Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 54 (2015), No. 1, 109–119

Persistent URL: http://dml.cz/dmlcz/144371

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **54**, 1 (2015) 109–119

Stability and Boundedness of Solutions of a Certain System of Third-order Nonlinear Delay Differential Equations

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(Received July 26, 2014)

Abstract

In this paper a number of known results on the stability and boundedness of solutions of some scalar third-order nonlinear delay differential equations are extended to some vector third-order nonlinear delay differential equations.

Key words: Lyapunov functional, third-order vector delay differential equation, boundedness, stability.

2010 Mathematics Subject Classification: 34K20

1 Introduction

The delay differential equation considered here is of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X(t-r(t))) = P(t), \tag{1.1}$$

in which $X \in \mathbb{R}^n$, $P\colon \mathbb{R} \to \mathbb{R}^n$, A and B are real $n \times n$ constant matrices, $0 \le r(t) \le \gamma$, γ is a positive constant which will be determined later, and the dots indicate differentiation with respect to t. The equation is the vector version for the system of real third-order delay differential equations

$$\dddot{x}_i + \sum_{k=1}^{n} a_{ik}\ddot{x}_k + \sum_{k=1}^{n} b_{ik}\dot{x}_k + h_i(x_1(t-r(t)), x_2(t-r(t)), \dots, x_n(t-r(t))) = p_i(t),$$

$i = 1, 2, \dots, n$, in which a_{ik}, b_{ik} are constants. It will be assumed as basic throughout what follows that $H \in \mathcal{C}'(\mathbb{R}^n)$ and $P \in \mathcal{C}(\mathbb{R})$ are such that solutions of (1.1) exist corresponding to any pre-assigned initial conditions. Here, $\mathcal{C}'(\mathbb{R}^n)$

is the family of all functions once continuously differentiable on $\mathbb{R}^n$ and $\mathcal{C}(\mathbb{R})$ is the family of all functions continuous on $\mathbb{R}$.

The study of (1.1) is concerned primarily with the problems of stability and boundedness of solutions of (1.1).

In the case $n = 1$, these problems have been investigated (see [3, 4, 5, 6, 10, 11, 14]) for a general scalar delay differential equation of the form

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x(t - r(t))) = p(t)$$

$(a, b$ constants). Their investigation shows that the stability and the ultimate boundedness of solutions can be established if $h'(x)$ is bounded and if $a, b, h(x)$ satisfy some suitable generalization of the Routh–Hurwitz conditions

$$a > 0, \quad b > 0, \quad ab - c > 0$$

for the asymptotic stability of the solution $x = 0$ of the linear system

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$$

with constant coefficients. The object of the present paper is to provide analogous results for n-dimensional equation (1.1) following the arguments used in some of the papers mentioned above.

Notation and definitions

Given any X, Y in $\mathbb{R}^n$ the symbol $\langle X, Y\rangle$ will be used to denote the usual scalar product in $\mathbb{R}^n$, that is $\langle X, Y\rangle = \sum_{i=1}^n x_i y_i$; thus $\|X\|^2 = \langle X, X\rangle$. The matrix A is said to be positive definite when $\langle AX, X\rangle > 0$ for all nonzero X in $\mathbb{R}^n$.

The following notations (see [5, 15]) will be useful in subsequent sections. For $x \in \mathbb{R}^n$, $|x|$ is the norm of x. For a given $r > 0$, $t_1 \in \mathbb{R}$,

$$C(t_1) = \{\phi\colon [t_1 - r, t_1] \to \mathbb{R}^n/\phi \text{ is continuous}\}.$$

In particular, $C = C(0)$ denotes the space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ and for $\phi \in C$, $\|\phi\| = \sup_{-r\le\theta\le 0} |\phi(0)|$. $C_{\mathbf{H}}$ will denote the set of ϕ such that $\|\phi\| \le \mathbf{H}$. For any continuous function $x(u)$ defined on $-h \le u < A$, $A > 0$, and any fixed t, $0 \le t < A$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t - r, t]$, that is, x_t is an element of C defined by $x_t(\theta) = x(t + \theta)$, $-r \le \theta \le 0$.

