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# RATIONALITY PRINCIPLES FOR PREFERENCES ON BELIEF FUNCTIONS

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A generalized notion of lottery is considered, where the uncertainty is expressed by a belief function. Given a partial preference relation on an arbitrary set of generalized lotteries all on the same finite totally ordered set of prizes, conditions for the representability, either by a linear utility or a Choquet expected utility are provided. Both the cases of a finite and an infinite set of generalized lotteries are investigated.

*Keywords:* generalized lottery, preference relation, belief function, linear utility, Choquet expected utility, rationality conditions

*Classification:* 91B06, 91B16

## 1. INTRODUCTION

The classical von Neumann–Morgenstern's model for *decision under risk* refers to a total *preference relation* on a suitable class of random quantities (also called *lotteries*) each equipped with a probability distribution. In this setting, the aim is to find necessary and sufficient conditions for the *representability* of a preference relation by a *linear utility* [16, 28]. In particular, assuming some additional conditions on the class of lotteries the linear utility can be expressed as an *expected utility* (EU) [28].

Nevertheless, in several situations uncertainty cannot be expressed through a probability but it is unavoidable to refer to non-additive uncertainty measures, such as Dempster-Shafer *belief functions* [8, 23]. Recall that in some probabilistic inferential problems belief functions can be obtained as lower envelopes of a family of probabilities [6, 7, 8, 12, 19].

For the above motivation, decision models have been generalized in a way to express decision maker's uncertainty by a non-additive measure: the resulting models look for a representation either by a *linear utility* (LU) [17] or a *Choquet expected utility* (CEU) which has to be maximized [2, 21, 29, 30, 31].

In any case, the quoted models still refer to a total preference relation on a set of objects (lotteries or their generalizations) closed with respect to suitable operations (such as convex combination) and containing particular sub-classes (such as the class of degenerate lotteries).

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In fact, to find the utility $u$ the classical methods ask for comparisons between "lotteries" and "certainty equivalent" or, anyway, comparisons among particular large classes of lotteries (for a discussion in the EU framework see [18]). Hence, the decision maker is often forced to provide comparisons which have little or nothing to do with the given problem, having to choose between risky prospects and certainty.

On the other hand, in case of partial information (i. e., when the preference relation is not total and is assessed on an arbitrary set of lotteries) it can be difficult to construct the utility function, and even to test if the preferences agree with the model of reference.

In [5], referring to the EU model, a different approach (based on a "rationality principle") is proposed: it does not need all these non-natural comparisons but, instead, it can work by considering only the (few) lotteries and comparisons of interest. The mentioned "rationality principle" can be summarized as follows: *it is not possible to obtain the same lottery by combining in the same way two groups of lotteries, if every lottery of the first group is not preferred to the corresponding one of the second group, and at least a preference is strict.*

Such a principle allows to assign a preference relation $(\precsim, \prec)$, just among *some* lotteries and it permits to check that the given relation does not contain inconsistencies, in the sense that it guarantees the existence of a utility function $u$ on the set of prizes whose expected value agrees with $(\precsim, \prec)$.

In this paper we cope with a generalization of lotteries already introduced in [17], where a *generalized lottery* $L$ (or *g-lottery* for short) is intended to be a random quantity with a finite support $X_L$ endowed with a *belief function* $Bel_L$ [8, 23, 26] (or, equivalently, a *basic assignment* $m_L$) defined on the power set $\wp(X_L)$.

Our aim is to propose an approach similar to that in [5] for both the linear utility model and the Choquet expected utility model.

The first "rationality principle" proposed here (namely, axiom **(g-R)**) is a direct generalization of the "rationality principle" given in the case of probabilities, simply obtained by changing "lotteries" with "generalized lotteries". Given a preference relation on an arbitrary (possibly infinite) set of generalized lotteries $\mathcal{L}$, this principle is a necessary and sufficient condition for the existence, for every finite $\mathcal{F} \subseteq \mathcal{L}$, of a linear utility representing the restriction of the preference relation to $\mathcal{F}$.

Nevertheless this principle is only necessary for the representability of the preference by a Choquet expected utility.

On the contrary, the "Choquet rationality principle" (namely, axiom **(g-CR)**) is based on the following property of Choquet integral for belief functions defined on the power set $\wp(X)$ of a finite set $X = \{x_1, \ldots, x_n\}$ totally ordered as $x_1 <^* \ldots <^* x_n$: for every g-lottery $L$ with support $X_L \subseteq X$, the Choquet integral of any strictly increasing utility function $u : X \to \mathbb{R}$, not only is a weighted average (as observed in [15]), but the weights have a clear meaning. In fact, for the worst prize $x_1$ the weight is the sum of the values of the basic assignment $m_L$ on the events implied by the event $\{L = x_1\}$, for $x_2$ it is the sum of the values of $m_L$ on the events implied by the event $\{L = x_2\}$ but not by $\{L = x_1\}$, and so on. This allows to map every g-lottery $L$ to a "standard" lottery whose probability distribution is constructed (following a pessimistic approach) through the *aggregated basic assignment* $M_L$. We note that $M_L$ is, in some sense, a "pessimistic" probability distribution on $X$ induced by a belief function.

Then axiom **(g-CR)** requires that *it is not possible to obtain two g-lotteries $L$ and $L'$, with $M_L = M_{L'}$ by combining in the same way the aggregated basic assignments of two groups of g-lotteries, if every g-lottery of the first group is not preferred to the corresponding one of the second group, and at least a preference is strict.*

Given a preference relation on an arbitrary (possibly infinite) set of generalized lotteries $\mathcal{L}$, axiom **(g-CR)** is a necessary and sufficient condition for the existence, for every finite $\mathcal{F} \subseteq \mathcal{L}$, of a strictly increasing utility function $u_{\mathcal{F}}$ whose CEU represents the restriction of the preference relation to $\mathcal{F}$.

Both rationality principles **(g-R)** and **(g-CR)** are not sufficient to assure representability on an infinite set of generalized lotteries. However, if the preference relation is total and assessed on the set of *all* belief functions on $\wp(X)$ it is sufficient to add an Archimedean axiom to **(g-R)** or **(g-CR)** in order to obtain, respectively, the representability by a linear utility functional or a Choquet expected utility functional.

## 2. PRELIMINARIES

### 2.1. Belief functions

Let $X$ be a finite set and denote by $\wp(X)$ the power set of $X$. We recall that a *belief function Bel* [8, 23, 26] on a field of subsets $\mathcal{A} \subseteq \wp(X)$ is a function such that $Bel(\emptyset) = 0$, $Bel(X) = 1$ and satisfying the $n$-monotonicity property for every $n \geq 2$, i. e., for every $A_1, \ldots, A_n \in \mathcal{A}$,

$$Bel\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right). \tag{1}$$

Previous properties imply the monotonicity of *Bel* with respect to set inclusion $\subseteq$, hence belief functions are particular *normalized capacities* [3].

A belief function *Bel* on $\mathcal{A}$ is completely singled out by its *Möbius inverse* defined for every $A \in \mathcal{A}$ as

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} Bel(B).$$

Such a function, usually called *basic (probability) assignment*, is a function $m : \mathcal{A} \to [0,1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A \in \mathcal{A}} m(A) = 1$, and is such that for every $A \in \mathcal{A}$

$$Bel(A) = \sum_{B \subseteq A} m(B). \tag{2}$$

A set $A$ in $\mathcal{A}$ is a *focal element* for $m$ (and so also for the corresponding *Bel*) whenever $m(A) > 0$.

Let $\mathcal{A}$ be a field of subsets of a non-empty set $X$ and $\varphi : \mathcal{A} \to [0,1]$ a normalized capacity, the *Choquet integral* of an $\mathcal{A}$-measurable function $f : X \to \mathbb{R}$ is defined as

$$\oint f \,\mathrm{d}\varphi = \int_{-\infty}^{0} (\varphi(\{x : f(x) \geq t\}) - 1) \,\mathrm{d}t + \int_{0}^{+\infty} \varphi(\{x : f(x) \geq t\}) \,\mathrm{d}t, \tag{3}$$

where both integrals on the right-side are of Riemann type.

If $X = \{x_1, \ldots, x_n\}$ and $f(x_1) \leq \ldots \leq f(x_n)$ (see [9]), then

$$\oint f \,\mathrm{d}\varphi = \sum_{i=1}^{n} f(x_i)(\varphi(E_i) - \varphi(E_{i+1})),$$

where $E_i = \{x_i, \ldots, x_n\}$ for $i = 1, \ldots, n$, and $E_{n+1} = \emptyset$.

## 2.2. Classical linear utility model

In the classical von Neumann–Morgenstern theory [28] a total preference relation $\precsim$ on a set $\mathcal{L}$ of *lotteries* is given, where a *lottery* $L$ is a *random quantity* endowed with a *probability distribution* $P_L$ on $X = \{x_1, \ldots, x_n\}$ with support $X_L \subseteq X$.

