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BIFURCATIONS OF INVARIANT MEASURES IN DISCRETE-TIME PARAMETER DEPENDENT COCYCLES

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Abstract. We consider parameter-dependent cocycles generated by nonautonomous difference equations. One of them is a discrete-time cardiac conduction model. For this system with a control variable a cocycle formulation is presented. We state a theorem about upper Hausdorff dimension estimates for cocycle attractors which includes some regulating function. We also consider the existence of invariant measures for cocycle systems using some elements of Perron-Frobenius theory and discuss the bifurcation of parameter-dependent measures.

Keywords: discrete-time parameter-dependent cocycles; Hausdorff dimension estimate; invariant measure

MSC 2010: 35B15, 35K20

1. Introduction

In this paper we consider nonautonomous discrete-time systems depending on a parameter. Such systems arise, for example, if we introduce a control into an autonomous discrete-time system, in order to stabilize the given system. The resulting control system can be analysed using the cocycle theory ([1], [7]). Thus the complete system consists of a driving or base system (control variables) and a cocycle over this base system (phase variables). We consider invariant sets and invariant measures, which depend on the control variables.

The paper is organized as follows. In Section 2 we recall some definitions on discrete-time cocycles and their nonautonomous invariant sets. In Section 3 we

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estimate from above the Hausdorff dimension of nonautonomous invariant sets. Our main result in that section is a generalization of the known Douady-Oesterlé estimate for Hausdorff dimension ([10]) to the case of discrete-time cocycles in Hilbert spaces including a regulating or Lyapunov type function. In Section 5 and Section 6 we show how to construct invariant parametrized measures for discrete-time cocycles and give a definition of bifurcations of such measures.

2. Some facts on discrete-time cocycle theory

Let (Q, d) be a complete metric space.

A *(discrete-time) base flow* $(\{\sigma^k\}_{k\in\mathbb{Z}}, (Q, d))$ is defined by a mapping

$$\sigma^{(\cdot)}(\cdot)\colon\ \mathbb{Z}\times Q \to Q,\ \ (k, q)\mapsto \sigma^k(q)$$

possessing the following properties:

1) $\sigma^0(\cdot) = \mathrm{id}_Q$;
2) $\sigma^{k+j}(\cdot) = \sigma^k(\cdot)\circ\sigma^j(\cdot)$ for each $k, j\in\mathbb{Z}$.

A *(discrete-time) cocycle over the base flow* $(\{\sigma^k\}_{k\in\mathbb{Z}}, (Q, d))$ is defined by the pair $(\{\varphi^k(q,\cdot)\}_{k\in\mathbb{Z}_+, q\in Q}, (M,\rho))$, where (M,ρ) is a metric space and the following conditions are satisfied:

1) $\varphi^k(q,\cdot)\colon\ M\to M,\ \forall\, k\in\mathbb{Z}_+,\ \forall\, q\in Q$;
2) $\varphi^0(q,\cdot) = \mathrm{id}_M,\ \forall\, q\in Q$;
3) $\varphi^{k+j}(q,\cdot) = \varphi^k(\sigma^j(q), \varphi^j(q,\cdot)),\ \forall\, k, j\in\mathbb{Z}_+,\ \forall\, q\in Q$.

Shortly we denote the cocycle over the base flow by (σ,φ). This definition and the subsequent properties were introduced in [7]. If $q\in Q\mapsto Z(q)\subset M$ is a map, we call $\widehat{Z} = \{Z(q)\}_{q\in Q}$ a *nonautonomous* set. The nonautonomous set $\widehat{Z} = \{Z(q)\}_{q\in Q}$ is said to be *invariant* for the cocycle (σ,φ) if $\varphi^k(q, Z(q)) = Z(\sigma^k(q))$ for all $k\in\mathbb{Z}_+$ and $q\in Q$.

The nonautonomous set $\widehat{Z} = \{Z(q)\}_{q\in Q}$ is called *bounded (closed, compact)* if for any $q\in Q$ the set $Z(q)$ is bounded (closed, compact) in M. A bounded nonautonomous set $\widehat{Z} = \{Z(q)\}_{q\in Q}$ is said to be *globally $\mathcal{B}$-pullback attracting* for (σ,φ) if for any $q\in Q$ and any bounded set $B\subset M$ we have $\lim\limits_{k\to\infty}\mathrm{dist}(\varphi^k(\sigma^{-k}(q), B), Z(q)) = 0$, where dist is the Hausdorff semidistance in (M,ρ).