2 Some preliminary results

In this section, we shall state the algebraic results required in the proofs of our main results. The proofs are not given since they are found in [1, 2, 7, 8, 9, 13].

Lemma 2.1 [1, 2, 7, 8, 9, 13] *Let D be a real symmetric positive definite $n \times n$ matrix, then for any X in $\mathbb{R}^n$, we have*

$$\delta_d\|X\|^2 \le \langle DX, X\rangle \le \Delta_d\|X\|^2$$

where δ_d, Δ_d are the least and the greatest eigenvalues of D, respectively.

Lemma 2.2 [1, 2, 7, 8, 9, 13] *Let Q, D be any two real $n \times n$ commuting symmetric matrices. Then*

(i) *the eigenvalues $\lambda_i(QD)$ $(i = 1, 2, \dots, n)$ of the product matrix QD are all real and satisfy*

$$\min_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D) \le \lambda_i(QD) \le \max_{1\le j,k\le n} \lambda_j(Q)\lambda_k(D);$$

(ii) *the eigenvalues $\lambda_i(Q + D)(i = 1, 2, \dots, n)$ of the sum of matrices Q and D are real and satisfy*

$$\{\min_{1\le j\le n} \lambda_j(Q)+\min_{1\le k\le n} \lambda_k(D)\} \le \lambda_i(Q+D) \le \{\max_{1\le j\le n} \lambda_j(Q)+\max_{1\le k\le n} \lambda_k(D)\}.$$

Lemma 2.3 [1, 2, 7, 8, 9, 13] *Let $H(X)$ be a continuous vector function and that $H(0) = 0$ then*

$$\frac{d}{dt}\int_0^1 \langle H(\sigma X), X\rangle \, d\sigma = \langle H(X), \dot{X}\rangle.$$

Lemma 2.4 *Let $H(X)$ be a continuous vector function and that $H(0) = 0$ then*

$$\delta_h\|X\|^2 \le 2\int_0^1 \langle H(\sigma X), X\rangle \, d\sigma \le \Delta_h\|X\|^2$$

where δ_h, Δ_h are the least and the greatest eigenvalues of $J_h(X)$ (Jacobian matrix of H), respectively.

3 Stability

First, we will give the stability criteria for the general autonomous delay differential system. We consider

$$x' = f(x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0,\ t \ge 0, \tag{3.1}$$

where $f\colon C_{\mathbf{H}} \to \mathbb{R}^n$ is a continuous mapping, $f(0) = 0$,

$$C_{\mathbf{H}} := \{\phi \in C([-r, 0], \mathbb{R}^n)\colon \|\phi\| \le \mathbf{H}\}$$

and for $\mathbf{H_1} < \mathbf{H}$, there exists $L > 0$, with $|f(\phi)| \le L$ when $\|\phi\| \le \mathbf{H_1}$. Here, $C([-r, 0], \mathbb{R}^n)$ is the family of all vector functions mapping $[-r, 0]$ into $\mathbb{R}^n$.

Definition 3.1 [3, 5, 6, 11, 12] An element $\psi \in C$ is in the $\omega-$limit set of ϕ, if $x(t, 0, \phi)$ is defined on $[0, \infty)$ and there is a sequence $\{t_n\}$, $t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)$ for $-r \le \theta \le 0$. $x(t; 0, \phi)$ is a motion of a system at $t \in \mathbb{R}$ if and only if $x(0) = \phi$.

Definition 3.2 [3, 5, 6, 11, 12] A set $Q \subset C_{\mathbf{H}}$ is an invariant set if for any $\phi \in Q$, the solution of (3.1), $x(t, 0, \phi)$, is defined on $[0, \infty)$, and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.