The theory requires that the set of lotteries $\mathcal{L}$ is a *mixture set* [16], i. e., for every $L, G \in \mathcal{L}$ and $\alpha \in [0, 1]$ the *convex combination* defined as

$$\alpha L + (1-\alpha)G = \begin{pmatrix} x_1 & \cdots & x_n \\ \alpha P_L(x_1) + (1-\alpha)P_G(x_1) & \cdots & \alpha P_L(x_n) + (1-\alpha)P_G(x_n) \end{pmatrix}, \quad (4)$$

is a lottery belonging to $\mathcal{L}$.

**Remark 2.1.** Every $L \in \mathcal{L}$ can simply be viewed as a vector $(P_L(x_1), \ldots, P_L(x_n)) \in [0, 1]^n$, thus $\mathcal{L}$ is a mixture set if and only if $\{(P_L(x_1), \ldots, P_L(x_n)) \in [0, 1]^n : L \in \mathcal{L}\}$ is a convex subset of $[0, 1]^n$.

The goal is to determine a set of axioms that the relation $\precsim$ has to satisfy in order to have a linear utility function $U : \mathcal{L} \to \mathbb{R}$ representing $\precsim$, i. e., such that for every $L, G \in \mathcal{L}$

$$L \precsim G \text{ if and only if } U(L) \leq U(G).$$

An axiom system characterizing the above representation is the following one:

**(VM1)** the preference relation $\precsim$ is a weak order;

**(VM2)** for every $L, G, H \in \mathcal{L}$ and $\lambda \in [0, 1]$, if $L \precsim G$ then

$$\lambda L + (1 - \lambda)H \precsim \lambda G + (1 - \lambda)H;$$

**(VM3)** for every $L, G \in \mathcal{L}$ and $0 \leq \beta < \alpha \leq 1$, if $L \prec G$ then

$$\alpha L + (1 - \alpha)G \prec \beta L + (1 - \beta)G;$$

**(VM4)** for every $L, G, H \in \mathcal{L}$ such that $L \prec G \prec H$ there exists an $\alpha \in ]0, 1[$ such that

$$\alpha L + (1 - \alpha)H \sim G.$$

We recall that in the literature there are many other axiom systems equivalent to **(VM1)**–**(VM4)** as the one proposed in [16]. In particular, axioms **(VM1)**–**(VM4)** do not rely on the structure of the lotteries in $\mathcal{L}$ but only on the fact that $\mathcal{L}$ is a mixture set.

It is well-known that, denoting with $\delta_x$ the *degenerate lottery* assigning probability 1 to $x \in X$, if $\mathcal{L}_0 = \{\delta_x : x \in X\} \subseteq \mathcal{L}$ and $X = \{x_1, \dots, x_n\}$, then there exists $u : X \to \mathbb{R}$ such that $u(x) = U(\delta_x)$ for $x \in X$, and for every $L \in \mathcal{L}$

$$U(L) = \mathbb{E}(u(L)) = \sum_{i=1}^{n} u(x_i) P_L(x_i),$$

i. e., $U$ can be expressed as an *expected utility.*

The classical expected utility theory has been object of many criticisms introduced through famous paradoxes, where preferences assessed by means of human experiments cannot be modelled through a linear functional of a utility function and a probability measure. We consider the following example, inspired by the well-known Ellsberg's paradox [11], which will be developed in the rest of the paper.

**Example 2.2.** Consider the following hypothetical experiment. Let us take two urns, $U_1$ and $U_2$, from which we draw a ball each. $U_1$ contains $\frac{1}{3}$ of white ($w$) balls and the remaining balls are black ($b$) and red ($r$), but in a ratio entirely unknown to us, analogously, $U_2$ contains $\frac{1}{4}$ of green ($g$) balls and the remaining balls are yellow ($y$) and orange ($o$), but in a ratio entirely unknown to us.

In light of the given information, the composition of $U_1$ singles out a class of probability measures $\mathbf{P}^1 = \{P^\theta\}$ on the power set $\wp(S_1)$ of $S_1 = \{w, b, r\}$ such that

$$P^\theta(\{w\}) = \frac{1}{3}, P^\theta(\{b\}) = \theta, P^\theta(\{r\}) = \frac{2}{3} - \theta,$$

with $\theta \in \left[0, \frac{2}{3}\right]$. Analogously, for the composition of $U_2$ consider the class $\mathbf{P}^2 = \{P^\lambda\}$ on $\wp(S_2)$ with $S_2 = \{g, y, o\}$ such that

$$P^\lambda(\{g\}) = \frac{1}{4}, P^\lambda(\{y\}) = \lambda, P^\lambda(\{o\}) = \frac{3}{4} - \lambda,$$

with $\lambda \in \left[0, \frac{3}{4}\right]$.

Concerning the ball drawn from $U_1$ and the one drawn from $U_2$, the following gambles are considered:

| | $w$ | $b$ | $r$ |
|---|---|---|---|
| $L_1$ | 100€ | 0€ | 0€ |
| $L_2$ | 0€ | 0€ | 100€ |
| $L_3$ | 0€ | 100€ | 100€ |
| $L_4$ | 100€ | 100€ | 0€ |

| | $g$ | $y$ | $o$ |
|---|---|---|---|
| $G_1$ | 100€ | 10€ | 10€ |

Consider the strict preferences $L_2 \prec L_1$, $L_4 \prec L_3$ assessed through a human experiment. Since the given preferences involve only gambles related to the first urn, we can restrict to the set of prizes $\{0, 100\}$. It holds that for no value of $\theta$ there exists a function $u : \{0, 100\} \to \mathbb{R}$ such that its expected value on the $L_i$'s with respect to $P^\theta$ represents our preferences on the $L_i$'s. Indeed, putting $w_1 = u(0)$ and $w_2 = u(100)$, both the following inequalities must hold

$$\frac{1}{3}w_1 + \theta w_1 + \left(\frac{2}{3} - \theta\right) w_2 < \frac{1}{3}w_2 + \theta w_1 + \left(\frac{2}{3} - \theta\right) w_1,$$

$$\frac{1}{3}w_2 + \theta w_2 + \left(\frac{2}{3} - \theta\right) w_1 < \frac{1}{3}w_1 + \theta w_2 + \left(\frac{2}{3} - \theta\right) w_2,$$

from which, summing memberwise, we get $w_1 + w_2 < w_1 + w_2$, i.e., a contradiction.

## 2.3. Linear utility model for belief functions

In [17] a generalized notion of lottery $L$ is given by assuming that a *belief function* $Bel_L$ is assigned on the power set $\wp(X_L)$ of a finite set $X_L$.

**Definition 2.3.** A **generalized lottery**, or **g-lottery** for short, on a finite set $X_L$ is a pair $L = (\wp(X_L), Bel_L)$ where $Bel_L$ is a belief function on $\wp(X_L)$.

Let us notice that, a g-lottery $L = (\wp(X_L), Bel_L)$ could be equivalently defined as $L = (\wp(X_L), m_L)$, where $m_L$ is the basic assignment associated to $Bel_L$. We stress that this definition of g-lottery generalizes the classical one in which $m_L(A) = 0$ for any $A \in \wp(X_L)$ with $\operatorname{card} A > 1$.

For example, a g-lottery $L$ on $X_L = \{x_1, x_2, x_3\}$ can be expressed as

$$L = \left(\begin{array}{ccccccc} \{x_1\} & \{x_2\} & \{x_3\} & \{x_1, x_2\} & \{x_1, x_3\} & \{x_2, x_3\} & X_L \\ b_{\{x_1\}} & b_{\{x_2\}} & b_{\{x_3\}} & b_{\{x_1,x_2\}} & b_{\{x_1,x_3\}} & b_{\{x_2,x_3\}} & b_{X_L} \end{array}\right)$$

where the belief function $Bel_L$ on $\wp(X_L)$ is such that $b_A = Bel_L(A)$ for every $A \in \wp(X_L) \setminus \{\emptyset\}$. Notice that since $Bel_L(\emptyset) = m_L(\emptyset) = 0$, the empty set is not reported in the tabular expression of $L$. An equivalent representation of previous g-lottery is obtained through the basic assignment $m_L$ associated to $Bel_L$ (where $m_A = m_L(A)$ for every $A \in \wp(X_L) \setminus \{\emptyset\}$)

$$L = \left(\begin{array}{ccccccc} \{x_1\} & \{x_2\} & \{x_3\} & \{x_1, x_2\} & \{x_1, x_3\} & \{x_2, x_3\} & X_L \\ m_{\{x_1\}} & m_{\{x_2\}} & m_{\{x_3\}} & m_{\{x_1,x_2\}} & m_{\{x_1,x_3\}} & m_{\{x_2,x_3\}} & m_{X_L} \end{array}\right).$$

Given a set $\mathcal{L}$ of g-lotteries, let $X = \bigcup\{X_L : L \in \mathcal{L}\}$. In the case $X$ is a finite set, any g-lottery $L$ on $X_L$ with belief function $Bel_L$ can be rewritten as a g-lottery on $X$ by defining a suitable extension $Bel'_L$ of $Bel_L$.

**Proposition 2.4.** Let $L = (\wp(X_L), Bel_L)$ be a g-lottery on $X_L$. Then, for any finite $X \supseteq X_L$ there exists a unique belief function $Bel'_L$ on $\wp(X)$ with the same focal elements as $Bel_L$ and such that $Bel'_{L|\wp(X_L)} = Bel_L$.

