e \widetilde{\mathcal{J}}_e(v)\|_{0,e}^2, \tag{3.7}$$

where $\mathbb{P}_T$ and $\mathbb{P}_e$ are the L^2-projection operators to polynomials of some degree on T and e, respectively.

Given a subset $\omega \subset \Omega$, we define the error estimate $\tilde{\eta}_k^2(v,\omega)$ and the oscillation $\widetilde{\operatorname{osc}}_k^2(v,\omega)$ by

$$\tilde{\eta}_k^2(v,\omega) = \sum_{T \in \mathcal{T}_k, T \subset \omega} \tilde{\eta}_k^2(v,T) \quad \text{and} \quad \widetilde{\operatorname{osc}}_k^2(v,\omega) = \sum_{T \in \mathcal{T}_k, T \subset \omega} \widetilde{\operatorname{osc}}_k^2(v,T). \tag{3.8}$$

The procedure *Refine* used in the adaptive algorithm is Dörfler's marking strategy which is introduced in [9]:

Algorithm 3.1. *Marking Strategy E_0*:
Given a parameter $\theta \in (0,1)$.

(1) Construct a minimal subset $\mathcal{M}_k$ of $\mathcal{T}_k$ by selecting some elements in $\mathcal{T}_k$ such that

$$\sum_{T \in \mathcal{M}_k} \tilde{\eta}_k^2(u_k,T) \geqslant \theta \tilde{\eta}_k^2(u_k,\Omega).$$

(2) Mark all the elements in $\mathcal{M}_k$.

We now recall some well-known results of the AFEM for linear elliptic equation.

Lemma 3.1 ([7], Lemma 2.2, Global upper and lower bounds). *Let $u \in H_0^1(\Omega)$ be the solution of (3.2) and $u_k \in V_{h_k}$ the corresponding finite element solution. Then there exist constants $\widetilde{C}_1$, $\widetilde{C}_2$ and $\widetilde{C}_3 > 0$ depending only on the shape regularity C_a and c_a such that*

$$\|u - u_k\|_{a,\Omega} \leqslant \widetilde{C}_1 \tilde{\eta}_k(u_k, \mathcal{T}_k), \tag{3.9}$$

$$\widetilde{C}_2 \tilde{\eta}_k^2(u_k, \mathcal{T}_k) \leqslant \|u - u_k\|_{a,\Omega}^2 + \widetilde{C}_3 \widetilde{\operatorname{osc}}_k^2(u_k, \mathcal{T}_k). \tag{3.10}$$

Lemma 3.2 ([7], Proposition 3.3, Local perturbation). *Let $\mathcal{T} \in \mathbb{T}$. For all $T \in \mathcal{T}$ and any pair of discrete functions $v, w \in V(\mathcal{T})$ we have*

$$\operatorname{osc}(v, T) \leqslant \operatorname{osc}(w, T) + C_L \|v - w\|_{1,w_T}, \tag{3.11}$$

where C_L is a constant depending on the coefficient $\mathcal{A}$ and the mesh regularity.

Lemma 3.3 ([7], Theorem 4.1). *Let $\{u_k\}_{k\in\mathbb{N}_0}$ be the sequence of finite element solutions corresponding to the sequence of nested finite element spaces $\{V_{h_k}\}_{k\in\mathbb{N}_0}$ produced by the AFEM. Then there exist constants $\tilde{\gamma} > 0$ and $\xi \in (0, 1)$ depending only on the shape regularity of the meshes and the marking parameter θ, such that any two consecutive iterates satisfy the inequality*

$$\|u - u_{k+1}\|_{a,\Omega}^2 + \tilde{\gamma} \tilde{\eta}_{k+1}^2(u_{k+1}, \mathcal{T}_{k+1}) \leqslant \xi^2 (\|u - u_k\|_{a,\Omega}^2 + \tilde{\gamma} \tilde{\eta}_k^2(u_k, \mathcal{T}_k)), \tag{3.12}$$

where the constant $\tilde{\gamma}$ has the form

$$\tilde{\gamma} = \frac{1}{(1 + \delta_1^{-1}) C_L^2}. \tag{3.13}$$

Let us define $\mathcal{R}_{\mathcal{T}_k \to \mathcal{T}_{k+1}} := \mathcal{T}_k \setminus (\mathcal{T}_k \cap \mathcal{T}_{k+1})$ to be the set of refined elements in $\mathcal{T}_k$. Thus $\mathcal{M}_k \subset \mathcal{R}_{\mathcal{T}_k \to \mathcal{T}_{k+1}}$.

Lemma 3.4 ([7], Lemma 5.9, Optimal marking). *Let $u_k \in V_{h_k}$ and $u_{k+1} \in V_{h_{k+1}}$ be the finite element solutions of (3.2) over a conforming mesh $\mathcal{T}_k$ and its refinement $\mathcal{T}_{k+1}$ with marking element set $\mathcal{M}_k$. Suppose that they satisfy the decrease inequality*

$$\|u - u_{k+1}\|_{a,\Omega}^2 + \tilde{\gamma}_* \widetilde{\operatorname{osc}}_{k+1}^2(u_{k+1}, \mathcal{T}_{k+1}) \leqslant \xi_*^2 (\|u - u_k\|_{a,\Omega}^2 + \tilde{\gamma}_* \widetilde{\operatorname{osc}}_k^2(u_k, \mathcal{T}_k))$$

with constants $\tilde{\gamma}_* > 0$ and $\xi_*^2 \in (0, \frac{1}{2})$. Then the set $\mathcal{R} := \mathcal{R}_{\mathcal{T}_k \to \mathcal{T}_{k+1}}$ satisfies the inequality

$$\tilde{\eta}_k^2(u_k, \mathcal{R}) \geqslant \tilde{\theta} \tilde{\eta}_k^2(u_k, \mathcal{T}_k),$$

where

$$\tilde{\theta} = \frac{\widetilde{C}_2(1-2\xi_*^2)}{\widetilde{C}_0\big(\widetilde{C}_1^2 + (1+2C_L^2\widetilde{C}_1^2)\tilde{\gamma}_*\big)} \quad \textit{with } \widetilde{C}_0 = \max\Big\{1, \frac{\widetilde{C}_3}{\tilde{\gamma}_*}\Big\}.$$

3.2. Multilevel correction adaptive algorithm for semilinear elliptic equation. In this subsection, we present a type of multilevel correction adaptive method for the semilinear elliptic equation which is the key contribution of this paper.

Similarly to the element residual $\widetilde{\mathcal{R}}_T(v)$ and the jump residual $\widetilde{\mathcal{J}}_e(v)$ for (3.3), we define the element residual $\mathcal{R}_T(v)$ and the jump residual $\mathcal{J}_e(v)$ for (2.5) as follows:

$$\mathcal{R}_T(v) := f - b(\cdot, v) - Lv \quad \text{in } T \in \mathcal{T}_k, \tag{3.14}$$

$$\mathcal{J}_e(v) := -\mathcal{A}\nabla v^+ \cdot \nu^+ - \mathcal{A}\nabla v^- \cdot \nu^- := [\mathcal{A}\nabla v]_e \cdot \nu_e \quad \text{on } e \in \mathcal{E}_k. \tag{3.15}$$

For any $T \in \mathcal{T}_k$, we define the local error indicator $\eta_k^2(v, T)$ by

$$\eta_k^2(v,T) := h_T^2\|\mathcal{R}_T(v)\|_{0,T}^2 + \sum_{e\in\mathcal{E}_k, e\subset\partial T} h_e\|\mathcal{J}_e(v)\|_{0,e}^2 \tag{3.16}$$

and the oscillation $\mathrm{osc}_k^2(v, T)$ by

$$\mathrm{osc}_k^2(v,T) := h_T^2\|\mathcal{R}_T(v) - \mathbb{P}_T\mathcal{R}_T(v)\|_{0,T}^2 + \sum_{e\in\mathcal{E}_k, e\subset\partial T} h_e\|\mathcal{J}_e(v) - \mathbb{P}_e\mathcal{J}_e(v)\|_{0,e}^2. \tag{3.17}$$

Given a subset $\omega \subset \Omega$, we define the error estimate $\eta_k^2(v, \omega)$ and the oscillation $\mathrm{osc}_k^2(v, \omega)$ by

$$\eta_k^2(v,\omega) = \sum_{T\in\mathcal{T}_k, T\subset\omega} \eta_k^2(v,T) \quad \text{and} \quad \mathrm{osc}_k^2(v,\omega) = \sum_{T\in\mathcal{T}_k, T\subset\omega} \mathrm{osc}_k^2(v,T). \tag{3.18}$$

Similarly to *Marking Strategy* E_0 defined in Algorithm 3.1, we also define *Marking Strategy E* for (2.5) to enforce the error reduction as follows:

Algorithm 3.2. *Marking Strategy E*

Given a parameter $\theta \in (0, 1)$.

