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HYERS-ULAM STABILITY OF FRACTIONAL LINEAR DIFFERENTIAL EQUATIONS INVOLVING CAPUTO FRACTIONAL DERIVATIVES

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Abstract. The aim of this paper is to study the stability of fractional differential equations in Hyers-Ulam sense. Namely, if we replace a given fractional differential equation by a fractional differential inequality, we ask when the solutions of the fractional differential inequality are close to the solutions of the strict differential equation. In this paper, we investigate the Hyers-Ulam stability of two types of fractional linear differential equations with Caputo fractional derivatives. We prove that the two types of fractional linear differential equations are Hyers-Ulam stable by applying the Laplace transform method. Finally, an example is given to illustrate the theoretical results.

Keywords: Hyers-Ulam stability; Laplace transform method; fractional differential equation; Caputo fractional derivative

MSC 2010: 26D10, 34A08

1. Introduction and preliminaries

For some equations (differential equations, functional equations, etc.) describing physical models and practical problems, finding exact solutions of these equations is very difficult, and the form of the exact solutions (if they exist) is often so complicated that it is not convenient for numerical calculation. In view of this, it is necessary to discuss approximate solutions with relatively simple form, and ask whether the approximate solutions lie near the exact solutions.

Generally, we say that a differential equation is stable in Hyers-Ulam sense if for every solution of the perturbed equation there exists a solution of the equation that is close to it. In other words, if we replace a given differential equation by

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differential inequality, when can one assert that the solutions of the inequality lie near the solutions of the equation?

In recent years, many researchers have focused on the study of Hyers-Ulam stability of differential equations, and gained a series of results (see [1]–[7] and [9]–[14] and the references therein).

Recently, by applying Laplace transform method, Rezaei, Jung, and Rassias discussed Hyers-Ulam stability of linear differential equations (see [11]). Popa and Raşa proved the generalized Hyers-Ulam stability of linear differential equations in a Banach space (see [10]). András and Mészáros presented Hyers-Ulam stability of dynamic equations on time scales via Picard operators (see [1]). Regarding partial differential equations, in [9], Lungu and Popa discussed Hyers-Ulam stability of a first order partial differential equation, and in [2], Gordji, Cho, Ghaemi, and Alizadeh investigated stability of second order partial differential equations. Hegyi and Jung discussed the stability of Laplace's equation (see [3]). As for fractional differential equations, Wang, Zhou et al. (see [12], [13], [14]) proved the stability of fractional evolution equations and the stability of nonlinear differential equations with fractional integrable impulses, and they also introduced some new concepts concerning the stability of fractional differential equations. In [4], Ibrahim presented Hyers-Ulam stability of Cauchy differential equation of fractional order in the unit disk. However, the theory of Hyers-Ulam stability of fractional differential equations is still in its initial stages.

The main aim of this paper is to prove the Hyers-Ulam stability of the following two types of fractional linear differential equations:

$$({}^C D^{\alpha}_{0+} y)(x) - \lambda y(x) = f(x), \tag{1.1}$$

and

$$({}^C D^{\alpha}_{0+} y)(x) - \lambda ({}^C D^{\beta}_{0+} y)(x) = g(x), \tag{1.2}$$

where $x > 0, \lambda \in \mathbb{R}$, $n - 1 < \alpha \leqslant n$, $m - 1 < \beta \leqslant m$, $0 < \beta < \alpha$, $m, n \in \mathbb{N}$, $m \leqslant n$, $f(x)$ and $g(x)$ are real functions defined on $\mathbb{R}_+$, and ${}^C D^{\alpha}_{0+}$ is the Caputo fractional derivative of order α defined by

$$({}^C D^{\alpha}_{0+} y)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} y^{(n)}(t)\,\mathrm{d}t. \tag{1.3}$$

In order to prove our main results, we recall the definition of the Laplace transform and some basic properties of the Laplace transform for fractional derivatives.