Lemma 3.3 (see [3, 5, 6, 11, 12]) *If $\phi \in C_{\mathbf{H}}$ is such that the solution $x_t(\theta)$ of* (3.1) *with $x_0(\phi) = \phi$ is defined on $[0, \infty)$ and $\|x_t(\phi)\| \leq \mathbf{H_1} < \mathbf{H}$ for $t \in [0, \infty)$, then $\Omega(\phi)$ (the $\omega-$limit set of ϕ) is a nonempty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0, \quad as \quad t \to \infty.$$

Lemma 3.4 (see [3, 5, 6, 11, 12]) *Let $V(\phi)\colon C_{\mathbf{H}} \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(0) = 0$ and such that*

(i) $W_1(|\phi(0)|) \leq V(\phi) \leq W_2(\|\phi(0)\|)$ where $W_1(r), W_2(r)$ are wedges.

(ii) $V'_{(3.1)}(\phi) \leq 0$, for $\phi \in C_{\mathbf{H}}$.

Then the zero solution of (3.1) *is uniformly stable. If we define*

$$Z = \{\phi \in C_{\mathbf{H}}\colon V'_{(3.1)}(\phi) = 0\},$$

then the zero solution of (3.1) *is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.*

Before we state our result in this section, we write equation (1.1) with $P \equiv 0$ as

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -AZ - BY - H(X) + \int_{t-r(t)}^{t} J_h(X)Y\, ds. \end{aligned} \tag{3.2}$$

We shall constantly refer to (3.2) subsequently in our discussion.

The following will be our main stability result for (3.2).

Theorem 3.5 *Consider* (3.2), *let $H(0) = 0$ and suppose that*

(i) $0 \leq r(t) \leq \gamma$ $(\gamma > 0)$, $r'(t) \leq \xi$, and $0 < \xi < 1$;

(ii) the matrices A, B and $J_h(X)$ (Jacobian matrix of $H(X)$) are symmetric and positive definite, and furthermore the eigenvalues $\lambda_i(A)$, $\lambda_i(B)$ and $\lambda_i(J_h(X))$ $(i = 1, 2, \ldots, n)$ of A, B and $J_h(X)$, respectively satisfy,

$$0 < \delta_a \leq \lambda_i(A) \leq \Delta_a \tag{3.3}$$

$$0 < \delta_b \leq \lambda_i(B) \leq \Delta_b \tag{3.4}$$

$$0 < \delta_h \leq \lambda_i(J_h(X)) \leq \Delta_h, \text{ for } X \in \mathbb{R}^n, \tag{3.5}$$

where $\delta_a, \delta_b, \delta_h, \Delta_a, \Delta_b, \Delta_h$ are finite constants;

(iii) the matrices A, B and $J_h(X)$ commute pairwise.

Then the zero solution of (3.2) *is asymptotically stable, provided*

$$\gamma < \min\left\{\frac{2(\beta\delta_a - 1)}{\beta\Delta_h}, \frac{2(\delta_b - \beta\Delta_h)(1-\xi)}{\Delta_h(2 + \beta - \xi)}\right\}.$$

Proof Using the equivalent system form (3.2), our main tool is the following Lyapunov functional, $V_1(X_t, Y_t, Z_t)$ defined as

$$2V_1(X_t,Y_t,Z_t) = 2\int_0^1 \langle H(\sigma X), X\rangle d\sigma + \langle AY, Y\rangle + \beta\langle Y, BY\rangle \\ + \beta\langle Z, Z\rangle + 2\langle Y, Z\rangle + 2\beta\langle Y, H(X)\rangle + \mu\int_{-r(t)}^{0}\int_{t+s}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta ds, \quad (3.6)$$

where

$$\frac{1}{\delta_a} < \beta < \frac{\delta_b}{\Delta_h}.$$

Since

$$\mu\int_{-r(t)}^{0}\int_{t+s}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta ds$$