(1) Construct a minimal subset $\mathcal{M}_k$ from $\mathcal{T}_k$ by selecting some elements in $\mathcal{T}_k$ such that

$$\sum_{T\in\mathcal{M}_k} \eta_k^2(u_k, T) \geqslant \theta\eta_k^2(u_k, \Omega).$$

(2) Mark all the elements in $\mathcal{M}_k$.

Then we propose the following multilevel correction adaptive method which is based on Marking Strategy E in Algorithm 3.2 and the multilevel correction idea in [13], [19]. As in the multilevel correction method for eigenvalue problems [13], [19], we do not solve the semilinear problem directly in the refined mesh which is different from the normal AFEM for semilinear problem. In the new AFEM, solving the semilinear problem in the refined mesh is replaced by solving a linear elliptic problem in the refined mesh and a semilinear problem in a very low dimensional space.

Algorithm 3.3. *Multilevel Correction Adaptive Algorithm*

(1) Generate a coarse triangulation $\mathcal{T}_H$ for the computing domain Ω. Based on the mesh $\mathcal{T}_H$, build a linear finite element space V_H. Pick up an initial mesh $\mathcal{T}_1$ which is produced by refining $\mathcal{T}_H$ several times in the uniform way (refine all the elements). Then build the initial finite element space V_{h_1} on the triangulation $\mathcal{T}_1$ and solve the following semilinear elliptic equation: Find $u_1 \in V_{h_1}$ such that

$$a(u_1, v_1) + (b(\cdot, u_1), v_1) = (f, v_1) \quad \forall\, v_1 \in V_{h_1}. \tag{3.19}$$

(2) Set $k = 1$.

(3) Compute the local error indicators $\eta_k(u_k, T)$.

(4) Construct the submesh $\mathcal{M}_k \subset \mathcal{T}_k$ by *Marking Strategy E* defined in Algorithm 3.2 with parameter θ and refine $\mathcal{T}_k$ to generate a new conforming mesh $\mathcal{T}_{k+1}$.

(5) Solve the linearized equation: Find $\widehat{u}_{k+1} \in V_{h_{k+1}}$ such that

$$a(\widehat{u}_{k+1}, v_{k+1}) = (f - b(\cdot, u_k), v_{k+1}) \quad \forall\, v_{k+1} \in V_{h_{k+1}}. \tag{3.20}$$

(6) Define a finite element space $V_{H,h_{k+1}} := V_H \oplus \mathrm{span}\{\widehat{u}_{k+1}\}$ and solve the following semilinear elliptic equation: Find $u_{k+1} \in V_{H,h_{k+1}}$ such that

$$a(u_{k+1}, v_{H,k+1}) + (b(\cdot, u_{k+1}), v_{H,k+1}) = (f, v_{H,k+1}) \quad \forall\, v_{H,k+1} \in V_{H,h_{k+1}}. \tag{3.21}$$

(7) Let $k := k + 1$ and go to Step (3).

Since the large-scale semilinear problem is replaced by the linear elliptic problem (3.20) in the refined mesh (which can be solved by many fast solvers) and a low-dimensional semilinear problem (3.21), the overfull computational efficiency can be improved.

4. Convergence of adaptive finite element method

In this section, we give the convergence analysis of Algorithm 3.3 for the semilinear elliptic problem (2.2) based on the existing conclusions in Section 3.1 for the linear elliptic problem (3.1).

4.1. The relationship between semilinear and linear elliptic equations. In order to analyze the convergence and complexity of Algorithm 3.3, we establish the relationship between the solutions of linear and semilinear elliptic equations.

Let $w_k \in V$ be the exact solution of the following problem: Find $w_k \in V$ such that

$$a(w_k, v) = (f - b(\cdot, u_k), v) \quad \forall\, v \in V. \tag{4.1}$$

Define the Galerkin-projection operator $\Pi_k\colon V \to V_{h_k}$ by

$$a(w - \Pi_k w, v) = 0 \quad \forall\, v \in V_{h_k}. \tag{4.2}$$

For any $w \in V$, we apparently have the inequality

$$\|\Pi_k w\|_{a,\Omega} \leqslant \|w\|_{a,\Omega}. \tag{4.3}$$

From the fifth step of Algorithm 3.3 and (4.1), we have

$$\widehat{u}_k = \Pi_k w_{k-1}. \tag{4.4}$$

For the finite element solution u_k of (2.2) and the solution w_k of linear elliptic equation (4.1), we obtain the following relationships.

Lemma 4.1. *Let w_k be the solution of* (4.1), *u_k the solution of* (2.2) *obtained by Algorithm* 3.3 *and let Assumptions* A *and* B *hold. We have the estimates*

$$\|u - u_k\|_{a,\Omega} = \|w_k - \Pi_k w_k\|_{a,\Omega} + O(\eta_a(V_H))(\|u - u_k\|_{a,\Omega} + \|u - u_{k-1}\|_{a,\Omega}), \tag{4.5}$$

$$\|u - u_k\|_{a,\Omega} = \|w_{k-1} - \Pi_k w_{k-1}\|_{a,\Omega} + O(\eta_a(V_H))(\|u - u_k\|_{a,\Omega} + \|u - u_{k-1}\|_{a,\Omega}). \tag{4.6}$$

P r o o f. First, $u - u_k$ can be decomposed as follows

$$u - u_k = u - w_k + w_k - \Pi_k w_k + \Pi_k w_k - \Pi_k w_{k-1} + \Pi_k w_{k-1} - u_k. \tag{4.7}$$

Combining (2.2), (2.7), (2.10), and (4.1) leads to

$$\begin{aligned}(4.8)\quad \|u-w_k\|_{a,\Omega}^2 &= a(u-w_k,u-w_k) = (b(\cdot,u)-b(\cdot,u_k),u-w_k)\\ &\lesssim \|u-u_k\|_{0,\Omega}\|u-w_k\|_{a,\Omega} \lesssim \eta_a(V_H)\|u-u_k\|_{a,\Omega}\|u-w_k\|_{a,\Omega}.\end{aligned}$$

Using (4.3) and (4.8), we have the estimates

$$\begin{aligned}(4.9)\quad \|\Pi_k w_k - \Pi_k w_{k-1}\|_{a,\Omega} &\leqslant \|w_k - w_{k-1}\|_{a,\Omega} \lesssim \|u-w_k\|_{a,\Omega} + \|u-w_{k-1}\|_{a,\Omega}\\ &\lesssim \eta_a(V_H)(\|u-u_k\|_{a,\Omega} + \|u-u_{k-1}\|_{a,\Omega}).\end{aligned}$$

Since $\Pi_k w_{k-1} - u_k \in V_{H,h_k}$ and $\Pi_k w_{k-1} = \widehat{u}_k$, the following inequality holds:

$$\begin{aligned}(4.10)\quad \|\widehat{u}_k - u_k\|_{a,\Omega}^2 &= a(\widehat{u}_k - u_k, \widehat{u}_k - u_k) = (b(\cdot,u_{k-1}) - b(\cdot,u_k), \widehat{u}_k - u_k)\\ &\lesssim \|u_{k-1}-u_k\|_{0,\Omega}\|\widehat{u}_k - u_k\|_{a,\Omega}.\end{aligned}$$

That is

$$(4.11)\quad \|\Pi_k w_{k-1} - u_k\|_{a,\Omega} \lesssim \|u_{k-1}-u_k\|_{0,\Omega} \lesssim \eta_a(V_H)(\|u-u_{k-1}\|_{a,\Omega} + \|u-u_k\|_{a,\Omega}).$$

Combining (4.7) with (4.8)–(4.11) leads to the desired result (4.5).