A function $y\colon (0,\infty)\to\mathbb{R}$ is said to be of exponential order if there are constants $A, B\in\mathbb{R}$ such that $|y(x)|\leqslant A\mathrm{e}^{Bx}$ for all $x>0$. For each function $y\colon (0,\infty)\to\mathbb{R}$ of exponential order, the Laplace transform of $y(x)$ is defined by

$$\mathcal{L}\{y(x)\}(s) := \int_0^\infty \mathrm{e}^{-sx} y(x)\,\mathrm{d}x, \quad s\in\mathbb{C}. \tag{1.4}$$

If the integral (1.4) is convergent at the point $s_0\in\mathbb{C}$, then it converges absolutely for $s\in\mathbb{C}$ such that $\Re(s)>\Re(s_0)$. One of the most useful properties of the Laplace transform is the convolution property

$$\mathcal{L}\{y_1(x)*y_2(x)\} = \mathcal{L}\{y_1(x)\}\mathcal{L}\{y_2(x)\}, \tag{1.5}$$

where $y_1(x)*y_2(x)=\int_0^x y_1(x-\xi)y_2(\xi)\,\mathrm{d}\xi$.

The following results are some basic properties of the Laplace transform of the Caputo fractional derivatives.

Lemma 1.1 ([8]). *Let $\alpha>0$, $n-1<\alpha\leqslant n$, $n\in\mathbb{N}$ be such that $y\in C^n(\mathbb{R}_+)$, $y^{(n)}\in L_1(0,b)$ for any $b>0$, the estimates $|y^{(n)}(x)|\leqslant B\mathrm{e}^{q_0x}$ (for $x>b>0$, B and q_0 are all constants, $B>0$, $q_0>0$) holds, the Laplace transforms $\mathcal{L}\{y(x)\}$ and $\mathcal{L}\{D^n y(x)\}$ exist, and $\lim\limits_{x\to\infty}(D^k y)(x)=0$ for $k=0,1,\dots,n-1$. Then the following relation holds:*

$$\mathcal{L}\{{}^C D_{0+}^\alpha y(x)\}(s) = s^\alpha\mathcal{L}\{y(x)\}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}(D^k y)(0). \tag{1.6}$$

In particular, if $0<\alpha\leqslant 1$, then

$$\mathcal{L}\{{}^C D_{0+}^\alpha y(x)\}(s) = s^\alpha\mathcal{L}\{y(x)\}(s) - s^{\alpha-1}y(0). \tag{1.7}$$

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta} = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z,\beta\in\mathbb{C},\ \Re(\alpha)>0, \tag{1.8}$$

when $\alpha=\beta=1$, we can see that $E_{1,1}(z)=\mathrm{e}^z$. More detailed information about the function can be found in [8].

Lemma 1.2 ([8]). *If* $\Re(s) > 0$, $\lambda \in \mathbb{C}$, $|\lambda s^{-\alpha}| < 1$, *then*

$$\mathcal{L}\{x^{\beta-1}E_{\alpha,\beta}(\lambda x^{\alpha})\}(s) = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \tag{1.9}$$

where $E_{\alpha,\beta}(\lambda x^{\alpha})$ *is the Mittag-Leffler function.*

Remark 1.3. When $\alpha = \beta$, we have $\mathcal{L}\{x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^{\alpha})\}(s) = 1/(s^{\alpha}-\lambda)$.

2. Hyers-Ulam stability of fractional differential equation (1.1)

In this section, we will prove that the fractional differential equation (1.1) is Hyers-Ulam stable.