Proof. The extension $Bel'_L$ is defined through the corresponding $m'_L$. For every $A \in \wp(X)$ we put $m'_L(A) = m_L(A)$ if $A \in \wp(X_L)$ and $m'_L(A) = 0$ otherwise. The function $m'_L$ is easily seen to be a basic assignment on $\wp(X)$, moreover, the corresponding belief function $Bel'_L$ on $\wp(X)$ is an extension of $Bel_L$ and has the same focal elements. □

Given $L_1, \ldots, L_t \in \mathcal{L}$, all rewritten on $X$, and a real vector $\mathbf{k} = (k_1, \ldots, k_t)$ with $k_i \geq 0$ $(i = 1, \ldots, t)$ and $\sum_{i=1}^t k_i = 1$, the *convex combination* of $L_1, \ldots, L_t$ according to $\mathbf{k}$ is the g-lottery on $X$

$$\mathbf{k}(L_1, \ldots, L_t) = \left(\begin{array}{c} A \\ \sum_{i=1}^t k_i m_{L_i}(A) \end{array} : A \in \wp(X) \setminus \{\emptyset\}\right). \tag{5}$$

Since the convex combination of belief functions is a belief function, it immediately follows that $\mathbf{k}(L_1, \ldots, L_t)$ is a g-lottery [17].

**Remark 2.5.** Every $L \in \mathcal{L}$ can simply be viewed as a vector $(m_L(A) : A \in \wp(X)) \in [0,1]^{2^n}$, thus $\mathcal{L}$ is a mixture set if and only if $\{(m_L(A) : A \in \wp(X)) \in [0,1]^{2^n} : L \in \mathcal{L}\}$ is a convex subset of $[0,1]^{2^n}$, for a finite set $X = \{x_1, \ldots, x_n\}$.

For every $A \in \wp(X) \setminus \{\emptyset\}$, there exists a *degenerate g-lottery* $\delta_A$ on $X$ such that $m_{\delta_A}(A) = 1$, and, moreover, every g-lottery $L$ with focal elements $A_1, \ldots, A_k$ can be expressed as $\mathbf{k}(\delta_{A_1}, \ldots, \delta_{A_k})$ with $\mathbf{k} = (m_L(A_1), \ldots, m_L(A_k))$.

For the above considerations, a set of g-lotteries $\mathcal{L}$, all written on the same finite set $X$, which is closed under convex combination, is again a mixture set. Hence, as proved in [17] by using results in [16], **(VM1)–(VM4)** are necessary and sufficient conditions for the existence of a linear utility function $\mathrm{LU} : \mathcal{L} \to \mathbb{R}$ representing $\precsim$. In particular, if $\mathcal{L}_0^* = \{\delta_B : B \in \wp(X) \setminus \{\emptyset\}\} \subseteq \mathcal{L}$ then there exists $v : \wp(X) \to \mathbb{R}$ such that for every $L \in \mathcal{L}$

$$\mathrm{LU}(L) = \sum_{B \in \wp(X)} v(B) m_L(B), \tag{6}$$

where $\mathrm{LU}(\delta_B) = v(B)$, for every $B \in \wp(X) \setminus \{\emptyset\}$.

A drawback of the above functional is that the "utility" $v(B)$ with $B \in \wp(X)$ and $\operatorname{card} B > 1$ has not a clear semantic interpretation.

As already noticed in [17], there is also a computational difficulty due to the number of parameters to specify, which is $2^{\operatorname{card} X}$, since $v$ must be assessed on $\wp(X)$. This is why in [17] a more restrictive functional is proposed which is based only on the "best" and "worst" alternatives in every set $B \in \wp(X)$, according to the relation on $X$ obtained restricting $\precsim$ on $\{\delta_{\{x\}} : x \in X\}$. To get a representation with such a functional, the axiom system has to be reinforced by introducing a further axiom of dominance [17].

Another possible choice for the function $U$ is based on the *Choquet expected utility* [15, 24],

$$\mathrm{CEU}(L) = \oint u \, \mathrm{d}Bel_L, \tag{7}$$

where $u : X \to \mathbb{R}$ is a utility function and $\mathrm{CEU}(\delta_{\{x\}}) = u(x)$, for every $x \in X$.

It is known [24] that $\mathrm{CEU}(L)$ coincides with a lower expected utility with respect to the class $\mathcal{P}_{Bel_L}$ of probabilities on $\mathcal{P}(X_L)$ dominating $Bel_L$ and so the maximization of (7) corresponds to the well-known *maxmin criterion* of choice [14, 15].

In the following example we show that the preferences in Example 2.2 are representable both by a linear utility defined as in (6) and by a Choquet expected utility defined as in (7).

**Example 2.6. (Example 2.2 continued)** Now take $\underline{P}^1 = \min \mathbf{P}^1$ and $\underline{P}^2 = \min \mathbf{P}^2$, where the minimum is intended pointwise on the elements of $\wp(S_1)$ and $\wp(S_2)$, obtaining:

| $\wp(S_1)$ | $\emptyset$ | $\{w\}$ | $\{b\}$ | $\{r\}$ | $\{w,b\}$ | $\{w,r\}$ | $\{b,r\}$ | $S_1$ |
|---|---|---|---|---|---|---|---|---|
| $\underline{P}^1$ | 0 | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 |

| $\wp(S_2)$ | $\emptyset$ | $\{g\}$ | $\{y\}$ | $\{o\}$ | $\{g,y\}$ | $\{g,o\}$ | $\{y,o\}$ | $S_2$ |
|---|---|---|---|---|---|---|---|---|
| $\underline{P}^2$ | 0 | $\frac{1}{4}$ | 0 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | 1 |

It is easily verified that both $\underline{P}^1$ and $\underline{P}^2$ are belief functions giving rise to the following g-lotteries

$$L_1 = \left( \begin{array}{ccc} \{0\} & \{100\} & \{0,100\} \\ \frac{2}{3} & \frac{1}{3} & 1 \end{array} \right), \; L_2 = \left( \begin{array}{ccc} \{0\} & \{100\} & \{0,100\} \\ \frac{1}{3} & 0 & 1 \end{array} \right),$$

$$L_3 = \left( \begin{array}{ccc} \{0\} & \{100\} & \{0,100\} \\ \frac{1}{3} & \frac{2}{3} & 1 \end{array} \right), \; L_4 = \left( \begin{array}{ccc} \{0\} & \{100\} & \{0,100\} \\ 0 & \frac{1}{3} & 1 \end{array} \right),$$

$$G_1 = \left( \begin{array}{ccc} \{10\} & \{100\} & \{10,100\} \\ \frac{3}{4} & \frac{1}{4} & 1 \end{array} \right).$$

The preferences $L_2 \prec L_1$, $L_4 \prec L_3$, are representable both by a linear utility and by a Choquet expected utility. Again, since the given preferences involve only g-lotteries $L_i$'s, we can restrict to the set of prizes $\{0, 100\}$.

For the first representation, any function $v : \wp(\{0, 100\}) \to \mathbb{R}$ satisfying

$$2v(\{0, 100\}) < v(\{0\}) + v(\{100\}),$$

is such that the weighted average LU of $v$ on $\wp(\{0, 100\})$ with weights given by $m_{L_i}$, $i = 1, \ldots, 4$, represents the given preferences. Indeed, denoting with $q_1 = v(\emptyset)$, $q_2 = v(\{0\})$, $q_3 = v(\{100\})$ and $q_4 = v(\{0, 100\})$ we have

$$\mathrm{LU}(L_1) = \frac{2}{3}q_2 + \frac{1}{3}q_3, \; \mathrm{LU}(L_2) = \frac{1}{3}q_2 + \frac{2}{3}q_4, \; \mathrm{LU}(L_3) = \frac{1}{3}q_2 + \frac{2}{3}q_3, \; \mathrm{LU}(L_4) = \frac{1}{3}q_3 + \frac{2}{3}q_4,$$

thus $\mathrm{LU}(L_2) < \mathrm{LU}(L_1)$ and $\mathrm{LU}(L_4) < \mathrm{LU}(L_3)$ hold whenever $2q_4 < q_2 + q_3$.

For the second representation, instead, any strictly increasing function $u : \{0, 100\} \to \mathbb{R}$ is such that the Choquet integral of $u$ with respect to $Bel_{L_i}$, $i = 1, \ldots, 4$, represents our preferences. Indeed, denoting $w_1 = u(0)$ and $w_2 = u(100)$ we get

$$\mathrm{CEU}(L_1) = \mathrm{CEU}(L_4) = \frac{2}{3}w_1 + \frac{1}{3}w_2, \quad \mathrm{CEU}(L_2) = w_1, \quad \mathrm{CEU}(L_3) = \frac{1}{3}w_1 + \frac{2}{3}w_2,$$

thus $\mathrm{CEU}(L_2) < \mathrm{CEU}(L_1)$ and $\mathrm{CEU}(L_4) < \mathrm{CEU}(L_3)$ hold whenever $w_1 < w_2$.