In order to prove the second identity, we decompose $u - u_k$ to

$$u - u_k = u - w_{k-1} + w_{k-1} - \Pi_k w_{k-1} + \Pi_k w_{k-1} - u_k.$$

Due to the estimates (4.8), (4.11) in the first part, (4.6) can be proved similarly. □

For the approximate solution u_k of (2.2) and the solution w_k of the linear elliptic equation (4.1), we establish the following relationships for the a posteriori error estimators defined in (3.8) and (3.18).

Lemma 4.2. *Let w_k be the solution of* (4.1) *and u_k the solution of* (2.2) *obtained by Algorithm* 3.3. *The a posteriori error indicators defined in* (3.18) *and* (3.8), *respectively, satisfy the following relations:*

$$\begin{aligned}(4.12)\quad \eta_k(u_k,\mathcal{T}_k) &= \tilde{\eta}_k(\Pi_k w_{k-1},\mathcal{T}_k)\\ &\quad + O(\eta_a(V_H))(\|u-u_k\|_{a,\Omega} + \|u-u_{k-1}\|_{a,\Omega}),\\ (4.13)\quad \eta_k(u_k,\mathcal{T}_k) &= \tilde{\eta}_k(\Pi_k w_k,\mathcal{T}_k)\\ &\quad + O(\eta_a(V_H))(\|u-u_k\|_{a,\Omega} + \|u-u_{k-1}\|_{a,\Omega}).\end{aligned}$$

P r o o f. From the definitions of $\eta_k(u_k,T)$ and $\tilde\eta_k(u_k,T)$ in (3.18) and (3.8), we have

$$
\begin{aligned}
\text{(4.14)}\quad & \eta_k(u_k,T)-\tilde\eta_k(\Pi_k w_{k-1},T)\\
&=\Big(h_T^2\|f-b(\cdot,u_k)-Lu_k\|_{0,T}^2+\sum_{e\in\mathcal{E}_h,e\subset\partial T}h_e\|[\mathcal{A}\nabla u_k]_e\cdot\nu_e\|_{0,e}^2\Big)^{1/2}\\
&\quad-\Big(h_T^2\|f-b(\cdot,u_{k-1})-L\widehat u_k\|_{0,T}^2+\sum_{e\in\mathcal{E}_h,e\subset\partial T}h_e\|[\mathcal{A}\nabla\widehat u_k]_e\cdot\nu_e\|_{0,e}^2\Big)^{1/2}\\
&\leqslant\Big\{(h_T\|f-b(\cdot,u_k)-Lu_k\|_{0,T}-h_T\|f-b(\cdot,u_{k-1})-L\widehat u_k\|_{0,T})^2\\
&\quad+h_e\sum_{e\in\mathcal{E}_h,e\subset\partial T}(\|[\mathcal{A}\nabla u_k]_e\cdot\nu_e\|_{0,e}-\|[\mathcal{A}\nabla\widehat u_k]_e\cdot\nu_e\|_{0,e})^2\Big\}^{1/2}\\
&\leqslant\Big\{(h_T\|b(\cdot,u_k)-b(\cdot,u_{k-1})-Lu_k+L\widehat u_k\|_{0,T})^2\\
&\quad+h_e\sum_{e\in\mathcal{E}_h,e\subset\partial T}(\|[\mathcal{A}\nabla u_k]_e\cdot\nu_e-[\mathcal{A}\nabla\widehat u_k]_e\cdot\nu_e\|_{0,e})^2\Big\}^{1/2}.
\end{aligned}
$$

It is obvious that the inverse estimate implies

$$\|Lu_k\|_{0,T}\lesssim h_T^{-1}\|\nabla u_k\|_{0,T}\quad\forall\,T\in\mathcal{T}_h.$$

Then the following inequalities hold:

$$\sum_{T\in\mathcal{T}_h}h_T^2\|Lu_k\|_{0,T}^2\lesssim\sum_{T\in\mathcal{T}_k}\|u_k\|_{a,T}^2\lesssim\|u_k\|_{a,\Omega}^2. \tag{4.15}$$

From the inverse estimate and the trace inequality

$$\|v\|_{0,\partial T}\lesssim h_T^{-1/2}\|v\|_{0,T}+h_T^{s-1/2}\|v\|_{s,T}\quad\forall\,s>1/2,\ v\in H^s(T),\ T\in\mathcal{T}_k,$$

we have

$$h_e\|[\mathcal{A}\nabla v_k]_e\cdot\nu_e\|_{0,e}^2\lesssim\|\nabla v_k\|_{0,T}^2\lesssim\|v_k\|_{a,T}^2\quad\forall\,v_k\in V_{h_k}. \tag{4.16}$$

Combining (4.14)–(4.16) with (4.10) yields

$$\eta_k(u_k,T)-\tilde\eta_k(\Pi_k w_{k-1},T)\lesssim h_T\|b(\cdot,u_k)-b(\cdot,u_{k-1})\|_{0,T}+\|u_k-\widehat u_k\|_{a,T}. \tag{4.17}$$

From Assumption A, B and (4.17), we derive

$$\begin{aligned}\eta_k(u_k,\mathcal{T}_k)-\tilde{\eta}_k(\Pi_k w_{k-1},\mathcal{T}_k) &= \Big(\sum_{T\in\mathcal{T}_k}\eta_k^2(u_k,T)\Big)^{1/2}-\Big(\sum_{T\in\mathcal{T}_k}\tilde{\eta}_k^2(\Pi_k w_{k-1},T)\Big)^{1/2}\\ &\lesssim \Big(\sum_{T\in\mathcal{T}_k}(\eta_k(u_k,T)-\tilde{\eta}_k(\Pi_k w_{k-1},T))^2\Big)^{1/2}\\ &\lesssim \eta_a(V_H)\|b(\cdot,u_k)-b(\cdot,u_{k-1})\|_{0,\Omega}+\eta_a(V_H)\|u_k-u_{k-1}\|_{a,\Omega}\\ &\lesssim \eta_a(V_H)(\|u-u_k\|_{a,\Omega}+\|u-u_{k-1}\|_{a,\Omega}).\end{aligned}$$

This is the desired conclusion (4.12). The result (4.13) can be derived similarly and we complete the proof. □

Lemma 4.3. *Let w_k and u_k be the solutions of* (4.1) *the latter obtained by Algorithm* 3.3. *Then the oscillations defined in* (3.8) *and* (3.18) *have the estimates*

$$\operatorname{osc}_k(u_k,\mathcal{T}_k)=\widetilde{\operatorname{osc}}_k(\Pi_k w_{k-1},\mathcal{T}_k)+O(\eta_a(V_H))(\|u-u_k\|_{a,\Omega}+\|u-u_{k-1}\|_{a,\Omega}), \tag{4.18}$$

$$\operatorname{osc}_k(u_k,\mathcal{T}_k)=\widetilde{\operatorname{osc}}_k(\Pi_k w_k,\mathcal{T}_k)+O(\eta_a(V_H))(\|u-u_k\|_{a,\Omega}+\|u-u_{k-1}\|_{a,\Omega}). \tag{4.19}$$

Proof. By a procedure similar to that in Lemma 4.2 and the definition of oscillation, we can derive the desired results. □

4.2. Convergence of multilevel correction adaptive algorithm. In this subsection, we give the convergence analysis of Algorithm 3.3 for the semilinear elliptic equation.