Definition 2.1. The fractional differential equation $\varphi(f, y, D^{\alpha_1}y, \ldots, D^{\alpha_n}y) = 0$ has Hyers-Ulam stability if for a given $\varepsilon > 0$ and a function y such that

$$|\varphi(f, y, D^{\alpha_1}y, \ldots, D^{\alpha_n}y)| \leqslant \varepsilon,$$

there exists a solution y_a of the differential equation such that $|y(x) - y_a(x)| \leqslant K(\varepsilon)$ and $\lim\limits_{\varepsilon\to 0} K(\varepsilon) = 0$. If this statement is also true when we replace ε and $K(\varepsilon)$ by $F(x)$ and $C(x)$, where F, C are appropriate functions not depending on y and y_a explicitly, then we say that the differential equation has the generalized Hyers-Ulam stability.

More about stability of ordinary differential equations and fractional differential equations can be found in [7], [11], [12] and [13].

Theorem 2.2. *Let* $\lambda \in \mathbb{R}$, $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$, *and let* $f(x)$ *be a given real function defined on* $\mathbb{R}_+$. *If a function* $y\colon (0,\infty) \to \mathbb{R}$ *satisfies the inequality*

$$|({}^{C}D^{\alpha}_{0+}y)(x) - \lambda y(x) - f(x)| \leqslant \varepsilon \tag{2.1}$$

for all $x > 0$ *and for some* $\varepsilon > 0$, *then there exists a solution* $y_a\colon (0,\infty) \to \mathbb{R}$ *of the fractional differential equation*

$$({}^{C}D^{\alpha}_{0+}y)(x) - \lambda y(x) = f(x) \tag{2.2}$$

such that

$$|y - y_a| \leqslant \varepsilon x^{\alpha} E_{\alpha,\alpha+1}(|\lambda| x^{\alpha}). \tag{2.3}$$

P r o o f. Putting $y^{(k)}(0) = b_k$, for $k = 0, 1, \ldots, n-1$, and $Y(x) = ({}^C D_{0+}^{\alpha} y)(x) - \lambda y(x) - f(x)$, by Lemma 1.1 we obtain

$$\begin{aligned}
\mathcal{L}\{Y(x)\} &= \mathcal{L}\{({}^C D_{0+}^{\alpha} y)(x) - \lambda y(x) - f(x)\} \\
&= s^{\alpha}\mathcal{L}\{y(x)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} b_k - \lambda\mathcal{L}\{y(x)\} - \mathcal{L}\{f(x)\} \\
&= (s^{\alpha} - \lambda)\mathcal{L}\{y(x)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} b_k - \mathcal{L}\{f(x)\},
\end{aligned} \tag{2.4}$$

so

$$\mathcal{L}\{y(x)\} = \frac{\sum_{k=0}^{n-1} s^{\alpha-k-1} b_k + \mathcal{L}\{f(x)\}}{s^{\alpha} - \lambda} + \frac{\mathcal{L}\{Y(x)\}}{s^{\alpha} - \lambda}. \tag{2.5}$$

Setting

$$y_a(x) = \sum_{k=0}^{n-1} b_k x^k E_{\alpha,k+1}(\lambda x^{\alpha}) + \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}[\lambda(x-t)^{\alpha}] f(t)\,\mathrm{d}t, \tag{2.6}$$

by Lemma 1.2 and (1.5), we get

$$\begin{aligned}
\mathcal{L}\{y_a(x)\} &= \mathcal{L}\Big\{\sum_{k=0}^{n-1} b_k x^k E_{\alpha,k+1}(\lambda x^{\alpha})\Big\} \\
&\quad + \mathcal{L}\Big\{\int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}[\lambda(x-t)^{\alpha}] f(t)\,\mathrm{d}t\Big\} \\
&= \sum_{k=0}^{n-1} b_k \mathcal{L}\{x^k E_{\alpha,k+1}(\lambda x^{\alpha})\} + \mathcal{L}\{x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^{\alpha})\}\mathcal{L}\{f(x)\} \\
&= \frac{\sum_{k=0}^{n-1} b_k s^{\alpha-(k+1)} + \mathcal{L}\{f(x)\}}{s^{\alpha} - \lambda}.
\end{aligned} \tag{2.7}$$