is non-negative, we have

$$\begin{aligned} 2V_1(X_t,Y_t,Z_t) &\geq 2\int_0^1 \langle H(\sigma X), X\rangle d\sigma + \langle AY, Y\rangle + \beta\langle Y, BY\rangle \\ &\quad + \beta\langle Z, Z\rangle + 2\langle Y, Z\rangle + 2\beta\langle Y, H(X)\rangle \\ &= 2\int_0^1 \langle H(\sigma X), X\rangle d\sigma - \beta\langle B^{-1}H(X), H(X)\rangle \\ &\quad + \beta\|B^{\frac{1}{2}}Y + B^{-\frac{1}{2}}H(X)\|^2 + \beta\|Z + \beta^{-1}Y\|^2 + \langle (A - \beta^{-1}I)Y, Y\rangle. \end{aligned}$$

Using Lemmas 2.1,2.2 and 2.4, and since $\beta\|B^{\frac{1}{2}}Y + B^{-\frac{1}{2}}H(X)\|^2 \geq 0$, we have

$$\begin{aligned} 2V_1(X_t,Y_t,Z_t) &\geq 2\int_0^1\int_0^1 \sigma\langle\{I - \beta B^{-1}J_h(\sigma X)\}\, J_h(\sigma\tau X)X, X\rangle d\sigma d\tau \\ &\quad + \beta\|Z + \beta^{-1}Y\|^2 + \langle (A - \beta^{-1}I)Y, Y\rangle \\ &\geq (1 - \beta\delta_b^{-1}\Delta_h)\delta_h\|X\|^2 + \beta\|Z + \beta^{-1}Y\|^2 + (\delta_a - \beta^{-1})\|Y\|^2. \end{aligned}$$

Hence there is a constant $K > 0$ (small enough) such that

$$V_1(X_t,Y_t,Z_t) \geq K\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right).$$

Next, our target is to show that $V_1(X_t, Y_t, Z_t)$ satisfies the second condition of Lemma 3.4. From (3.2), (3.6) and using Lemma 2.3, we have

$$\begin{aligned} \frac{d}{dt}V_1(X_t,Y_t,Z_t) &= -\langle(\beta A - I)Z, Z\rangle - \langle\{B - \beta J_h(X)\}\, Y, Y\rangle \\ &\quad + \int_{t-r(t)}^{t}\langle Y, J_h(X)Y\rangle ds + \beta\int_{t-r(t)}^{t}\langle Z, J_h(X)Y\rangle\, ds \\ &\quad + \mu r(t)\langle Y, Y\rangle - \mu(1 - r'(t))\int_{t-r(t)}^{t}\langle Y(\theta), Y(\theta)\rangle\, d\theta. \end{aligned} \quad (3.7)$$

On using Lemmas 2.1 and 2.2, and the identity $2|\langle U,V\rangle| \leq \|U\|^2 + \|V\|^2$, we obtain,

$$\begin{aligned}\frac{d}{dt}V_1(X_t,Y_t,Z_t) \leq &-\left(\delta_b - \beta\Delta_h - \frac{1}{2}\Delta_h\gamma - \mu\gamma\right)\|Y\|^2\\ &-\left(\beta\delta_a - 1 - \tfrac{1}{2}\beta\gamma\Delta_h\right)\|Z\|^2\\ &+\left\{\tfrac{1}{2}\beta\Delta_h + \tfrac{1}{2}\Delta_h - \mu(1-\xi)\right\}\int_{t-r(t)}^{t}\langle Y(\theta),Y(\theta)\rangle\, d\theta. \end{aligned} \tag{3.8}$$

If we choose $\mu = \frac{(\beta+1)\Delta_h}{2(1-\xi)}$,

$$\begin{aligned}\frac{d}{dt}V_1(X_t,Y_t,Z_t) \leq &-\left\{\delta_b - \beta\Delta_h - \frac{(2+\beta-\xi)}{2(1-\xi)}\Delta_h\gamma\right\}\|Y\|^2\\ &-\left(\beta\delta_a - 1 - \frac{1}{2}\beta\gamma\Delta_h\right)\|Z\|^2,\end{aligned}$$