First, based on Lemma 3.1 and Lemmas 4.1–4.3, we analyze the reliability and efficiency of the a posteriori error estimator defined in (3.18).

Theorem 4.1. *Let $\eta_a(V_H)$ be small enough. Then for the finite element solution u_k of* (2.2), *there exist mesh independent constants C_1, C_2 and C_3 such that*

$$\|u-u_k\|_{a,\Omega}\leqslant C_1\eta_k(u_k,\mathcal{T}_k)+O(\eta_a(V_H))\|u-u_{k-1}\|_{a,\Omega} \tag{4.20}$$

and

$$C_2\eta_k^2(u_k,\mathcal{T}_k)\leqslant\|u-u_k\|_{a,\Omega}^2+C_3\operatorname{osc}_k(u_k,\mathcal{T}_k)+O(\eta_a^2(V_H))\|u-u_{k-1}\|_{a,\Omega}^2, \tag{4.21}$$

where

$$C_1=\frac{\widetilde{C}_1}{1-C'\eta_a(V_H)},\quad C_2=\frac{\widetilde{C}_2}{4+C\eta_a^2(V_H)},\quad C_3=\frac{4\widetilde{C}_3}{4+C\eta_a^2(V_H)}$$

with the constants C' and C only depending on the mesh regularity and $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3$.

P r o o f. Since w_{k-1} is the exact solution of the equation

$$a(w_{k-1}, v) = (f - b(\cdot, u_{k-1}), v) \quad \forall\, v \in V,$$

from Lemma 3.1 we obtain

$$\|w_{k-1} - \Pi_k w_{k-1}\|_{a,\Omega} \leqslant \widetilde{C}_1 \tilde{\eta}_k(\Pi_k w_{k-1}, \mathcal{T}_k)$$

and

$$\widetilde{C}_2 \tilde{\eta}_k^2(\Pi_k w_{k-1}, \mathcal{T}_k) \leqslant \|w_{k-1} - \Pi_k w_{k-1}\|_{a,\Omega}^2 + \widetilde{C}_3 \widetilde{\operatorname{osc}}_k^2(\Pi_k w_{k-1}, \mathcal{T}_k).$$

Then from Lemmas 4.1–4.3 we can get the desired results. □

Now we are in the position to present the error reduction of Algorithm 3.3.

Theorem 4.2. *Let $\theta \in (0,1)$ and $\{u_k\}_{k\in\mathbb{N}_0}$ be the sequence of finite element solutions of* (2.2) *corresponding to the sequence of nested finite element space $\{V_{h_k}\}_{k\in\mathbb{N}_0}$ produced by Algorithm* 3.3*. Assume that $\eta_a(V_H)$ is sufficiently small. Then there exist constants $\gamma > 0$ and $\theta_1 \in (0,1)$ depending only on the shape regularity of meshes and the marking parameter θ used by Algorithm* 3.3*, such that*

$$\begin{aligned}(4.22)\qquad \|u - u_k\|_{a,\Omega}^2 + \gamma\eta_k^2(u_k, \mathcal{T}_k) &\leqslant \theta_1^2(\|u - u_{k-1}\|_{a,\Omega}^2 + \gamma\eta_{k-1}^2(u_{k-1}, \mathcal{T}_{k-1})) \\ &\quad + \theta_0\eta_a^2(V_H)\|u - u_{k-2}\|_{a,\Omega}^2,\end{aligned}$$

where θ_0 is a constant independent of the mesh size.

P r o o f. From Lemma 3.3 and the definition of w_{k-1} in (4.1), we have the contraction property

$$\begin{aligned}(4.23)\qquad &\|w_{k-1} - \Pi_k w_{k-1}\|_{a,\Omega}^2 + \tilde{\gamma}\tilde{\eta}_k^2(\Pi_k w_{k-1}, \mathcal{T}_k) \\ &\leqslant \xi^2(\|w_{k-1} - \Pi_{k-1} w_{k-1}\|_{a,\Omega}^2 + \tilde{\gamma}\tilde{\eta}_{k-1}^2(\Pi_{k-1} w_{k-1}, \mathcal{T}_{k-1})) \\ &\quad + \widetilde{C}\eta_a^2(V_H)(\|u - u_{k-1}\|_{a,\Omega}^2 + \|u - u_{k-2}\|_{a,\Omega}^2).\end{aligned}$$

By Lemmas 4.1 and 4.2, there exists a constant $C > 0$ such that

$$\begin{aligned}(4.24)\qquad &\|u - u_k\|_{a,\Omega}^2 + \tilde{\gamma}\eta_k^2(u_k, \mathcal{T}_k) \\ &\leqslant (1+\delta_1)\|w_{k-1} - \Pi_k w_{k-1}\|_{a,\Omega}^2 + (1+\delta_1)\tilde{\gamma}\tilde{\eta}_k^2(\Pi_k w_{k-1}, \mathcal{T}_k) \\ &\quad + C\delta_1^{-1}\eta_a^2(V_H)(\|u - u_k\|_{a,\Omega}^2 + \|u - u_{k-1}\|_{a,\Omega}^2) \\ &\leqslant (1+\delta_1)\xi^2(\|w_{k-1} - \Pi_{k-1} w_{k-1}\|_{a,\Omega}^2 + \tilde{\gamma}\tilde{\eta}_{k-1}^2(\Pi_{k-1} w_{k-1}, \mathcal{T}_{k-1})) \\ &\quad + C\delta_1^{-1}\eta_a^2(V_H)(\|u - u_k\|_{a,\Omega}^2 + \|u - u_{k-1}\|_{a,\Omega}^2 + \|u - u_{k-2}\|_{a,\Omega}^2),\end{aligned}$$

where the Young inequality is used and $\delta_1 \in (0,1)$.

Using a similar argument on the right-hand side term of (4.25), we have

$$
\begin{aligned}
(4.25)\qquad & \|u-u_k\|_{a,\Omega}^2+\tilde{\gamma}\eta_k^2(u_k,\mathcal{T}_k)\\
&\leqslant (1+\delta_1)(1+\delta_2)\xi^2(\|u-u_{k-1}\|_{a,\Omega}^2+\tilde{\gamma}\eta_{k-1}^2(u_{k-1},\mathcal{T}_{k-1}))\\
&\quad + C^*\eta_a^2(V_H)(\|u-u_k\|_{a,\Omega}^2+\|u-u_{k-1}\|_{a,\Omega}^2+\|u-u_{k-2}\|_{a,\Omega}^2),
\end{aligned}
$$

where $\delta_2\in(0,1)$ satisfies $(1+\delta_2)(1+\delta_1)\xi^2<1$ and C^* depends on C in (4.24).