## 3. PREFERENCES OVER A SET OF GENERALIZED LOTTERIES

Consider a set $\mathcal{L}$ of g-lotteries with $X = \bigcup\{X_L : L \in \mathcal{L}\}$. Assume that $X$ is *totally ordered* and denote by $\leq^*$ this relation. Let $<^*$ be the total strict order on $X$ induced by $\leq^*$. The assumption that $X$ is totally ordered is quite natural when the elements of $X$ are real numbers (e. g., when they are money payoffs) and in this case $\leq^*$ simply coincides with the usual total order $\leq$ on $\mathbb{R}$. Furthermore, such condition is acceptable also in the case $X$ is composed of arbitrary objects. In this case, since $X$ is a set of prizes, the elicitation of the relation $<^*$ (and so of $\leq^*$) amounts to ask to the decision maker to rank the elements of $X$ by their strict desirability, assuming he/she could receive them with certainty. In the particular interpretation of $X$ as a set of money rewards, this encodes the economic idea of "more money is better".

In what follows the set $X$ is always assumed to be finite, i.e., $X = \{x_1, \ldots, x_n\}$ with $x_1 <^* \ldots <^* x_n$. Under previous assumption, we can define the *aggregated basic assignment* of a g-lottery $L$, for every $x_i \in X$, as

$$M_L(x_i) = \sum_{\{x_i\} \subseteq B \subseteq E_i} m_L(B), \tag{8}$$

where $E_i = \{x_i, \ldots, x_n\}$ for $i = 1, \ldots, n$.

Note that $M_L(x_i) \geq 0$ for every $x_i \in X$ and $\sum_{i=1}^n M_L(x_i) = 1$, thus $M_L$ determines a probability distribution on $X$.

Let $\mathcal{R}$ be a *partial* binary relation on $\mathcal{L}$. For every $(L, L') \in \mathcal{R}$ denote by $L \precsim L'$ the assertion *$L$ is not preferred to $L'$*. The assertion *$L$ is indifferent to $L'$*, denoted by $L \sim L'$, summarizes the two assertions $L \precsim L'$ and $L' \precsim L$, so $\mathcal{R}$ determines the symmetric relation $\mathcal{I} = \{(L, L') \in \mathcal{R} : (L', L) \in \mathcal{R}\}$. An additional strict preference relation $\mathcal{R}'$ can be elicited by assertions such as *$L'$ is strictly preferred to $L$*, denoted by $L \prec L'$. Let $\mathcal{R}^*$ be the asymmetric relation formally deduced from $\mathcal{R}$, namely $\mathcal{R}^* = \mathcal{R} \setminus \mathcal{I}$.

Since the pair of relations $(\mathcal{R}, \mathcal{R}')$ represents the opinion of the decision maker, it is natural to have $\mathcal{R}' \subseteq \mathcal{R}^*$: in fact, it is possible that, in the first approach to the decision problem, the decision maker is not able to evaluate yet whether $L \prec L'$ or $L \sim L'$ and he/she expresses his/her opinion only by $L \precsim L'$.

Obviously, if $\mathcal{R}$ is total on the set of g-lotteries $\mathcal{L}$ then $\mathcal{R}' = \mathcal{R}^*$ and for every $L, L' \in \mathcal{L}$: $L \prec L'$ or $L' \prec L$ or $L \sim L'$.

We call a pair $(\mathcal{R}, \mathcal{R}')$, denoted in the following by $(\precsim, \prec)$, a *strengthened preference relation* if $\emptyset \neq \mathcal{R}' \subseteq \mathcal{R}$ and $\mathcal{I} \cap \mathcal{R}' = \emptyset$.

Since the set $X$ is totally ordered by $\leq^*$, it is natural to require that the partial preference relation $(\precsim, \prec)$ agrees with $\leq^*$ on degenerate g-lotteries $\delta_{\{x\}}$, for $x \in X$, that correspond to decisions under certainty. For this the preference $(\precsim, \prec)$ is asked to satisfy the following assumption

**(A0)** $\mathcal{L}$ contains the set of degenerate g-lotteries on singletons $\mathcal{L}_0 = \{\delta_{\{x\}} : x \in X\}$ and if $x <^* x'$ then $\delta_{\{x\}} \prec \delta_{\{x'\}}$, for $x, x' \in X$.

Note that the decision maker is not required to provide comparisons between degenerate g-lotteries and non-degenerate g-lotteries, but just to accept the (natural) preferences considered in condition **(A0)**.

We say that a function $U : \mathcal{L} \to \mathbb{R}$ *represents* (or *agrees with*) $(\precsim, \prec)$ if, for every $L, L' \in \mathcal{L}$

$$L \precsim L' \Longrightarrow U(L) \leq U(L') \text{ and } L \prec L' \Longrightarrow U(L) < U(L'). \tag{9}$$

In the following we consider two different functionals for the utility $U$ representing the preference $(\precsim, \prec)$: a linear utility LU (as proposed in [17]) and a Choquet expected utility CEU (as proposed in [2, 24, 25]).

Since our main interest is to manage a finite set $\mathcal{L}$ of g-lotteries with a possibly partial preference relation $(\precsim, \prec)$, in analogy with [5], we search for a necessary and sufficient condition for the existence of either a linear utility function on $\mathcal{L}$ representing $(\precsim, \prec)$ or a utility function $u : X \to \mathbb{R}$ such that its Choquet expected value represents $(\precsim, \prec)$.

### 3.1. Rationality conditions for g-lotteries

The first axiom of rationality we introduce is formally equal to the one given in [5]: it requires that it is not possible to obtain the same g-lottery, by combining in the same way two groups of g-lotteries, if each g-lottery in the first group is not preferred to the corresponding one in the second group, and at least a preference is strict.

**Definition 3.1.** A strengthened preference relation $(\precsim, \prec)$ on a set $\mathcal{L}$ of g-lotteries is said to be **rational** if it satisfies the following condition:

**(g-R)** For all $h \in \mathbb{N}$ and $L_i, L_i' \in \mathcal{L}$ with $L_i \precsim L_i'$ $(i = 1, \dots, h)$, if

$$\mathbf{k}(L_1, \dots, L_h) = \mathbf{k}(L_1', \dots, L_h')$$

with $\mathbf{k} = (k_1, \dots, k_h)$, $k_i > 0$ $(i = 1, \dots, h)$, and $\sum_{i=1}^h k_i = 1$, then it cannot be $L_i \prec L_i'$ for any $i = 1, \dots, h$.

We stress that the convex combination referred to in condition **(g-R)** involves belief functions (or, equivalently, the corresponding basic assignments).

The second axiom requires that it is not possible to obtain the same probability distribution on $X$, by combining in the same way the aggregated basic assignments of two groups of g-lotteries, if each g-lottery in the first group is not preferred to the corresponding one in the second group, and at least a preference is strict.

**Definition 3.2.** A strengthened preference relation $(\precsim, \prec)$ on a set $\mathcal{L}$ of g-lotteries is said to be **Choquet rational** if it satisfies the following condition:

**(g-CR)** For all $h \in \mathbb{N}$ and $L_i, L_i' \in \mathcal{L}$ with $L_i \precsim L_i'$ $(i = 1, \dots, h)$, if

$$\mathbf{k}(M_{L_1}, \dots, M_{L_h}) = \mathbf{k}(M_{L_1'}, \dots, M_{L_h'})$$

with $\mathbf{k} = (k_1, \dots, k_h)$, $k_i > 0$ $(i = 1, \dots, h)$ and $\sum_{i=1}^h k_i = 1$, then it cannot be $L_i \prec L_i'$ for any $i = 1, \dots, h$.

Note that the convex combination referred to in condition **(g-CR)** is the usual one involving probability distributions on $X$, since aggregated basic assignments are probability distributions on $X$.

It is easily proven that if $\mathbf{k}(L_1, \dots, L_h) = \mathbf{k}(L_1', \dots, L_h')$ then $\mathbf{k}(M_{L_1}, \dots, M_{L_h}) = \mathbf{k}(M_{L_1'}, \dots, M_{L_h'})$, but the converse is generally not true, as shown in next example (so condition **(g-CR)** strictly implies condition **(g-R)**).

**Example 3.3.** Let $X = \{x_1, x_2\}$ with $x_1 <^* x_2$ and consider the g-lotteries

$$L_1 = \begin{pmatrix} \{x_1\} & \{x_2\} & \{x_1, x_2\} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}, L_1' = \begin{pmatrix} \{x_1\} & \{x_2\} & \{x_1, x_2\} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} \{x_1\} & \{x_2\} & \{x_1, x_2\} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}, L_2' = \begin{pmatrix} \{x_1\} & \{x_2\} & \{x_1, x_2\} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix},$$

with the preferences $L_1 \precsim L_1'$ and $L_2 \precsim L_2'$. There is no $k \in [0,1]$ such that $kL_1 + (1-k)L_2 = kL_1' + (1-k)L_2'$, indeed, the following system

$$\begin{cases} k\frac{1}{4} = k\frac{1}{3} \\ k\frac{3}{4} + (1-k)\frac{2}{3} = k\frac{2}{3} + (1-k)\frac{3}{4} \\ (1-k)\frac{1}{3} = (1-k)\frac{1}{4} \end{cases}$$

has not solution. Nevertheless, considering the aggregated basic assignments of $L_1, L_1', L_2, L_2'$ we have

$$M_{L_1} = \begin{pmatrix} x_1 & x_2 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}, M_{L_1'} = \begin{pmatrix} x_1 & x_2 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, M_{L_2} = \begin{pmatrix} x_1 & x_2 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, M_{L_2'} = \begin{pmatrix} x_1 & x_2 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix},$$

for which we have $\frac{1}{2}M_{L_1} + \frac{1}{2}M_{L_2} = \frac{1}{2}M_{L_1'} + \frac{1}{2}M_{L_2'}$.