and choosing

$$\gamma < \min\left\{\frac{2(\beta\delta_a - 1)}{\beta\Delta_h}, \frac{2(\delta_b - \beta\Delta_h)}{\Delta_h(2+\beta-\xi)}\right\},$$

there is a constant $D > 0$ such that

$$\frac{d}{dt}V_1(X_t,Y_t,Z_t) \leq -D\left(\|Y\|^2 + \|Z\|^2\right).$$

Hence the result follows. □

Example 1 As a special case of system (1.1) (for $P(t) = 0$), let us take $n = 2$ that

$$X = \begin{pmatrix} x_1(t)\\ x_2(t)\end{pmatrix}, \quad A = \begin{pmatrix} 8 & 0\\ 0 & 10\end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0\\ 0 & 3\end{pmatrix}$$

and

$$H(X(t-r(t))) = \begin{pmatrix} \tan^{-1} x_1(t-r(t)) + 2x_1(t-r(t))\\ \tan^{-1} x_2(t-r(t)) + 2x_2(t-r(t))\end{pmatrix}.$$

Thus,

$$H(X(t)) = \begin{pmatrix} \tan^{-1} x_1(t) + 2x_1(t)\\ \tan^{-1} x_2(t) + 2x_2(t)\end{pmatrix} \text{ and } J_h(X) = \begin{pmatrix} 2 + \frac{1}{1+x_1^2} & 0\\ 0 & 3 + \frac{1}{1+x_2^2}\end{pmatrix}.$$

If we take $r(t) = \frac{1}{22+t^2}$, then $0 \leq \frac{1}{22+t^2} < \gamma$, and that $r'(t) = \frac{-2t}{(22+t^2)^2} \leq \xi$, $0 < \xi < 1$. Clearly, A, B and $J_h(X)$ are symmetric and commute pairwise. That is,

$$AB = \begin{pmatrix} 8 & 0\\ 0 & 30\end{pmatrix} = BA,$$

$$AJ_h(X) = \begin{pmatrix} 16 + \frac{8}{1+x_1^2} & 0\\ 0 & 30 + \frac{10}{1+x_2^2}\end{pmatrix} = J_h(X)A$$

and

$$BJ_h(X) = \begin{pmatrix} 2+\frac{1}{1+x_1^2} & 0 \\ 0 & 9+\frac{3}{1+x_2^2} \end{pmatrix} = J_h(X)B.$$

Then, by easy calculation, we obtain eigenvalues of the matrices A, B and $J_h(X)$ as follows:

$$\lambda_1(A) = 8, \quad \lambda_2(A) = 10, \quad \lambda_1(B) = 1, \quad \lambda_2(B) = 3,$$

$$\lambda_1(J_h(X)) = 2 + \frac{1}{1+x_1^2}, \quad \lambda_2(J_h(X)) = 3 + \frac{1}{1+x_1^2}.$$

It is obvious that $\delta_a = 8$, $\Delta_a = 10$, $\delta_b = 1$, $\Delta_b = 3$, $\delta_h = 2$, $\Delta_h = 4$. If we choose $\beta = \frac{1}{6}$ and $\xi = \frac{1}{2}$, we must have that $\gamma < \min\{1, \frac{1}{20}\}$.

Thus, all the conditions of Theorem 3.5 are satisfied.

4 Boundedness

First, consider a system of delay differential equations

$$\dot{x} = F(t, x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0,\ t \ge 0, \tag{4.1}$$

where $F\colon \mathbb{R} \times C_{\mathbf{H}} \to \mathbb{R}^n$ is a continuous mapping and takes bounded set into bounded sets.

The following lemma is a well-known result obtained by Burton [5].

Lemma 4.1 [3, 5, 6, 11, 12] *Let $V(t, \phi)\colon \mathbb{R} \times C_{\mathbf{H}} \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If*

(i) $W(|x(t)|) \le V(t, x_t) \le W_1(|x(t)|) + W_2\left(\int_{t-r(t)}^{t} W_3(|x(s)|)ds\right)$, *and*

(ii) $\dot{V}_{(4.1)} \le -W_3(|x(s)|) + M$,

for some $M > 0$, where $W(r)$, $W_i (i = 1, 2, 3)$ are wedges, then the solutions of (4.1) are uniformly bounded and uniformly ultimately bounded for bound $\boldsymbol{B}$.