From (4.25), we have

$$
\begin{aligned}
&(1-C^*\eta_a^2(V_H))\|u-u_k\|_{a,\Omega}^2+\tilde{\gamma}\eta_k^2(u_k,\mathcal{T}_k)\\
&\leqslant ((1+\delta_2)(1+\delta_1)\xi^2+C^*\eta_a^2(V_H))\|u-u_{k-1}\|_{a,\Omega}^2\\
&\quad +(1+\delta_2)(1+\delta_1)\xi^2\tilde{\gamma}\eta_{k-1}^2(u_{k-1},\mathcal{T}_{k-1})+C^*\eta_a^2(V_H)\|u-u_{k-2}\|_{a,\Omega}^2.
\end{aligned}
$$

Denote

$$
(4.26)\qquad \gamma=\frac{\tilde{\gamma}}{1-C^*\eta_a^2(V_H)},\quad \theta_1^2=\frac{(1+\delta_2)(1+\delta_1)\xi^2+C^*\eta_a^2(V_H)}{1-C^*\eta_a^2(V_H)},\quad \theta_0=\frac{C^*}{1-C^*\eta_a^2(V_H)}.
$$

Since

$$
\frac{(1+\delta_2)(1+\delta_1)\xi^2\tilde{\gamma}}{(1+\delta_2)(1+\delta_1)\xi^2+C^*\eta_a^2(V_H)}<\tilde{\gamma}<\gamma,
$$

we obtain the desired conclusion (4.22) and $\theta_1\in(0,1)$ when $\eta_a(V_H)$ is sufficiently small. Then we complete the proof. □

Theorem 4.3. *Let u and u_k be the exact solution and the corresponding approximation by Algorithm 3.3, respectively. When $\eta_a(V_H)$ is small enough, there exist constants $\beta>0$ and $\alpha\in(0,1)$, depending on the shape regularity of meshes and the parameter θ, such that any two consecutive iterates k and $k-1$ have the error contraction properties*

$$
(4.27)\qquad d_{h_k}^2\leqslant\theta_1^2 d_{h_{k-1}}^2+\theta_0\eta_a^2(V_H)\|u-u_{k-2}\|_{a,\Omega}^2
$$

and

$$
(4.28)\qquad d_{h_k}^2+\beta^2\eta_a^2(V_H)d_{h_{k-1}}^2\leqslant\alpha^2(d_{h_{k-1}}^2+\beta^2\eta_a^2(V_H)d_{h_{k-2}}^2),
$$

where $d_{h_k}^2=\|u-u_k\|_{a,\Omega}^2+\gamma\eta_k^2(u_k,\mathcal{T}_k)$.

Proof. It is obvious that the inequality (4.27) can be derived from Theorem 4.2 directly. We choose α and β such that

$$\alpha^2 - \beta^2\eta_a^2(V_H) = \theta_1^2, \quad \alpha^2\beta^2 = \theta_0.$$

The above two equations lead to

$$\alpha^2 = \frac{\theta_1^2 + \sqrt{\theta_1^4 + 4\theta_0\eta_a^2(V_H)}}{2} \quad \text{and} \quad \beta^2 = \frac{2\theta_0}{\theta_1^2 + \sqrt{\theta_1^4 + 4\theta_0\eta_a^2(V_H)}}.$$

Then we have $\alpha < 1$ provided $\eta_a(V_H)$ is small enough and $\theta_1 < 1$. It means the desired result (4.28) is obtained with the chosen α and β. □

5. Complexity analysis

Due to the complexity analysis results for the linear boundary value problem, we are able to analyze the complexity of Algorithm 3.3 for the semilinear elliptic problem.

In this section, we assume that $\eta_a(V_H)$ is small enough such that

$$\eta_a(V_H)\|u - u_{k-2}\|_{a,\Omega}^2 \leqslant \|u - u_{k-1}\|_{a,\Omega}^2. \tag{5.1}$$

Then from Theorem 4.2, we have the following error reduction property: for Algorithm 3.3

$$\|u - u_k\|_{a,\Omega}^2 + \gamma\eta_k^2(u_k, \mathcal{T}_k) \leqslant \tilde{\theta}_1^2(\|u - u_{k-1}\|_{a,\Omega}^2 + \gamma\eta_{k-1}^2(u_{k-1}, \mathcal{T}_{k-1})) \tag{5.2}$$

with $\tilde{\theta}_1^2 = \theta_1^2 + \theta_0\eta_a(V_H) < 1$.

As in the normal analysis of AFEM for the linear boundary value problem, we shall study the complexity in a class of functions defined by

$$\mathcal{A}_\gamma^s := \{v \in H_0^1(\Omega)\colon\ |v|_{s,\gamma} < \infty\},$$

where $\gamma > 0$ is a constant and

$$|v|_{s,\gamma} = \sup_{\varepsilon>0} \varepsilon \inf_{\{\mathcal{T}_k \subset \mathcal{T}_1\colon \inf(\|v-v_k\|_{a,\Omega}^2 + (\gamma+1)\operatorname{osc}_k^2(v_k,\mathcal{T}_k))^{1/2} \leqslant \varepsilon\}} (\#\mathcal{T}_k - \#\mathcal{T}_1)^s$$

and $\mathcal{T}_k \subset \mathcal{T}_1$ means $\mathcal{T}_k$ is a refinement of $\mathcal{T}_1$. It is seen from the definition that, for all $\gamma > 0$, $\mathcal{A}_\gamma^s = \mathcal{A}_1^s$, and we denote $\mathcal{A}^s$ as $\mathcal{A}_1^s$, $|v|_s$ as $|v|_{s,\gamma}$ for simplicity. Hence, the symbol $\mathcal{A}^s$ is the class of functions that can be approximated within a given tolerance ε by continuous piecewise polynomial functions over a partition $\mathcal{T}_k$ with the number of degrees of freedom $\#\mathcal{T}_k - \#\mathcal{T}_1 \lesssim \varepsilon^{-1/s}|v|_s^{1/s}$.

In our analysis, we also need the following two results (see, e.g., [7], [17], [18]).

Lemma 5.1 (Complexity of refinements). *For $k \geqslant 0$, let $\{\mathcal{T}_k\}_{k\geqslant 1}$ be any sequence of refinements of $\mathcal{T}_1$, where $\mathcal{T}_{k+1}$ is generated from $\mathcal{T}_k$ by Dörfler's Marking Strategy with a subset $\mathcal{M}_k \subset \mathcal{T}_k$. Then*

$$\#\mathcal{T}_k - \#\mathcal{T}_1 \leqslant C_0 \sum_{j=1}^{k-1} \#\mathcal{M}_j \quad \forall\, k \geqslant 1. \tag{5.3}$$

Here and hereafter in this paper, we use $\#\mathcal{T}$ to denote the number of elements in the mesh $\mathcal{T}$.

For the smallest common refinement of $\mathcal{T}_s$, $\mathcal{T}_t$, that is $\mathcal{T} := \mathcal{T}_s \oplus \mathcal{T}_t$, we have the following property.

Lemma 5.2 (Overlay of mesh). *For $\mathcal{T}_s$, $\mathcal{T}_t \subset \mathcal{T}_1$, the overlay $\mathcal{T} := \mathcal{T}_s \oplus \mathcal{T}_t$ is conforming and satisfies*

$$\#\mathcal{T} \leqslant \#\mathcal{T}_s + \#\mathcal{T}_t - \#\mathcal{T}_1. \tag{5.4}$$

In order to give the proof of the optimal complexity of Algorithm 3.3 for solving the semilinear elliptic problem, we should give some preparations.

Using the assumption (5.1) and the procedure similar to that in the proof of Theorem 4.2 when (4.12) is replaced by (4.18), we have

Lemma 5.3. *Let u_{k-1} and u_k be finite element solutions of (2.2) over a conforming mesh $\mathcal{T}_{k-1}$ and its refinement $\mathcal{T}_k$ with marked set $\mathcal{M}_{k-1}$. Suppose that they satisfy the estimate*

$$\|u-u_k\|_{a,\Omega}^2 + \gamma_* \operatorname{osc}_k^2(u_k,\mathcal{T}_k) \leqslant \xi_*^2(\|u-u_{k-1}\|_{a,\Omega}^2 + \gamma_* \operatorname{osc}_{k-1}^2(u_{k-1},\mathcal{T}_{k-1})), \tag{5.5}$$

where γ_* and ξ_* are some positive constants. Then for problem (4.1), we have

$$\begin{aligned} &\|w_{k-1} - \Pi_k w_{k-1}\|_{a,\Omega}^2 + \gamma_* \widetilde{\operatorname{osc}}_k^2(\Pi_k w_{k-1}, \mathcal{T}_k) \\ &\quad \leqslant \tilde{\xi}_*^2(\|w_{k-1} - \Pi_{k-1} w_{k-1}\|_{a,\Omega}^2 + \gamma_* \widetilde{\operatorname{osc}}_{k-1}^2(\Pi_{k-1} w_{k-1}, \mathcal{T}_{k-1})) \end{aligned} \tag{5.6}$$

with

$$\tilde{\xi}_*^2 = (1+\delta_2)\frac{(1+\delta_1)\xi_*^2 + C^*\delta_1^{-1}\eta_a^2(V_H)(1+\xi_*^2)}{1 - C\delta_1^{-1}\eta_a(V_H)(1+\eta_a(V_H))},$$

where the constants C and C^* depend on δ_1, $\delta_2 \in (0,1)$ as in the proof of Theorem 4.2.