The following Propositions 3.4 and 3.5 have a straightforward proof.

**Proposition 3.4.** Let $\mathcal{L}$ be a set of g-lotteries on a finite set $X$ totally ordered by $\leq^*$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$ satisfying **(A0)**. If $(\precsim, \prec)$ satisfies **(g-R)** then it satisfies the following properties:

*(i)* $(\precsim, \prec)$ has no intransitive cycles, that is: if $L_i \precsim L_j$ and $L_j \precsim L_k$ then it cannot be $L_k \prec L_i$ and similarly if $L_i \prec L_j$ and $L_j \precsim L_k$ it cannot be $L_k \precsim L_i$;

*(ii)* for every $L_i, L_j, L_k \in \mathcal{L}$ and $\lambda \in [0,1]$, if $L_i \precsim L_j$ then it cannot be

$$\lambda L_j + (1-\lambda)L_k \prec \lambda L_i + (1-\lambda)L_k;$$

*(iii)* for every $L_i, L_j \in \mathcal{L}$ and $0 \leq \alpha < \beta \leq 1$ if $L_i \prec L_j$ then it cannot be

$$\beta L_j + (1-\beta)L_i \precsim \alpha L_j + (1-\alpha)L_i.$$

**Proposition 3.5.** Let $\mathcal{L}$ be a set of g-lotteries on a finite set $X$ totally ordered by $\leq^*$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$ satisfying **(A0)**. If $(\precsim, \prec)$ satisfies **(g-CR)** then it satisfies conditions *(i)*, *(ii)*, *(iii)* and the following:

*(iv)* if $M_{L_i} = M_{L_j}$ it cannot be $L_i \prec L_j$ or $L_j \prec L_i$.

We stress that conditions *(i)*–*(iii)* are equivalent to axioms **(VM1)**–**(VM3)** in the case $(\precsim, \prec)$ is a total preference relation defined on the set of *all* $g$-lotteries on $X$.

## 4. REPRESENTABILITY OF RATIONAL PREFERENCE RELATIONS

In this section we study representability of a strengthened preference relation $(\precsim, \prec)$ on a set of g-lotteries $\mathcal{L}$ all having support on a finite set $X$ totally ordered by $\leq^*$. In detail, we investigate the consequences of axioms **(g-R)** and **(g-CR)**, respectively, both when $\mathcal{L}$ is finite and when $\mathcal{L}$ is infinite.

### 4.1. Representability of rational preference relations on finite sets of g-lotteries

We first consider the representability by a linear utility of a strengthened preference relation $(\precsim, \prec)$ on a finite set of g-lotteries.

**Theorem 4.1.** Let $\mathcal{L}$ be a finite set of g-lotteries, $X = \bigcup\{X_L : L \in \mathcal{L}\}$ a finite set totally ordered by $\leq^*$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$ satisfying **(A0)**. The following statements are equivalent:

*(i)* $(\precsim, \prec)$ satisfies **(g-R)** condition;

*(ii)* there exists a linear utility function $\mathrm{LU} : \mathcal{L} \to \mathbb{R}$ representing $(\precsim, \prec)$;

*(iii)* there exists a utility function $v : \wp(X) \to \mathbb{R}$ such that, for $x, x' \in X$, if $x <^* x'$ then $v(\{x\}) < v(\{x'\})$ and the function LU on $\mathcal{L}$, defined for every $L \in \mathcal{L}$ as

$$\mathrm{LU}(L) = \sum_{B \in \wp(X)} v(B) m_L(B)$$

represents $(\precsim, \prec)$.

Proof. Equivalence between *(ii)* and *(iii)* has been essentially proved in [17]. We prove the equivalence between *(i)* and *(iii)*. Let $X = \{x_1, \ldots, x_n\}$ with $x_1 <^* \ldots <^* x_n$ and assume all g-lotteries in $\mathcal{L}$ are rewritten on $X$. Introduce the collections $S = \{(L_j, L'_j) : L_j \prec L'_j, L_j, L'_j \in \mathcal{L}\}$ and $R = \{(G_h, G'_h) : G_h \precsim G'_h, G_h, G'_h \in \mathcal{L}\}$ with $s = \operatorname{card} S$ and $r = \operatorname{card} R$.

*(iii)* $\Rightarrow$ *(i)*. Let us consider the events $A_i \in \wp(X)$ with $i = 1, \ldots, 2^n$. Condition *(iii)* is equivalent to the existence of a $(2^n \times 1)$ column vector $\mathbf{q}$, with $q_i = v(A_i)$, for $i = 1, \ldots, 2^n$, which is a solution of the following system

$$\mathcal{S} : \begin{cases} A\mathbf{q} > \mathbf{0} \\ B\mathbf{q} \geq \mathbf{0} \\ \mathbf{q} \geq \mathbf{0} \end{cases}$$

where $A = (a^j)$ and $B = (b^h)$ are, respectively, $(s \times 2^n)$ and $(r \times 2^n)$ real matrices with rows $a^j = m_{L'_j} - m_{L_j}$ for $j = 1, \ldots, s$, and $b^h = m_{G'_h} - m_{G_h}$ for $h = 1, \ldots, r$. Notice that in $\mathcal{S}$ we can restrict to a non-negative $\mathbf{q}$ because of the homogeneity of first two constraints.

By a well-known alternative theorem (see, e. g., [13]) the solvability of $\mathcal{S}$ is equivalent to the non-solvability of the following system

$$\mathcal{S}' : \begin{cases} \mathbf{y}A + \mathbf{z}B \leq \mathbf{0} \\ \mathbf{y}, \mathbf{z} \geq \mathbf{0} \\ \mathbf{y} \neq \mathbf{0} \end{cases}$$

where $\mathbf{y}$ and $\mathbf{z}$ are, respectively, $(1 \times s)$ and $(1 \times r)$ unknown row vectors. Since the system $\mathcal{S}'$ is homogeneous, it is solvable if and only if it admits a solution $(\mathbf{y}, \mathbf{z})$ with $L^1$-norm equal to 1. By taking into account that a convex combination of basic assignments is a basic assignment and that for two basic assignment vectors $m_L$ and $m_G$ one has $m_L \leq m_G$ if and only if $m_L = m_G$, system $\mathcal{S}'$ is solvable if and only if the following system $\mathcal{S}''$ is solvable

$$\mathcal{S}'' : \begin{cases} \mathbf{y}A + \mathbf{z}B = \mathbf{0} \\ \mathbf{y}, \mathbf{z} \geq \mathbf{0} \\ \mathbf{y} \neq \mathbf{0} \\ ||(\mathbf{y}, \mathbf{z})||_1 = 1. \end{cases}$$

The non-solvability of $\mathcal{S}''$ implies that for every $\mathbf{k} = (\mathbf{y}, \mathbf{z}) \geq \mathbf{0}$ with $||\mathbf{k}||_1 = 1$ and $k_i > 0$ for at least one index $i \in \{1, \ldots, s\}$ it holds

$$\mathbf{k}(L_1, \ldots, L_s, G_1, \ldots, G_r) \neq \mathbf{k}(L'_1, \ldots, L'_s, G'_1, \ldots, G'_r),$$

and this implies condition **(g-R)** is satisfied.

*(i)* $\Rightarrow$ *(iii)*. Suppose **(g-R)** holds. In this case, considering a convex combination $\mathbf{k}(L_1, \ldots, L_s, G_1, \ldots, G_r) = \mathbf{k}(L'_1, \ldots, L'_s, G'_1, \ldots, G'_r)$ where some of the $k_i$'s can be 0, it must be that $k_i = 0$ for $i = 1, \ldots, s$, thus system $\mathcal{S}''$ cannot have solution, while $\mathcal{S}$ has solution $\mathbf{q}$. Moreover, assumption **(A0)** assures that, for $x, x' \in X$, if $x <^* x'$ then $v(\{x\}) < v(\{x'\})$. $\square$

**Remark 4.2.** Notice that the utility function $v$ in condition *(iii)* is not unique up to a positive linear transformation. Indeed, there can exist a utility function $v' = \varphi(v)$ still satisfying condition *(iii)*, where $\varphi$ is a strictly increasing non-linear transformation as the following trivial Example 4.3 shows. Every utility function $v$ induces a weak order on the convex closure of the set $\mathcal{L}$ which is the same induced by $v' = \varphi(v)$ only when $\varphi$ is a positive linear transformation.