To study the boundedness of solutions of (1.1) for which $P(t) \ne 0$, we would need to write Eq. (1.1) in the form

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -AZ - BY - H(X) + \int_{t-r(t)}^{t} J_h(X)Y ds + P(t). \end{aligned} \tag{4.2}$$

Thus our main theorem in this section is stated with respect to Eq.(4.2) as follows:

Theorem 4.2 *If the conditions of Theorem 3.5 hold, and*

$$\|P(t)\| \leq m, \tag{4.3}$$

where m *is a positive constant, then the solutions of equation* (4.2) *are uniformly bounded and uniformly ultimately bounded provided* γ *satisfies*

$$\gamma < \min\left\{\frac{2\delta_h(\delta_a\delta_b-\Delta_h)}{\Delta_h(\Delta_a\Delta_b-\Delta_h)}, \frac{2(\beta\delta_a-1)}{\Delta_h(\beta+\Delta_a)}, \frac{2(\delta_b-\beta\Delta_h)(1-\xi)}{\Delta_h\left\{(1+\Delta_a^2)(1-\xi)+2+\beta+\Delta_a^2+(\Delta_a\Delta_b-\Delta_h)\right\}}\right\},$$

Proof Consider the function

$$V = V_1(X_t, Y_t, Z_t) + V_2(X_t, Y_t, Z_t), \tag{4.4}$$

where $V_1(X_t, Y_t, Z_t)$ is defined as (3.6) and $V_2(X_t, Y_t, Z_t)$ defined as

$$\begin{aligned} 2V_2(X_t, Y_t, Z_t) &= 2\int_0^1 \langle AH(\sigma X), AX\rangle d\sigma + \langle B(AB-\Delta_h I)X, X\rangle \\ &+ 2\langle AY, H(X)\rangle + \langle \Delta_h Y, Y\rangle \\ &+ 2\langle (AB-\Delta_h I)X, Z+AY\rangle + \langle A(Z+AY), Z+AY\rangle. \end{aligned} \tag{4.5}$$

This we can rewrite

$$\begin{aligned} 2V_2(X_t, Y_t, Z_t) &= 2\int_0^1 \langle AH(\sigma X), AX\rangle d\sigma - \langle A\Delta_h^{-1}H(X), AH(X)\rangle \\ &+ \langle \Delta_h Y + H(X), Y + A\Delta_h^{-1}H(X)\rangle + \langle \Delta_h A^{-1}(AB-\Delta_h I)X, X\rangle \\ &+ \langle (AB-\Delta_h I)X + A^2Y + AZ, A^{-1}(AB-\Delta_h I)X + AY + Z\rangle \\ &= 2\int_0^1\int_0^1 \sigma\langle\left\{I-\Delta_h^{-1}J_h(\sigma X)\right\}A^2J(\tau\sigma X)X, X\rangle\, d\sigma d\tau \\ &+ \|A^{-\frac{1}{2}}(AB-\Delta_h I)X + A^{\frac{3}{2}}Y + A^{\frac{1}{2}}Z\|^2 \\ &+ \|\Delta_h^{\frac{1}{2}}(Y+\Delta_h^{-1}AH(X))\|^2 + \langle \Delta_h A^{-1}(AB-\Delta_h I)X, X\rangle. \end{aligned}$$