Theorem 5.1. *Let u_{k-1} and u_k be as in Lemma 5.3. Suppose that they have the decrease property*

$$\|u-u_k\|_{a,\Omega}^2+\gamma_*\operatorname{osc}_k^2(u_k,\mathcal{T}_k)\leqslant\xi_*^2(\|u-u_{k-1}\|_{a,\Omega}^2+\gamma_*\operatorname{osc}_{k-1}^2(u_{k-1},\mathcal{T}_{k-1}))$$

with constants $\gamma_*>0$ and $\xi_*^2\in(0,1/2)$. When $\eta_a(V_H)$ is sufficiently small, the set $\mathcal{R}:=\mathcal{R}_{\mathcal{T}_{k-1}\to\mathcal{T}_k}$ satisfies the inequality

$$\eta_{k-1}^2(u_{k-1},\mathcal{R})\geqslant\widehat{\beta}\eta_{k-1}^2(u_{k-1},\mathcal{T}_{k-1})$$

with

$$\widehat{\beta}=\frac{\widetilde{C}_2(1-2\tilde{\xi}_*^2)}{4C_0(\widetilde{C}_1^2+(1+2C_L^2\widetilde{C}_1^2)\gamma_*)}\quad\text{and}\quad C_0=\max\Bigl(1,\frac{\widetilde{C}_3}{\gamma_*}\Bigr),$$

where $\tilde{\xi}_*$ is defined in Lemma 5.3 with δ_1 and δ_2 being chosen such that $\tilde{\xi}_*^2\in(0,1/2)$.

Proof. The result is a direct consequence of Lemmas 3.4, 4.2, and 5.3. □

Lemma 5.4 (Cardinality of $\mathcal{M}_k$). *Let $u\in\mathcal{A}^s$ be the solution of (2.2), $\mathcal{T}_k$ the conforming partition obtained from $\mathcal{T}_1$, and let θ satisfy*

$$\theta\in\Bigl(0,\frac{C_2\gamma}{C_3(C_1^2+(1+2C_L^2C_1^2)\gamma)}\Bigr).$$

Then the following estimate is valid:

$$\#\mathcal{M}_k\leqslant C(\|u-u_k\|_{a,\Omega}^2+\gamma\operatorname{osc}_k^2(u_k,\mathcal{T}_k))^{-1/(2s)}|u|_s^{1/s},\tag{5.7}$$

where the constant C depends on the discrepancy between θ and

$$\frac{C_2\gamma}{C_3(C_1^2+(1+2C_L^2C_1^2)\gamma)}.$$

Proof. We choose β, $\beta_1\in(0,1)$ such that $\beta_1\in(0,\beta)$ and

$$\theta<\frac{C_2\gamma}{C_3(C_1^2+(1+2C_L^2C_1^2)\gamma)}(1-\beta^2).$$

Set

$$\varepsilon=\frac{1}{\sqrt{2}}\beta_1(\|u-u_k\|_{a,\Omega}^2+\gamma\operatorname{osc}_k^2(u_k,\mathcal{T}_k))^{1/2}.$$

Let $\delta_1,\delta_2\in(0,1)$ be constants such that

$$(1+\delta_1)(1+\delta_2)\beta_1^2\leqslant 1.\tag{5.8}$$

And let $\mathcal{T}_\varepsilon$ be a refinement of $\mathcal{T}_1$ with minimum degrees of freedom satisfying

$$\|u - u_\varepsilon\|_{a,\Omega}^2 + (\gamma + 1)\operatorname{osc}_\varepsilon(u_\varepsilon, \mathcal{T}_\varepsilon)^2 \leqslant \varepsilon^2,$$

where u_ε denotes the solution over the mesh $\mathcal{T}_\varepsilon$. By the definition of $\mathcal{A}^s$, we can get

$$\#\mathcal{T}_\varepsilon - \#\mathcal{T}_1 \lesssim \Big(\frac{1}{\sqrt{2}}\beta_1\Big)^{-1/s}(\|u - u_k\|_{a,\Omega}^2 + \gamma\operatorname{osc}_k(u_k, \mathcal{T}_k)^2)^{-1/(2s)}|u|_s^{1/s}.$$

Let $\mathcal{T}_*$ be the smallest common refinement of $\mathcal{T}_k$ and $\mathcal{T}_\varepsilon$. For w_ε defined in (4.1), from Lemma 3.2 and Young's inequality we have

$$\widetilde{\operatorname{osc}}_*^2(\Pi_* w_\varepsilon, \mathcal{T}_*) \leqslant 2\widetilde{\operatorname{osc}}_\varepsilon^2(\Pi_\varepsilon w_\varepsilon, \mathcal{T}_\varepsilon) + 2C_L^2\|\Pi_\varepsilon w_\varepsilon - \Pi_* w_\varepsilon\|_{a,\Omega}^2.$$

Due to the orthogonality

$$\|w_\varepsilon - \Pi_* w_\varepsilon\|_{a,\Omega}^2 = \|w_\varepsilon - \Pi_\varepsilon w_\varepsilon\|_{a,\Omega}^2 - \|\Pi_* w_\varepsilon - \Pi_\varepsilon w_\varepsilon\|_{a,\Omega}^2,$$

we arrive at

$$\|w_\varepsilon - \Pi_* w_\varepsilon\|_{a,\Omega}^2 + \frac{1}{2C_L^2}\widetilde{\operatorname{osc}}_*^2(\Pi_* w_\varepsilon, \mathcal{T}_*) \leqslant \|w_\varepsilon - \Pi_\varepsilon w_\varepsilon\|_{a,\Omega}^2 + \frac{1}{C_L^2}\widetilde{\operatorname{osc}}_\varepsilon^2(\Pi_\varepsilon w_\varepsilon, \mathcal{T}_\varepsilon).$$

From the definition of $\tilde{\gamma}$ in (3.13), $\tilde{\gamma} := 1/((1 + \delta_1^{-1})C_L^2)$, we have $\tilde{\gamma} \leqslant 1/(2C_L^2)$. Then we derive that

$$\begin{aligned}
(5.9)\qquad & \|w_\varepsilon - \Pi_* w_\varepsilon\|_{a,\Omega}^2 + \tilde{\gamma}\widetilde{\operatorname{osc}}_*^2(\Pi_* w_\varepsilon, \mathcal{T}_*) \\
&\leqslant \|w_\varepsilon - \Pi_* w_\varepsilon\|_{a,\Omega}^2 + \frac{1}{2C_L^2}\widetilde{\operatorname{osc}}_*^2(\Pi_* w_\varepsilon, \mathcal{T}_*) \\
&\leqslant \|w_\varepsilon - \Pi_\varepsilon w_\varepsilon\|_{a,\Omega}^2 + \frac{1}{C_L^2}\widetilde{\operatorname{osc}}_\varepsilon^2(\Pi_\varepsilon w_\varepsilon, \mathcal{T}_\varepsilon) \\
&\leqslant \|w_\varepsilon - \Pi_\varepsilon w_\varepsilon\|_{a,\Omega}^2 + (\tilde{\gamma} + \sigma)\widetilde{\operatorname{osc}}_\varepsilon^2(\Pi_\varepsilon w_\varepsilon, \mathcal{T}_\varepsilon)
\end{aligned}$$

with $\sigma = 1/C_L^2 - \tilde{\gamma} \in (0, 1)$. Applying an argument similar to that in the proof of Theorem 4.2, when Lemma 4.2 is replaced by Lemma 4.3, we obtain

$$\|u - u_*\|_{a,\Omega}^2 + \gamma\operatorname{osc}_*(u_*, \mathcal{T}_*) \leqslant \beta_0^2(\|u - u_\varepsilon\|_{a,\Omega}^2 + (\gamma + 1)\operatorname{osc}_\varepsilon(u_\varepsilon, \mathcal{T}_\varepsilon)),$$

where

$$(5.10)\qquad \beta_0^2 = \frac{(1 + \delta_1)(1 + \delta_2) + C_5\delta_1^{-1}\eta_a^2(V_H)}{1 - C_5\delta_1^{-1}\eta_a^2(V_H)}$$

with C_5 being the constant which depends on C_L.