**Example 4.3.** Let $X = \{x_1, x_2, x_3\}$ with $x_1 <^* x_2 <^* x_3$ and $\mathcal{L} = \{\delta_{\{x_1\}}, \delta_{\{x_2\}}, \delta_{\{x_3\}}, L\}$ where $L = (\wp(X), Bel_L)$ with $Bel_L(\{x_i\}) = \frac{1}{3}$ for $i = 1, 2, 3$. Consider the preferences $\delta_{\{x_1\}} \prec \delta_{\{x_2\}} \prec \delta_{\{x_3\}}$, $\delta_{\{x_1\}} \prec L \prec \delta_{\{x_3\}}$.

The functions $v_1$ and $v_2$ on $\wp(X)$ defined as $v_1(\{x_1\}) = 0$, $v_1(\{x_2\}) = \frac{1}{2}$, $v_1(\{x_3\}) = 1$, and $v_2(\{x_1\}) = 0$, $v_2(\{x_2\}) = \frac{1}{4}$, $v_2(\{x_3\}) = 1$ and zero otherwise, are such that $v_2 = v_1^2$.

The LU functional corresponding to $v_1$ and that corresponding to $v_2$ both represent the given preferences. Denote $\precsim_{v_1}$ and $\precsim_{v_2}$ the weak orders induced by $v_1$ and $v_2$ through the corresponding LU functional, respectively, on the convex closure of $\mathcal{L}$, for which it holds that $\delta_{\{x_2\}} \sim_{v_1} L$ while $\delta_{\{x_2\}} \prec_{v_2} L$.

Notice that introducing $v_3 = v_2^2$ we obtain a weak order $\precsim_{v_3}$ which coincides with $\precsim_{v_2}$ on $\mathcal{L}$, while the same does not hold on the whole convex closure of $\mathcal{L}$.

Consider now the representability of $(\precsim, \prec)$ by a CEU. The following theorem (already proved in [4]) shows that **(g-CR)** is a necessary and sufficient condition for the existence of a strictly increasing utility function $u$ on $X$ whose Choquet expected value represents $(\precsim, \prec)$, moreover its proof provides a procedure to compute such a $u$.

**Theorem 4.4.** Let $\mathcal{L}$ be a finite set of g-lotteries, $X = \bigcup\{X_L : L \in \mathcal{L}\}$ a finite set totally ordered by $\leq^*$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$ satisfying **(A0)**. The following statements are equivalent:

*(i)* $(\precsim, \prec)$ satisfies **(g-CR)** condition;

*(ii)* there exists a strictly increasing utility function $u : X \to \mathbb{R}$ whose Choquet expected value on $\mathcal{L}$ represents $(\precsim, \prec)$.

Proof. Let $X = \{x_1, \ldots, x_n\}$ with $x_1^* < \ldots < x_n^*$ and assume all g-lotteries in $\mathcal{L}$ are rewritten on $X$. Introduce the collections $S = \{(L_j, L'_j) : L_j \prec L'_j, L_j, L'_j \in \mathcal{L}\}$ and $R = \{(G_h, G'_h) : G_h \precsim G'_h, G_h, G'_h \in \mathcal{L}\}$ with $s = \operatorname{card} S$ and $r = \operatorname{card} R$.

*(ii)* $\Rightarrow$ *(i)*. Condition *(ii)* holds if and only if there are $n$ real numbers $w_i = u(x_i)$, with $w_1 < \ldots < w_n$, s.t. for all $(L_j, L'_j) \in S$ we have $\mathrm{CEU}(L_j) < \mathrm{CEU}(L'_j)$, and for all $(G_h, G'_h) \in R$ we have $\mathrm{CEU}(G_h) \leq \mathrm{CEU}(G'_h)$. Setting $E_i = \{x_i, \ldots, x_n\}$ $(i = 1, \ldots, n)$ and $E_{n+1} = \emptyset$, for every g-lottery $L \in \mathcal{L}$ it holds

$$
\begin{aligned}
\mathrm{CEU}(L) &= \sum_{i=1}^{n} w_i \left[Bel_L(E_i) - Bel_L(E_{i+1})\right] \\
&= \sum_{i=1}^{n} w_i \left[\sum_{B \subseteq E_i} m_L(B) - \sum_{B \subseteq E_{i+1}} m_L(B)\right] = \sum_{i=1}^{n} w_i M_L(x_i).
\end{aligned}
$$

Hence, condition *(i)* is equivalent to the existence of an $(n \times 1)$ column vector $\mathbf{w}$ which is solution of the following system

$$
\mathcal{S} : \begin{cases} A\mathbf{w} > \mathbf{0} \\ B\mathbf{w} \geq \mathbf{0} \\ \mathbf{w} \geq \mathbf{0} \end{cases}
$$

where $A = (a^j)$ and $B = (b^h)$ are, respectively, $(s \times n)$ and $(r \times n)$ real matrices with rows $a^j = M_{L'_j} - M_{L_j}$ for $j = 1, \ldots, s$, and $b^h = M_{G'_h} - M_{G_h}$ for $h = 1, \ldots, r$. Notice that in $\mathcal{S}$ we can restrict to a non-negative $\mathbf{w}$ because of the homogeneity of first two constraints.

By the same alternative theorem (see, e. g., [13]) and taking into account the same considerations made in the proof of Theorem 4.1 the solvability of $\mathcal{S}$ is equivalent to the non-solvability of the following system

$$
\mathcal{S}'' : \begin{cases} \mathbf{y}A + \mathbf{z}B = \mathbf{0} \\ \mathbf{y}, \mathbf{z} \geq \mathbf{0} \\ \mathbf{y} \neq \mathbf{0} \\ ||(\mathbf{y}, \mathbf{z})||_1 = 1 \end{cases}
$$

where $\mathbf{y}$ and $\mathbf{z}$ are, respectively, $(1 \times s)$ and $(1 \times r)$ unknown row vectors.

The non-solvability of $\mathcal{S}''$ implies that for every $\mathbf{k} = (\mathbf{y}, \mathbf{z}) \geq \mathbf{0}$ with $||\mathbf{k}||_1 = 1$ and $k_i > 0$ for at least one index $i \in \{1, \ldots, s\}$ it holds

$$
\mathbf{k}(M_{L_1}, \ldots, M_{L_s}, M_{G_1}, \ldots, M_{G_r}) \neq \mathbf{k}(M_{L'_1}, \ldots, M_{L'_s}, M_{G'_1}, \ldots, M_{G'_r}),
$$

and this implies condition **(g-CR)** is satisfied.

*(i) ⇒ (ii).* Suppose **(g-CR)** holds. In this case, considering a convex combination $\mathbf{k}(M_{L_1},\ldots,M_{L_s},M_{G_1},\ldots,M_{G_r}) = \mathbf{k}(M_{L'_1},\ldots,M_{L'_s},M_{G'_1},\ldots,M_{G'_r})$ where some of the $k_i$'s can be 0, it must be that $k_i = 0$ for $i = 1,\ldots,s$, thus system $\mathcal{S}''$ cannot have solution, while $\mathcal{S}$ has solution $\mathbf{w}$. Assumption **(A0)** assures that $w_1 < \ldots < w_n$, thus, $u(x_i) = w_i$, $i = 1,\ldots,n$, is a strictly increasing utility function on $X$ whose CEU represents $(\precsim,\prec)$. □

Analogous considerations as those in Remark 4.2 and the relevant Example 4.3 hold for the CEU representation.

The following example shows a relation that violates both **(g-CR)** and **(g-R)**, and so admits neither a CEU representation nor a LU representation.

**Example 4.5. (Example 2.2 continued)** Consider again Example 2.2 and suppose to toss a fair coin and to choose among $L_1$ and $G_1$ depending on the result of the toss. In analogy, suppose to choose among $L_2$ and $G_1$ with a totally similar experiment. Let us denote with $F_1$ and $F_2$ the results of the two experiments.

In order to express $F_1$ and $F_2$ we need to properly combine $L_1$, $L_2$ and $G_1$ and for this we have to rewrite them on the same set of prizes $\{0,10,100\}$

$$L_1 = \begin{pmatrix} \{0\} & \{10\} & \{100\} & \{0,10\} & \{0,100\} & \{10,100\} & \{0,10,100\} \\ \frac{2}{3} & 0 & \frac{1}{3} & \frac{2}{3} & 1 & \frac{1}{3} & 1 \end{pmatrix},$$

$$L_2 = \begin{pmatrix} \{0\} & \{10\} & \{100\} & \{0,10\} & \{0,100\} & \{10,100\} & \{0,10,100\} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 1 & 0 & 1 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} \{0\} & \{10\} & \{100\} & \{0,10\} & \{0,100\} & \{10,100\} & \{0,10,100\} \\ 0 & \frac{3}{4} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & 1 & 1 \end{pmatrix}.$$

$L_1$, $L_2$ and $G_1$ can be simply regarded as belief functions on the same field, thus $F_1$ and $F_2$ can be defined as the convex combinations $F_1 = \frac{1}{2}L_1 + \frac{1}{2}G_1$ and $F_2 = \frac{1}{2}L_2 + \frac{1}{2}G_1$, obtaining

$$F_1 = \begin{pmatrix} \{0\} & \{10\} & \{100\} & \{0,10\} & \{0,100\} & \{10,100\} & \{0,10,100\} \\ \frac{8}{24} & \frac{9}{24} & \frac{7}{24} & \frac{17}{24} & \frac{15}{24} & \frac{16}{24} & 1 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} \{0\} & \{10\} & \{100\} & \{0,10\} & \{0,100\} & \{10,100\} & \{0,10,100\} \\ \frac{4}{24} & \frac{9}{24} & \frac{3}{24} & \frac{13}{24} & \frac{15}{24} & \frac{12}{24} & 1 \end{pmatrix}.$$

Taking into account assumption **(A0)**, it is easily proven that for every strictly increasing $u : \{0,10,100\} \to \mathbb{R}$ the strict preferences $L_2 \prec L_1$, $L_4 \prec L_3$, are represented by a CEU functional. Analogously, the above preferences are representable also by a LU functional for every $v : \wp(\{0,10,100\}) \to \mathbb{R}$ such that $v(\{0\}) < v(\{10\}) < v(\{100\})$ and $2v(\{0,100\}) < v(\{0\}) + v(\{100\})$.