However,

$$2\int_0^1\int_0^1 \sigma\langle\left\{I-\Delta_h^{-1}J_h(\sigma X)\right\}A^2J(\tau\sigma X)X, X\rangle\, d\sigma d\tau \geq 0$$

and $\langle \Delta_h A^{-1}(AB-\Delta_h I)X, X\rangle \geq \Delta_h\Delta_a^{-1}(\delta_a\delta_b-\Delta_h)\|X\|^2$. Thus,

$$\begin{aligned} 2V_2(X_t, Y_t, Z_t) &\geq \Delta_h\Delta_a^{-1}(\delta_a\delta_b-\Delta_h)\|X\|^2 + \|\Delta_h^{\frac{1}{2}}(Y+\Delta_h^{-1}AH(X))\|^2 \\ &+ \|A^{-\frac{1}{2}}(AB-\Delta_h I)X + A^{\frac{3}{2}}Y + A^{\frac{1}{2}}Z\|^2. \end{aligned}$$

It follows that $V_2(X_t, Y_t, Z_t)$ is positive definite.

From (3.6), (3.7), (3.8), (4.2) and Lemma2.3, we find

$$\frac{d}{dt}V_1(X_t,Y_t,Z_t) \le -\left(\delta_b - \beta\Delta_h - \frac{1}{2}\Delta_h\gamma - \mu\gamma\right)\|Y\|^2 - \left(\beta\delta_a - 1 - \frac{1}{2}\beta\gamma\Delta_h\right)\|Z\|^2 + \left\{\frac{1}{2}\beta\Delta_h + \frac{1}{2}\Delta_h - \mu(1-\xi)\right\} \times \int_{t-r(t)}^{t}\langle Y(\theta),Y(\theta)\rangle d\theta + (\|Y\| + \beta\|Z\|)m. \tag{4.6}$$

Also from (4.2), (4.5) and Lemma 2.3 we obtain,

$$\frac{d}{dt}V_2(X_t,Y_t,Z_t) = -\langle (AB-\Delta_h I)X, H(X)\rangle - \langle A(\Delta_h I - J_h(X))Y, Y\rangle + \int_{t-r(t)}^{t}\langle (AB-\Delta_h I)X, J_h(X)Y\rangle\, ds + \int_{t-r(t)}^{t}\langle AZ, J_h(X)Y\rangle\, ds + \int_{t-r(t)}^{t}\langle A^2Y, J_h(X)Y\rangle\, ds + \langle P(t), (AB-\Delta_h I)X + A^2Y + AZ\rangle$$

Also using Lemmas 2.1 and 2.2, the identity $2|\langle U,V\rangle| \le (\|U\|^2 + \|V\|^2)$, and the fact that $\langle A(\Delta_h I - J_h(X))Y, Y\rangle \ge 0$ we find that

$$\frac{d}{dt}V_2(X_t,Y_t,Z_t) \le -\left\{\delta_h(\delta_a\delta_b - \Delta_h) - \frac{1}{2}\Delta_h(\Delta_a\Delta_b - \Delta_h)r(t)\right\}\|X\|^2 + \frac{1}{2}\Delta_a\Delta_h r(t)\|Z\|^2 + \frac{1}{2}\Delta_a^2\Delta_h r(t)\|Y\|^2 + \frac{1}{2}\Delta_h\left\{(\Delta_a\Delta_b - \Delta_h) + \Delta_a + \Delta_a^2\right\}\int_{t-r(t)}^{t}\langle Y(s),Y(s)\rangle\, ds + \left\{(\Delta_a\Delta_b - \Delta_h)\|X\| + \Delta_a^2\|Y\| + \Delta_a\|Z\|\right\}m \tag{4.7}$$

Therefore, from (4.4), (4.6) and (4.7), we obtain

$$\frac{d}{dt}V(X_t,Y_t,Z_t) \le -\left\{\delta_h(\delta_a\delta_b - \Delta_h) - \frac{1}{2}\Delta_h\gamma(\Delta_a\Delta_b - \Delta_h)\right\}\|X\|^2 - \left\{\delta_b - \beta\Delta_h - \gamma\left(\frac{1}{2}\Delta_h + \frac{1}{2}\Delta_a^2\Delta_h + \mu\right)\right\}\|Y\|^2 - \left\{\beta\delta_a - 1 - \frac{1}{2}\gamma\Delta_h(\beta + \Delta_a)\right\}\|Z\|^2 + \left\{\frac{1}{2}\Delta_h\left(2 + \beta + \Delta_a^2 + (\Delta_a\Delta_b - \Delta_h)\right) - \mu(1-\xi)\right\}\int_{t-r(t)}^{t}\langle Y(\theta),Y(\theta)\rangle\, d\theta + \left\{(\Delta_a\Delta_b - \Delta_h)\|X\| + (1+\Delta_a^2)\|Y\| + (\beta + \Delta_a)\|Z\|\right\}m.$$