Then we arrive at

$$\|u-u_*\|_{a,\Omega}^2+\gamma\,\mathrm{osc}_*^2(u_*)\leqslant\overline{\xi}^2(\|u-u_k\|_{a,\Omega}^2+\gamma\,\mathrm{osc}_k^2(u_k,\mathcal{T}_k))$$

with $\overline{\xi}^2=\frac{1}{2}\beta_0^2\beta_1^2$. If $\eta_a(V_H)$ is small enough, we have $\overline{\xi}^2\in(0,\frac{1}{2})$. Thus by Theorem 5.1, we have

$$\sum_{T\in\mathcal{M}(\mathcal{T}_*)}\eta_k^2(u_k,T)\geqslant\overline{\theta}\sum_{T\in\mathcal{T}_k}\eta_k^2(u_k,\mathcal{T}_k),$$

where

$$\overline{\theta}=\frac{\widetilde{C}_2(1-2\tilde{\xi}^2)}{4C_0(\widetilde{C}_1^2+(1+2C_L^2\widetilde{C}_1^2)\gamma)},\quad C_0=\max\Big(\frac{\widetilde{C}_3}{\gamma},1\Big)$$

and

$$\tilde{\xi}^2=(1+\delta_2)\frac{(1+\delta_1)\overline{\xi}^2+C^*\delta_1^{-1}\eta_a^2(V_H)(1+\overline{\xi}^2)}{1-C\eta_a(H)(1+\eta_a(V_H))}.$$

It follows from the definition of γ (see (4.26)) and $\tilde{\gamma}$ (see (3.13)) that $\gamma<1$ when $\eta_a(V_H)$ is small enough. On the other hand, we have $\widetilde{C}_3>1$ and hence $C_0=\widetilde{C}_3/\gamma$.

Since $\eta_a(V_H)\ll 1$, we obtain that $\tilde{\xi}^2\in(0,\frac{1}{2}\beta^2)$. And using Theorem 4.1, we have

$$\begin{aligned}\overline{\theta}&=\frac{\widetilde{C}_2(1-2\tilde{\xi}^2)}{4C_0(\widetilde{C}_1^2+(1+2C_L^2\widetilde{C}_1^2)\gamma)}\geqslant\frac{\widetilde{C}_2}{(4\widetilde{C}_3/\gamma)(\widetilde{C}_1^2+(1+2C_L^2\widetilde{C}_1^2)\gamma)}(1-\beta^2)\\&\geqslant\frac{\widetilde{C}_2}{4\widetilde{C}_3\big(\widetilde{C}_1^2/\gamma+1+2C_L^2\widetilde{C}_1^2\big)}(1-\beta^2)\\&=\frac{4C_2(4+C\eta_a^2(V_H))\gamma(1-\beta^2)}{4C_3(4+C\eta_a^2(V_H))\{C_1^2(1-C'\eta_a(V_H))^2+(1+2C_L^2C_1^2(1-C'\eta_a(V_H))^2)\gamma\}}\\&\geqslant\frac{C_2\gamma}{C_3(C_1^2+(1+2C_L^2C_1^2)\gamma)}(1-\beta^2)>\theta.\end{aligned}$$

Thus

$$\begin{aligned}\#\mathcal{M}_k&\leqslant\#\mathcal{R}\leqslant\#\mathcal{T}_*-\#\mathcal{T}_k\leqslant\#\mathcal{T}_\varepsilon-\#\mathcal{T}_1\\&\leqslant\Big(\frac{1}{\sqrt{2}}\beta_1\Big)^{-1/s}(\|u-u_k\|_{a,\Omega}^2+\gamma\,\mathrm{osc}_k^2(u_k,\mathcal{T}_k))^{-1/(2s)}|u|_s^{1/s}.\end{aligned}$$

Then we derive the desired estimate (5.7) with an explicit dependence on the discrepancy between θ and

$$\frac{C_2\gamma}{C_3(C_1^2+(1+2C_L^2C_1^2)\gamma)}.$$

□

As a consequence, we obtain the optimal complexity as follows.

Theorem 5.2. *Let $u \in \mathcal{A}^s$ be the solution of* (2.2) *and* $\{u_k\}_{k\in\mathbb{N}_+}$ *be a sequence of finite element solutions produced by Algorithm* 3.3. *Under the assumption* (5.1), *the k-th approximate solution satisfies the optimal bound*

$$\|u-u_k\|_{a,\Omega}^2 + \gamma\,\mathrm{osc}_k^2(u_k,\mathcal{T}_k) \lesssim (\#\mathcal{T}_k - \#\mathcal{T}_1)^{-2s}|u|_s^2, \tag{5.11}$$

where the hidden constant depends on the exact solution u and the discrepancy between θ and

$$\frac{C_2\gamma}{C_3(C_1^2+(1+2C_L^2C_1^2)\gamma)}.$$

P r o o f. From Lemmas 5.1 and 5.4 we obtain

$$\#\mathcal{T}_k - \#\mathcal{T}_1 \lesssim \sum_{j=1}^{k-1}\#\mathcal{M}_j \lesssim \sum_{j=1}^{k-1}(\|u-u_j\|_{a,\Omega}^2 + \gamma\,\mathrm{osc}_j^2(u_j,\mathcal{T}_j))^{-1/(2s)}|u|_s^{1/s}. \tag{5.12}$$

And from the efficiency of the a posteriori estimator, we have

$$\|u-u_j\|_{a,\Omega}^2 + \gamma\eta_j^2(u_j,\mathcal{T}_j) \leqslant C(\|u-u_j\|_{a,\Omega}^2 + \gamma\,\mathrm{osc}_j^2(u_j,\mathcal{T}_j)). \tag{5.13}$$

Combining (5.12) and (5.13) leads to

$$\#\mathcal{T}_k - \#\mathcal{T}_1 \lesssim \sum_{j=1}^{k-1}(\|u-u_j\|_{a,\Omega}^2 + \gamma\eta_j^2(u_j,\mathcal{T}_j))^{-1/(2s)}|u|_s^{1/s}.$$

By the contraction property (5.2)

$$\|u-u_k\|_{a,\Omega}^2 + \gamma\eta_k^2(u_k,\mathcal{T}_k) \leqslant \tilde\theta_1^{2(k-j)}(\|u-u_j\|_{a,\Omega}^2 + \gamma\eta_j^2(u_j,\mathcal{T}_j)),$$

we have

$$\begin{aligned}\#\mathcal{T}_k - \#\mathcal{T}_1 &\lesssim (\|u-u_k\|_{a,\Omega}^2 + \gamma\eta_k^2(u_k,\mathcal{T}_k))^{-1/(2s)}|u|_s^{1/s}\sum_{j=1}^{k-1}\tilde\theta_1^{(k-j)/s}\\ &\lesssim (\|u-u_k\|_{a,\Omega}^2 + \gamma\eta_k^2(u_k,\mathcal{T}_k))^{-1/(2s)}|u|_s^{1/s}.\end{aligned}$$

Since $\mathrm{osc}_k(u_k,\mathcal{T}_k) \leqslant \eta_k(u_k,\mathcal{T}_k)$, we arrive at the desired result (5.11). □

R e m a r k 5.1. Step (5) of Algorithm 3.3 means we use the fixed-point iteration for the nonlinear elliptic equation. Of course, for strong nonlinear equations, we can use the Newton iteration to design the corresponding multilevel correction adaptive algorithm.