A nonautonomous set is called a *global $\mathcal{B}$-pullback attractor* for the cocycle (σ,φ) if it is compact, invariant and globally $\mathcal{B}$-pullback attracting.

Denote by $\tilde\rho$ the metric on the product space $W := Q\times M$ given by

$$\tilde\rho((q,u),(q',u')) := \sqrt{d^2(q,q') + \rho^2(u,u')},\quad (q,u),(q',u')\in W.$$

Introduce the *skew product system* $(\{S^k\}_{k\in\mathbb{Z}_+}, (W, \tilde{\rho}))$ defined by

$$(q, u) \in W \mapsto S^k(q, u) := (\sigma^k(q), \varphi^k(q, u)).$$

3. Hausdorff dimension estimates for invariant sets of cocycles

Suppose that H is a separable Hilbert space with the scalar product $(\cdot, \cdot)$ and the associated norm $\|\cdot\|$, $K \subset H$ is a compact set, $L \in \mathcal{L}(H)$ is a linear operator. Let us introduce the *singular values* of L by

$$\alpha_n(L) = \sup_{\substack{M\subset H \\ \dim M = n}} \inf_{\substack{u\in M \\ \|u\|=1}} \|Lu\|, \quad n = 1, 2, \ldots$$

Suppose that $d \geqslant 0$ is an arbitrary number. It can be represented as $d = d_0 + s$, where $d_0 \in \mathbb{N}_0$ and $s \in [0, 1]$. Now we put

$$\omega_d(L) := \begin{cases} \alpha_1(L) \cdot \alpha_2(L) \cdot \ldots \cdot \alpha_{d_0}(L) \cdot \alpha_{d_0+1}^s(L), & \text{for } d > 0, \\ 1, & \text{for } d = 0 \end{cases}$$

and call $\omega_d(L)$ the *singular value function of L of order d*, see [4].

Suppose that (σ, φ) is a discrete-time cocycle, given for $k \in \mathbb{Z}_+$ by

$$\sigma^k\colon\ Q \to Q, \quad \varphi^k(\cdot,\cdot)\colon\ Q \times H \to H$$

and let us introduce the following assumptions:

(A1) The nonautonomous set $\widehat{Z} = \{Z(q)\}_{q\in Q}$ is invariant for the cocycle (σ, φ).

(A2) For each $q \in Q$ and $k \in \mathbb{N}$ let $\partial_2\varphi^k(q, \cdot)\colon\ H \to H$ be the Fréchet derivative of $\varphi^k(q, \cdot)$ with respect to the second argument u, which has the following properties:

a) For each $\varepsilon > 0$ and $k \in \mathbb{N}$ the function

$$g_\varepsilon(k, q) := \sup_{\substack{u,v\in Z(q) \\ 0<\|v-u\|\leqslant\varepsilon}} \frac{\|\varphi^k(q, v) - \varphi^k(q, u) - \partial_2\varphi^k(q, u)(v - u)\|}{\|v - u\|}$$

is bounded on Q and converges to zero as $\varepsilon \to 0$.

b) For each $k \in \mathbb{N}$

$$\sup_{q\in Q} \sup_{u\in Z(q)} \|\partial_2\varphi^k(q, u)\|_{\text{op}} < \infty.$$

Theorem 3.1. *Suppose that the assumptions* (A1) *and* (A2) *are satisfied and the following conditions hold:*

1) *There exists a compact set* $\widetilde{K} \subset H$*, such that*

$$\overline{\bigcup_{q\in Q} Z(q)} \subset \widetilde{K}.$$

2) *There exists a continuous function with respect to the second variable* $\kappa\colon\ Q \times H \to \mathbb{R}_+$*, a time* $j \in \mathbb{N}$ *and a number* $d > 0$ *such that*

$$Z(q) \subset Z(\sigma^j(q)), \quad q \in Q$$

and

$$\sup_{(q,u)\in Q\times\widetilde{K}} \frac{\kappa(\sigma^j(q), \varphi^j(q,u))}{\kappa(q,u)}\, \omega_d(\partial_2\varphi^j(q,u)) < 1. \tag{3.1}$$

Then $\dim_H Z(q) \leqslant d,\ \forall\, q \in Q$.

Here $\dim_H(\cdot)$ *denotes the Hausdorff dimension of a set.*

Note that a finite-dimensional version of Theorem 3.1 ($H = \mathbb{R}^n$) was shown in [10]. A stochastic version of the theorem (without a regulating function $\kappa(\cdot,\cdot)$) was derived in [5].