If we choose

$$\mu = \frac{\Delta_h \left\{2 + \beta + \Delta_a^2 + (\Delta_a \Delta_b - \Delta_h)\right\}}{2(1-\xi)},$$

we obtain

$$\frac{d}{dt}V(X_t, Y_t, Z_t) \leq -\left\{\delta_h(\delta_a\delta_b - \Delta_h) - \frac{1}{2}\Delta_h\gamma(\Delta_a\Delta_b - \Delta_h)\right\}\|X\|^2$$
$$- \frac{1}{2(1-\xi)}\big\{2(\delta_b - \beta\Delta_h)(1-\xi)$$
$$- \gamma\Delta_h\left\{(1+\Delta_a^2)(1-\xi) + 2 + \beta + \Delta_a^2 + (\Delta_a\Delta_b - \Delta_h)\right\}\big\}\|Y\|^2$$
$$- \left\{\beta\delta_a - 1 - \frac{1}{2}\gamma\Delta_h(\beta + \Delta_a)\right\}\|Z\|^2$$
$$+ \left\{(\Delta_a\Delta_b - \Delta_h)\|X\| + (1+\Delta_a^2)\|Y\| + (\beta + \Delta_a)\|Z\|\right\} m.$$

When

$$\gamma < \min\left\{\frac{2\delta_h(\delta_a\delta_b - \Delta_h)}{\Delta_h(\Delta_a\Delta_b - \Delta_h)}, \frac{2(\beta\delta_a - 1)}{\Delta_h(\beta + \Delta_a)}, \frac{2(\delta_b - \beta\Delta_h)(1-\xi)}{\Delta_h\left\{(1+\Delta_a^2)(1-\xi) + 2 + \beta + \Delta_a^2 + (\Delta_a\Delta_b - \Delta_h)\right\}}\right\},$$

we get

$$\frac{d}{dt}V(X_t, Y_t, Z_t) \leq -\alpha(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + k\alpha(\|X\| + \|Y\| + \|Z\|)$$
$$= -\frac{\alpha}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2) - \frac{\alpha}{2}\left\{(\|X\| - k)^2 + (\|Y\| - k)^2 + (\|Z\| - k)^2\right\} + \frac{3\alpha}{2}k^2$$
$$\leq -\frac{\alpha}{2}(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \frac{3\alpha}{2}k^2, \quad \text{for some } k, \alpha > 0.$$

This completes the proof. □

Example 2 Now, as a special case of system (1.1) (for $P(t) \neq 0$), let us take $n = 2$ that A, B, $H(X(t - r(t)))$ defined in Example 1 hold. If we take $r(t) = \frac{1}{2146+t^2}$, then $0 \leq \frac{1}{2146+t^2} < \gamma$, and $r'(t) = \frac{-2t}{(2146+t^2)^2} \leq \xi$, $0 < \xi < 1$. Let

$$P(t) = \begin{pmatrix} \frac{1}{1+t^2} \\ \frac{1}{1+t^2} \end{pmatrix}, \quad \beta = \frac{1}{6} \text{ and } \xi = \frac{1}{2},$$

we have that

$$\|P(t)\| = \frac{2}{1+t^2} \leq 2 \text{ and } \gamma < \min\left\{\frac{2}{13}, \frac{1}{6}, \frac{1}{2144}\right\}.$$

Thus, all the conditions of Theorem 4.2 are satisfied.

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