Nevertheless, by considering the further strict preference $F_1 \prec F_2$, there is neither a LU nor a CEU functional on $\mathcal{L} = \{L_1, L_2, L_3, L_4, G_1, F_1, F_2\}$ representing the given preferences. Indeed, by Propositions 3.4 and 3.5, in order to have such representations $L_2 \prec L_1$ implies that it cannot be $F_1 = \frac{1}{2}L_1 + \frac{1}{2}G_1 \prec \frac{1}{2}L_2 + \frac{1}{2}G_1 = F_2$.

### 4.2. Representability of rational preference relations on infinite sets of g-lotteries

Consider a strengthened preference relation $(\precsim, \prec)$ on an infinite set $\mathcal{L}$ of g-lotteries with support on a finite set $X$ totally ordered by $\leq^*$. Our aim is to study its representability under either **(g-R)** or **(g-CR)** conditions. For that, for every finite $\mathcal{F} \subseteq \mathcal{L}$ let $X_{\mathcal{F}} = \bigcup\{X_L : L \in \mathcal{F}\}$, and denote by $(\precsim_{\mathcal{F}}, \prec_{\mathcal{F}})$ the restriction of $(\precsim, \prec)$ to $\mathcal{F}$.

As a direct consequence of Theorem 4.1 the following result holds.

**Theorem 4.6.** Let $\mathcal{L}$ be an arbitrary set of g-lotteries with support on a finite set $X$ totally ordered by $\leq^*$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$ satisfying **(A0)**. The following statements are equivalent:

*(i)* $(\precsim, \prec)$ satisfies **(g-R)** condition;

*(ii)* for every finite $\mathcal{F} \subseteq \mathcal{L}$, there exists a linear utility function $\mathrm{LU}_{\mathcal{F}} : \mathcal{F} \to \mathbb{R}$ representing $(\precsim_{\mathcal{F}}, \prec_{\mathcal{F}})$;

*(iii)* for every finite $\mathcal{F} \subseteq \mathcal{L}$, there exists a utility function $v_{\mathcal{F}} : \wp(X_{\mathcal{F}}) \to \mathbb{R}$ such that, for $x, x' \in X_{\mathcal{F}}$, if $x <^* x'$ then $v(\{x\}) < v(\{x'\})$ and the function $\mathrm{LU}_{\mathcal{F}}$ on $\mathcal{F}$, defined for every $L \in \mathcal{F}$ as

$$\mathrm{LU}_{\mathcal{F}}(L) = \sum_{B \in \wp(X_{\mathcal{F}})} v_{\mathcal{F}}(B) m_L(B)$$

represents $(\precsim_{\mathcal{F}}, \prec_{\mathcal{F}})$.

**Remark 4.7.** For every finite subsets $\mathcal{F}, \mathcal{G}$ of $\mathcal{L}$ such that $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{L}$, the set of utility functions $\{v_{\mathcal{G}|\wp(X_{\mathcal{F}})}\}$ is contained in $\{v_{\mathcal{F}}\}$. Since the set $\{v_{\mathcal{L}}\}$ can be empty when $\mathcal{L}$ is infinite, as the following example shows, condition **(g-R)** is not sufficient for the existence of a linear utility function LU on the whole set $\mathcal{L}$ representing $(\precsim, \prec)$.

**Example 4.8.** Take $X = \{x_1, x_2, x_3\}$ with $x_1 <^* x_2 <^* x_3$ and consider the following infinite set of g-lotteries

$$\mathcal{L} = \{L_1 = \delta_{\{x_1\}}, L_2 = \delta_{\{x_2\}}, L_3 = \delta_{\{x_3\}}, \alpha L_1 + (1-\alpha)L_3 \,:\, \alpha \in ]0,1[\},$$

together with the preference relation on $\mathcal{L}$ satisfying **(A0)**

$$L_1 \prec L_2 \prec L_3, \quad \alpha L_1 + (1-\alpha)L_3 \prec L_2 \quad \text{for every } \alpha \in ]0,1[.$$

For every finite subset $\mathcal{F}$ of $\mathcal{L}$ we can find a linear utility function $\mathrm{LU}_{\mathcal{F}}$ on $\mathcal{F}$ representing $(\precsim_{\mathcal{F}}, \prec_{\mathcal{F}})$. By virtue of Remark 4.7, we can restrict to subfamilies of the form

$$\mathcal{F} = \{L_1, L_2, L_3, \alpha_j L_1 + (1-\alpha_j)L_3 \,:\, j = 1, \ldots, n\}.$$

For that, taking into account that $L_i$ is a degenerate g-lottery and so $\mathrm{LU}_{\mathcal{F}}(L_i) = v_{\mathcal{F}}(\{x_i\})$, for $i = 1, 2, 3$, and that

$$\mathrm{LU}_{\mathcal{F}}(\alpha_j L_1 + (1-\alpha_j)L_3) = \alpha_j v_{\mathcal{F}}(\{x_1\}) + (1-\alpha_j) v_{\mathcal{F}}(\{x_3\}),$$

for $j = 1, \ldots, n$, it is sufficient to consider any function $v_{\mathcal{F}}$ on $\wp(X_{\mathcal{F}})$ with $X_{\mathcal{F}} = X$ satisfying the constraints

$$v_{\mathcal{F}}(\{x_1\}) < v_{\mathcal{F}}(\{x_2\}) < v_{\mathcal{F}}(\{x_3\}),$$

$$\max_{j=1,\ldots,n}\{\alpha_j v_{\mathcal{F}}(\{x_1\}) + (1-\alpha_j)v_{\mathcal{F}}(\{x_3\})\} < v_{\mathcal{F}}(\{x_2\}) < v_{\mathcal{F}}(\{x_3\}).$$

Hence, from Theorem 4.6 $(\precsim, \prec)$ satisfies **(g-R)**.

On the other hand, there is no linear utility function representing $(\precsim, \prec)$ on $\mathcal{L}$. In fact, for every choice of $\mathrm{LU}(L_1) < \mathrm{LU}(L_2) < \mathrm{LU}(L_3)$ there exists an $\alpha \in ]0,1[$ such that $\alpha\mathrm{LU}(L_1) + (1-\alpha)\mathrm{LU}(L_3) \geq \mathrm{LU}(L_2)$ and so it should be $L_2 \precsim \alpha L_1 + (1-\alpha)L_3$, contradicting the given preferences.

It is clear that what is lacking in previous example is an Archimedean condition for $(\precsim, \prec)$. In fact such a condition is not implied by **(g-R)**. The following Theorem 4.9 shows that it is sufficient to add to **(g-R)** condition **(VM4)** to obtain representability when we consider the (infinite) set of *all* g-lotteries over a finite set $X$.

**Theorem 4.9.** Let $\mathcal{L}$ be the set of all g-lotteries with support on a finite set $X$ totally ordered by $\leq^*$, and $\precsim$ a total preference relation on $\mathcal{L}$ satisfying **(A0)**. The following statements are equivalent:

*(i)* $\precsim$ satisfies condition **(g-R)** and **(VM4)**;

*(ii)* there exists a linear utility function $\mathrm{LU} : \mathcal{L} \to \mathbb{R}$ representing $\precsim$;

*(iii)* there exists a utility function $v : \wp(X) \to \mathbb{R}$ such that, for $x, x' \in X$, if $x <^* x'$ then $v(\{x\}) < v(\{x'\})$ and the function LU on $\mathcal{L}$ defined for every $L \in \mathcal{L}$ as

$$\mathrm{LU}(L) = \sum_{B \in \wp(X)} v(B) m_L(B),$$

represents $\precsim$.

Moreover, LU is unique up to a positive linear transformation.

Proof. The set $\mathcal{L}$ is a mixture set, moreover, since the preference relation $\precsim$ is total, from Proposition 3.4 we have that axioms **(VM1)**–**(VM3)** are satisfied. Hence the validity of **(VM1)**–**(VM3)** and **(VM4)** is equivalent to *(ii)* and *(iii)* (see [17]), respectively, and, moreover, LU is unique up to a positive linear transformation. In particular, assumption **(A0)** implies, for $x, x' \in X$, if $x <^* x'$ then $v(\{x\}) < v(\{x'\})$. □

Next Theorems 4.10 and 4.12 are analogous to Theorems 4.6 and 4.9 for the CEU representation and are direct consequences of Theorem 4.1.