4. The cardiac conduction model

A cardiac conduction model is formulated in [11] as a two-dimensional piecewise smooth map. The model predicts a variety of experimentally observed complex rythms of nodal conduction. It is shown in this paper that for certain parameter values alternans, in which there is an alternation in conduction time from beat to beat, are associated with period-doubling bifurcation.

The model can be written as a discrete-time nonautonomous system in the following form:

$$\begin{cases} A_{k+1} = A_{\min} + R_k \exp\Big(-\dfrac{A_k + H_k}{\tau_{\text{fat}}}\Big) \\ \qquad\qquad + \gamma \exp\Big(-\dfrac{H_k}{\tau_{\text{fat}}}\Big) + \beta(A_k)\exp\Big(-\dfrac{H_k}{\tau_{\text{rec}}}\Big), \\ R_{k+1} = R_k \exp\Big(-\dfrac{A_k + H_k}{\tau_{\text{fat}}}\Big) + \gamma \exp\Big(-\dfrac{H_k}{\tau_{\text{fat}}}\Big), \quad k = 0, 1, \dots, \end{cases} \tag{4.1}$$

where $\beta\colon \mathbb{R} \to \mathbb{R}$ is the piecewise linear function

$$\beta(x) := \begin{cases} 201 - 0.7x, & \text{for } x < 130, \\ 500 - 3.0x, & \text{for } x \geqslant 130. \end{cases}$$

We suppose that $A_{\min}$, τ_{rec}, γ and τ_{fat} are positive constants. The variable A_k represents the conduction time of the kth beat; R_k represents a drift in the nodal conduction time; H_k represents the interval between the bundle of His activation and the subsequent activation (the AV nodal recovery time).

It is assumed that

$$H_k = \alpha + p_k, \quad k = 0, 1, 2, \dots, \tag{4.2}$$

where α is a parameter and $\{p_k\}_{k=0}^{\infty}$ is viewed as a control variable. Let us write system (4.1), (4.2) as

$$u_{k+1} = f(p_k, u_k, \alpha), \quad k = 0, 1, 2, \dots, \tag{4.3}$$

where $u = (A, R)$ and $f\colon \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is the right-hand side of (4.1).

We assume that $\{p_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}, \mathbb{R})$. Introduce the space

$$Q := \overline{\{p_{k+\cdot}\,;\ k \in \mathbb{Z}\}} \tag{4.4}$$

as the closure in the topology of $l^2(\mathbb{Z}, \mathbb{R})$. Suppose that $\sigma^k\colon Q \to Q$, $k \in \mathbb{Z}$ is the shift operator on Q which is defined for $q = \{q_j\}_{j\in\mathbb{Z}} \in Q$ by

$$\sigma^k(q) := \{q_{k+j}\}_{j\in\mathbb{Z}}. \tag{4.5}$$

Along with system (4.3)–(4.5) we consider the family of parameter-dependent control systems

$$u_{k+1} = f(q_k, u_k, \alpha), \quad k = 0, 1, 2, \dots, \tag{4.6}$$

where $q = \{q_k\}_{k\in\mathbb{Z}} \in Q$.

Suppose that $u_k(q, 0, u_0)$, $k \in \mathbb{N}_0$ is the solution of (4.6) for fixed α with $u_0(q, 0, u_0) = u_0$.

Let us define a cocycle with respect to the base flow (4.4), (4.5) by

$$\varphi^k(q, u_0) := u_k(q, 0, u_0), \quad k \in \mathbb{Z}_+. \tag{4.7}$$

5. Invariant measures for cocycles and the Perron-Frobenius operator

Suppose that (σ, φ) is a cocycle over the base flow in the sense of Section 2. Assume that in addition to the metric structure (Q, d) we have the structure of a measurable space $(Q, \mathfrak{A}, \mu)$, where $\mathfrak{A}$ is a σ-algebra over Q and μ is a probability measure on $\mathfrak{A}$. It is also assumed that μ is invariant for the base flow $\{\sigma^k\}_{k\in\mathbb{Z}}$, i.e.,

$$\mu(\sigma^{-k}(A)) = \mu(A), \quad A \in \mathfrak{A},\ k \in \mathbb{Z}.$$

Suppose that $\mathfrak{B}$ is the σ-algebra of Borel sets on M. An *invariant measure* ν for the cocycle (σ, φ) is a probability measure on $\mathfrak{A} \otimes \mathfrak{B}$ which is invariant with respect to the skew product semiflow $\{S^k\}_{k\in\mathbb{Z}_+}$, i.e.,