6. Numerical results

In this section, two numerical experiments are presented to verify the theoretical analysis and efficiency of Algorithm 3.3.

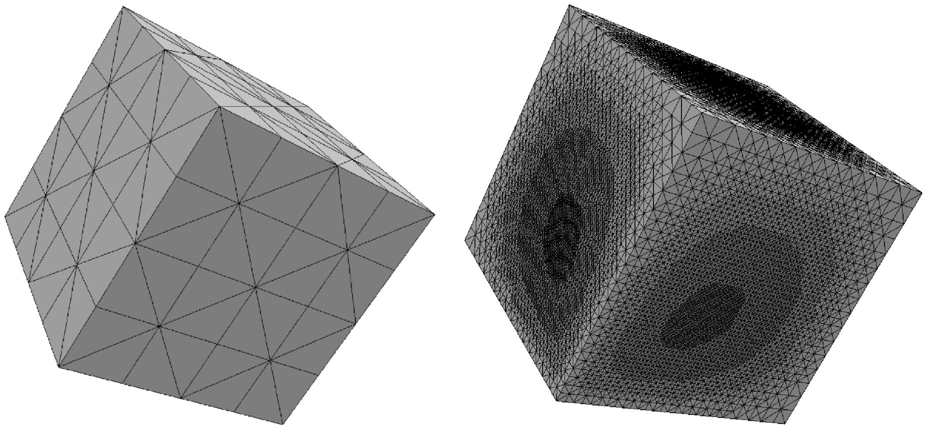

Figure 1. The initial triangulation and the one after adaptive iterations for Example 6.1.

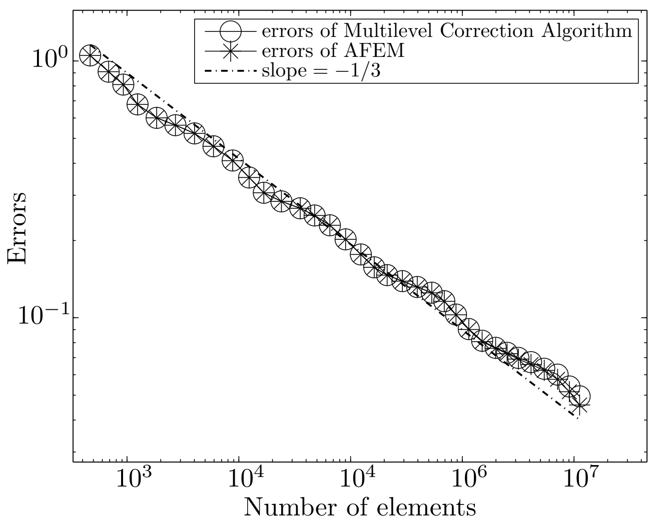

Figure 2. Errors by adaptive finite element method of Algorithm 3.3 for Example 6.1.

E x a m p l e 6.1. We consider the semilinear elliptic problem

$$\begin{cases} -\Delta u + u^3 = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $\Omega = (0,1)^3$. We choose the right-hand side term f such that the exact solution is given by

$$u = \frac{\sin(\pi x)\sin(\pi y)}{(x^2+y^2)^{1/2}}. \tag{6.2}$$

We give the numerical results for the approximate solutions by Algorithm 3.3 with the parameter $\theta = 0.4$. In order to show the efficiency more clearly, we compare the results with those obtained by the direct AFEM. Figure 1 shows the initial triangulation and the one after 20 adaptive iterations. It is shown in Figure 2 that the approximate solutions by Algorithm 3.3 have the optimal convergence rate which coincides with the theoretical results.

Example 6.2. In the second example, we solve the semilinear elliptic problem

$$\begin{cases} -\Delta u + u^{3/2} = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

where $\Omega = (0,1)\times(0,2)\times(0,2)\setminus(0,1)\times[1,2)\times[1,2)$. Due to the reentrant corner of Ω, the exact solution with singularities is expected. Since the exact solution is not known, we choose an adequately accurate approximate solution on a fine enough mesh as the exact one.

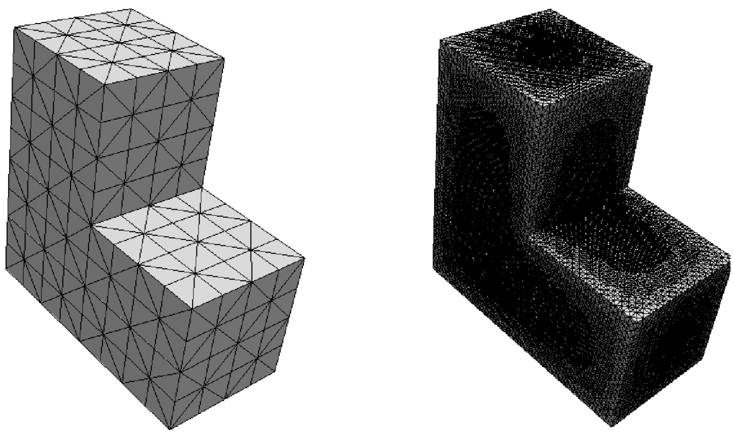

Figure 3. The initial triangulation and the one after adaptive iterations for Example 6.2.

Algorithm 3.3 is applied to this example with parameter $\theta = 0.4$. Figure 3 shows the initial mesh and the one after 18 adaptive iterations. In order to show the efficiency of Algorithm 3.3 more clearly, we also compare the results with those obtained

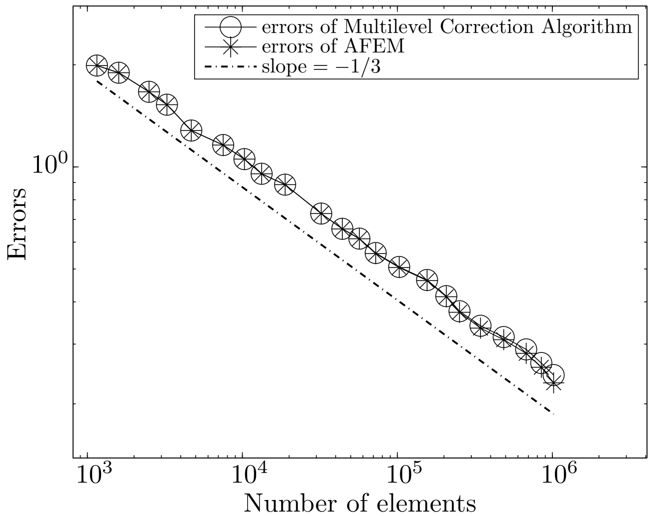

Figure 4. Errors by adaptive finite element method of Algorithm 3.3 for Example 6.2.

by the direct AFEM. Figure 4 gives the corresponding numerical results for the first 22 adaptive iterations which show the optimal convergence rate of Algorithm 3.3.

7. Concluding remarks

In this paper, we propose a type of multilevel correction type of AFEM for semilinear elliptic equations. Unlike the classical AFEM for semilinear equations, the proposed method transforms the semilinear equation solving to a series of linear elliptic equation solving and some semilinear elliptic solutions in a very low dimensional space. The high efficiency of linear elliptic equation solving can improve the overall efficiency of the AFEM for semilinear elliptic equations. The corresponding analysis of the convergence and optimal complexity has also been given.

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