**Theorem 4.10.** Let $\mathcal{L}$ be an arbitrary set of g-lotteries with support on a finite set $X$ totally ordered by $\leq^*$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$ satisfying **(A0)**. The following statements are equivalent:

*(i)* $(\precsim, \prec)$ satisfies **(g-CR)** condition;

*(ii)* for every finite $\mathcal{F} \subseteq \mathcal{L}$, there exists a strictly increasing utility function $u_{\mathcal{F}} : X_{\mathcal{F}} \to \mathbb{R}$ whose Choquet expected value on $\mathcal{F}$ represents $(\precsim_{\mathcal{F}}, \prec_{\mathcal{F}})$.

Nevertheless, also in this case, if $\mathcal{L}$ is not finite, condition **(g-CR)** is not sufficient for the existence of a strictly increasing $u : X \to \mathbb{R}$ whose CEU on $\mathcal{L}$ represents the given preferences.

**Example 4.11.** Consider the g-lotteries and preferences of Example 4.8. For every finite subset $\mathcal{F}$ of $\mathcal{L}$ we can find a strictly increasing utility function $u_{\mathcal{F}}$ on $X_{\mathcal{F}}$ such that its Choquet expected value represents $(\precsim_{\mathcal{F}}, \prec_{\mathcal{F}})$. By analogous considerations of those in Remark 4.7, we can restrict to subfamilies of the form

$$\mathcal{F} = \{L_1, L_2, L_3, \alpha_j L_1 + (1-\alpha_j)L_3 \, : \, j = 1, \ldots, n\}.$$

For that, taking into account that $L_i$ is a degenerate g-lottery and so $\mathrm{CEU}_{\mathcal{F}}(L_i) = u_{\mathcal{F}}(x_i)$, for $i = 1, 2, 3$, and that $\mathrm{CEU}_{\mathcal{F}}(\alpha_j L_1 + (1-\alpha_j)L_3) = \alpha_j u_{\mathcal{F}}(x_1) + (1-\alpha_j)u_{\mathcal{F}}(x_3)$, for $j = 1, \ldots, n$, it is sufficient to consider any function $u_{\mathcal{F}}$ on $X_{\mathcal{F}} = X$ satisfying the constraints

$$u_{\mathcal{F}}(x_1) < u_{\mathcal{F}}(x_2) < u_{\mathcal{F}}(x_3),$$

$$\max_{j=1,\ldots,n} \{\alpha_j u_{\mathcal{F}}(x_1) + (1-\alpha_j)u_{\mathcal{F}}(x_3)\} < u_{\mathcal{F}}(x_2) < u_{\mathcal{F}}(x_3).$$

Hence, from Theorem 4.10 $(\precsim, \prec)$ satisfies **(g-CR)**.

On the other hand, there is no utility function $u$ on $X$ whose Choquet expected value represents $(\precsim, \prec)$ on $\mathcal{L}$. In fact, for every choice of $u(x_1) < u(x_2) < u(x_3)$ there exists an $\alpha \in ]0,1[$ such that $\alpha u(x_1) + (1-\alpha)u(x_3) \geq u(x_2)$ and so it should be $L_2 \precsim \alpha L_1 + (1-\alpha)L_3$, contradicting the given preferences.

It is clear that also in this case axiom **(g-CR)** must be reinforced by an Archimedean condition to have representability of a total preference relation $\precsim$ on the set of all *g*-lotteries.

**Theorem 4.12.** Let $\mathcal{L}$ be the set of all g-lotteries with support on a finite set $X$ totally ordered by $\leq^*$, and let $\precsim$ be a total preference relation on $\mathcal{L}$ satisfying **(A0)**. The following statements are equivalent:

*(i)* $\precsim$ satisfies conditions **(g-CR)** and **(VM4)**;

*(ii)* there exists a strictly increasing utility function $u : X \to \mathbb{R}$ (unique up to a positive linear transformation) whose Choquet expected value on $\mathcal{L}$ represents $\precsim$.

Proof. The set $\mathcal{L}$ is a mixture set, moreover, since the preference relation $\precsim$ is total, from Proposition 3.5 we have that axioms **(VM1)**–**(VM3)** are satisfied together with condition

*(*)* for every $L, L' \in \mathcal{L}$, $M_L = M_{L'}$ implies $L \sim L'$.

Define the relation $\equiv_M$ on $\mathcal{L}$ as $L \equiv_M L'$ if and only if $M_L = M_{L'}$, for every $L, L' \in \mathcal{L}$, which is easily seen to be an equivalence relation, and consider the quotient set $\mathcal{L}_{/\equiv_M}$. Since $\mathcal{L}$ contains all belief functions on $\wp(X)$, in particular, it contains all probability distributions on $X$, then for every g-lottery $L \in \mathcal{L}$ there is a probabilistic g-lottery in $\mathcal{L}$ having $M_L$ as probability distribution, that can be chosen as representative of the corresponding equivalence class in $\mathcal{L}_{/\equiv_M}$.

Let $\mathcal{L}^P$ be the set of probabilistic lotteries formed by the representatives of equivalence classes in $\mathcal{L}_{/\equiv_M}$. The set $\mathcal{L}^P$ consists of all probability distributions on $X$ and is, therefore, a mixture set. Axioms **(VM1)**–**(VM4)** are necessary and sufficient for the existence of a function $u : X \to \mathbb{R}$ (unique up to a positive linear transformation) whose expected value represents the restriction of $\precsim$ on $\mathcal{L}^P$. The assumption **(A0)** assures that $u$ is strictly increasing. The proof immediately follows by taking into account what proved in Theorem 4.4, i. e., $\mathrm{CEU}(L) = \sum_{i=1}^{n} u(x_i) M_L(x_i)$, for every $L \in \mathcal{L}$. □

## 5. CONCLUSIONS

The paper copes with decisions under risk, where the uncertainty on consequences is modelled by a pre-assigned ("objective" in the jargon by von Neumann–Morgenstern) belief function.

Two rationality principles **(g-R)** and **(g-CR)** for preference relations among random quantities equipped with a belief function (g-lotteries) are introduced, following the line of the rationality principle given in [5]. Such principles allow to handle representability of preference relations when the set of random quantities is arbitrary (not necessarily closed under convex combinations) and possibly finite. Note that the probabilistic case is subsumed by both conditions **(g-R)** and **(g-CR)**, which coincide in case of probabilistic g-lotteries.

Concerning the probabilistic modelling of uncertainty, in the literature there are proposals for removing the hypothesis of totality on the preference relation both in the von Neumann–Morgenstern [10] and in the Anscombe-Aumann frameworks [20]. In such proposals, even if the preference relation is not asked to be total, the considered set of lotteries must possess a proper mathematical structure and the preference relation must be assessed on proper subsets of lotteries. In particular, none of the existing models allows to deal with partial preferences on finite arbitrary sets of lotteries.

At the same time, the possibility of dealing with possibly partial preference relations assessed on arbitrary finite sets of g-lotteries is significant in real decision problems.

This paper assumes that the set $X$ of prizes of g-lotteries is totally ordered and that the preferences expressed by the decision maker agree with this order. This is quite natural if $X$ is a subset of $\mathbb{R}$, for example when $X$ is a set of money payoffs. Otherwise, the total order on $X$ can be elicited by asking to the decision maker to rank the elements of $X$ by their strict desirability, assuming he/she could receive them with certainty. The assumption of a totally ordered set of prizes is not substantial and has been adopted in favour of a simpler exposition of mathematical results. Such assumption can be relaxed by requiring that $X$ is only totally preordered, obtaining results in line of those in this paper but with a little more complex presentation.

The principle **(g-R)** is a necessary and sufficient condition for the existence of a linear functional (as in [17]), determined by a utility function defined on all the elements of the power set of $X$. While, the principle **(g-CR)** is a necessary and sufficient condition for the existence of a strictly increasing utility function $u$ on $X$ whose Choquet integral on g-lotteries represents the preferences. Under both representations, the underlying choice criterion is intended to be the maximization of the resulting functional.

In [27], the author considers the case of uncertainty expressed by a general lower probability, presenting a comparison of functionals and choice criteria in terms of resulting decisions. Such paper does not cope with the determination of a utility on the prizes and tacitly assumes a pre-assigned utility function. In our setting, it could be interesting to provide a similar study and so to compare the effects on decisions of different choice criteria and also of the selected utility function (which is generally not unique).

In case of an infinite set of g-lotteries, the above conditions assure the representability of a preference relation only on finite subfamilies of g-lotteries because they do not impose any Archimedean condition on the preferences. Nevertheless, when the preference relation is total and defined on the set of all g-lotteries, it is sufficient to add an Archimedean condition to each of the two rationality principles to get a complete representation.

In the probabilistic framework the rationality principle (see [5]) assures and rules the extendibility of the relation to new lotteries, which is a useful property in order to apply the model. The extendibility of preferences on generalized lotteries, satisfying the two above rationality principles, is one of our future aims. Indeed, for both rationality conditions, as underlined in Remark 4.2, the extension of a rational preference relation is in general not unique.


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