$$\nu(S^{-k}(C)) = \nu(C), \quad C \in \mathfrak{A} \otimes \mathfrak{B},\ k \in \mathbb{Z}_+,$$

and satisfies $\pi_Q \nu = \mu$, where $\pi_Q \colon Q \times M \to Q$ denotes the projection on Q. We can characterize such invariant measures by their disintegration property

$$\nu(\mathrm{d}q, \mathrm{d}u) = \nu_q(\mathrm{d}u)\mu(\mathrm{d}q),$$

i.e., for any set $C \in \mathfrak{A} \otimes \mathfrak{B}$ we have

$$\nu(C) = \int_Q \nu_q(C(q))\,\mathrm{d}\mu(q). \tag{5.1}$$

Thus a probability measure ν on $Q \times M$ is $\{S^k\}$-invariant iff ν is of the form (5.1) and $\varphi^k(q, \cdot)\nu_q = \nu_{\sigma^k(q)}$ μ-a.s. for $k \in \mathbb{Z}$.

For the determination of such invariant measures one can use the *Perron-Frobenius operator* P, which is defined as

$$P\nu_q(C(q)) := \nu_q\bigl(\varphi^{-1}(q, C(\sigma^1(q)))\bigr), \quad q \in Q. \tag{5.2}$$

R e m a r k 5.1. Instead of the Perron-Frobenius operator (5.2) one can use transfer operators. Consider, for example, the map $\widehat{\varphi} \colon \hat{I} \to \hat{I}$, where $\hat{I} = \bigcup_{k\geqslant 0} (\{k\} \times B_k)$ and $B_0 = I$ is the unit interval, $\{B_k\}$ are subsets of I, $\widehat{\varphi}(k, u) = (k + 1, \varphi(u))$ is a tower construction, where the smooth map $\varphi \colon I \to I$ admits an invariant measure μ absolutely continuous with respect to the Lebesgue measure m ([2]).

Introduce a regulating function $\kappa\colon \hat{I} \to (0,\infty)$ and the *transfer operator* ([2])

$$\mathcal{L}(\hat{g})(k,y) := \sum_{\widehat{\varphi}(l,x)=(k,y)} \frac{\kappa(l,x)}{\kappa(k,y)} \frac{\hat{g}(l,x)}{|\varphi'(x)|} \tag{5.3}$$

acting on the Banach space $BV(\hat{I})$ of functions $\hat{g}\colon \hat{I} \to \mathbb{R}$.

If ϱ is an eigenfunction of $\mathcal{L}$ associated with the eigenvalue 1 then $\hat{\mu} = \varrho\kappa\,\mathrm{d}x$ is an invariant measure for $\widehat{\varphi}$. Suppose that $\widehat{\varphi}$ is invertible. Then (5.3) reduces with $q = k$, $u = x$ to

$$\mathcal{L}(\hat{g})(\widehat{\varphi}(q,u)) = \frac{\kappa(q,u)}{\kappa(\widehat{\varphi}(q,u))} \frac{\hat{g}(q,u)}{|\varphi'(u)|}.$$

For the existence of an invariant measure we need

$$\frac{\kappa(\widehat{\varphi}(q,u))}{\kappa(q,u)} |\varphi'(u)| = 1, \quad (q,u) \in Q \times I. \tag{5.4}$$

For $d = 1$ we have in (3.1) $\omega_1(\partial_2\varphi^1(q,u)) = |\det \partial_2\varphi^1(q,u)|$. Thus, if we consider (3.1) as equality, this condition coincides with (5.4).

6. Parametrized cocycles and bifurcations

Suppose that (Λ, ρ_Λ) is a metric space of parameters. Assume that $\{(Q_\alpha, d_\alpha)\}_{\alpha\in\Lambda}$ is a family of complete metric spaces,

$$\Big(\{\sigma_\alpha^k\}_{\substack{k\in\mathbb{Z}\\ \alpha\in\Lambda}}, (Q_\alpha, d_\alpha)\Big) \tag{6.1}$$

is a parametrized base flow and

$$\Big(\{\varphi_\alpha^k(q,\cdot)\}_{\substack{k\in\mathbb{Z}_+\\ q\in Q\\ \alpha\in\Lambda}}, (M,\rho)\Big) \tag{6.2}$$

is a parametrized cocycle over the base flow (6.1). Here (M,ρ) is a complete metric space.

According to the parametrized cocycle (6.2) over the parametrized base flow (6.1), which we shortly denote by $(\sigma_\alpha, \varphi_\alpha)$, we introduce the parametrized skew product semiflow

$$\Big(\{S_\alpha^k\}_{\substack{k\in\mathbb{Z}_+\\ \alpha\in\Lambda}}, (W_\alpha, \tilde{\rho}_\alpha)\Big), \tag{6.3}$$

defined for all $\alpha \in \Lambda$ by $W_\alpha := Q_\alpha \times M$, $\tilde{\rho}_\alpha((q,u),(q',u')) := \sqrt{d_\alpha^2(q,q') + \rho_\alpha^2(u,u')}$, $(q,u),(q',u') \in W_\alpha$, and $(q,u) \in W_\alpha \mapsto S_\alpha^k(q,u) := (\sigma_\alpha^k(q), \varphi_\alpha^k(q,u))$, $k \in \mathbb{Z}_+$.

Let $(Q_\alpha, \mathfrak{A}_\alpha, \mu_\alpha)$ be a family of probability spaces depending on a parameter $\alpha \in \Lambda$. The maps $\{\sigma_\alpha^k\}_{k\in\mathbb{Z},\alpha\in\Lambda}$ are assumed to be measure preserving, i.e., $\sigma_\alpha^k(\mu_\alpha) = \mu_\alpha$, $k \in \mathbb{Z}$, $\alpha \in \Lambda$. Suppose that $(M, \mathfrak{B})$ is a measurable space and that the maps $\varphi_\alpha^k(\cdot)\colon Q_\alpha \times M \to M$ are $(\mathfrak{A}_\alpha \otimes \mathfrak{B}, \mathfrak{B})$ measurable for $k \in \mathbb{Z}_+$, $\alpha \in \Lambda$.

Let $\{\nu_\alpha\}_{\alpha\in\Lambda}$ be a family of invariant maps for the parametrized skew product, i.e., $S_\alpha^k(\nu_\alpha) = \nu_\alpha$ and $\pi_{Q_\alpha}\nu_\alpha = \nu_\alpha$ for $k \in \mathbb{Z}$ and $\alpha \in \Lambda$.

A parameter value α_0 is called a *bifurcation point* of the family of invariant measures $\{\nu_\alpha\}_{\alpha\in\Lambda}$ if this family is not structurally stable at α_0, i.e., if in any neighborhood of α_0 there are parameter values $\alpha \in \Lambda$ such that $\{S_{\alpha_0}^k\}$ and $\{S_\alpha^k\}$ are not topologically equivalent ([1], [9]).

Example 6.1. The Rényi map $\varphi_\alpha\colon [0,1] \to [0,1]$, which can be viewed as a one-dimensional model of cardiac arrhythmias ([6]), is given by $\varphi_\alpha(x) = \alpha x \bmod 1$ with $\alpha > 1$. This map generates a metric dynamical system $(\{\varphi_\alpha^k\}, m)$, where m denotes the Lebesgue measure on the unit interval. Let us consider the transfer operator $\mathcal{L}_\alpha$ given by $\mathcal{L}_\alpha\colon L^2(m) \to L^2(m)$, where $\mathcal{L}_\alpha\eta := (\mathrm{d}/\mathrm{d}m)\int_{\varphi_\alpha^{-1}(\cdot)} \eta\,\mathrm{d}m$ and d/dm is the Radon-Nikodym derivative with respect to m. Spectral properties of this operator family are investigated for test functions in rigged Hilbert spaces in [3]. Depending on the choice of these rigged Hilbert spaces different parametrized invariant measures are derived and bifurcations of such measures are considered in [3], [8].

The map under perturbation q is $\varphi_\alpha(q,u) = \varphi_\alpha(u) + q$. Thus we can consider the associated skew product system $S_\alpha^{(\cdot)}(\cdot)\colon Q_\alpha \times I \to Q_\alpha \times I$. The spaces $\{Q_\alpha\}_{\alpha>1}$ can be defined by $Q_\alpha = l^2(\mathbb{Z}, \mathbb{R})$, where for $q \in Q_\alpha$ we define $\|q\|^2_{l^2_\alpha(\mathbb{Z},\mathbb{R})} := \sum_k |\alpha^{-k} q_k|^2$.

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