By Lemma 1.1, (2.7) and a simple computation, one can get

$$\begin{aligned}
\mathcal{L}\{({}^C D_{0+}^{\alpha} y_a)(x) - \lambda y_a(x)\} &= s^{\alpha}\mathcal{L}\{y_a(x)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} b_k - \lambda\mathcal{L}\{y_a(x)\} \\
&= \mathcal{L}\{f(x)\}.
\end{aligned} \tag{2.8}$$

Since $\mathcal{L}$ is one-to-one, it follows that $({}^C D_{0+}^{\alpha} y_a)(x) - \lambda y_a(x) = f(x)$, so $y_a(x)$ is a solution of (2.2).

By (2.5) and (2.7), we obtain

$$\mathcal{L}\{y(x) - y_a(x)\} = \frac{\mathcal{L}\{Y(x)\}}{s^\alpha - \lambda}. \tag{2.9}$$

Using the convolution property and Lemma 1.2, one can get

$$\mathcal{L}\{(x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^\alpha)) * Y(x)\} = \mathcal{L}\{x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^\alpha)\}\mathcal{L}\{Y(x)\} = \frac{\mathcal{L}\{Y(x)\}}{s^\alpha - \lambda}. \tag{2.10}$$

By (2.9) and (2.10), we have

$$y(x) - y_a(x) = (x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^\alpha)) * Y(x), \tag{2.11}$$

therefore, from (2.1), it follows that

$$\begin{aligned}
|y(x) - y_a(x)| &= |(x^{\alpha-1}E_{\alpha,\alpha}(\lambda x^\alpha)) * Y(x)| \\
&= \left|\int_0^x (x-t)^{\alpha-1}E_{\alpha,\alpha}[\lambda(x-t)^\alpha]Y(t)\,\mathrm{d}t\right| \\
&= \left|\int_0^x \sum_{k=0}^\infty \frac{\lambda^k(x-t)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}Y(t)\,\mathrm{d}t\right| \\
&= \left|\sum_{k=0}^\infty \int_0^x \frac{\lambda^k(x-t)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}Y(t)\,\mathrm{d}t\right| \\
&\leqslant \sum_{k=0}^\infty \int_0^x \left|\frac{\lambda^k(x-t)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}\right||Y(t)|\,\mathrm{d}t \\
&\leqslant \varepsilon \sum_{k=0}^\infty \frac{|\lambda|^k}{\Gamma(\alpha k+\alpha)} \int_0^x (x-t)^{\alpha k+\alpha-1}\,\mathrm{d}t \\
&= \varepsilon x^\alpha \sum_{k=0}^\infty \frac{(|\lambda|x^\alpha)^k}{\Gamma(\alpha k+\alpha+1)} \\
&= \varepsilon x^\alpha E_{\alpha,\alpha+1}(|\lambda|x^\alpha),
\end{aligned} \tag{2.12}$$

which completes the proof. □

Similarly, we can prove that the fractional differential equation (2.2) is generalized Hyers-Ulam stable.

Corollary 2.3. *Let $\lambda \in \mathbb{R}$, $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$, and let $f(x)$ be a given real function defined on $\mathbb{R}_+$. If a function $y\colon (0,\infty) \to \mathbb{R}$ satisfies the inequality*

$$|({}^C D^{\alpha}_{0+} y)(x) - \lambda y(x) - f(x)| \leqslant F(x) \tag{2.13}$$

for all $x > 0$ and for some function $F(x) > 0$, then there exists a solution $y_a\colon (0,\infty) \to \mathbb{R}$ of the fractional differential equation (2.2) such that

$$|y - y_a| \leqslant C(x), \tag{2.14}$$

where $C(x) = x^{\alpha} F(x) E_{\alpha,\alpha+1}(|\lambda| x^{\alpha})$.

3. Hyers-Ulam stability of fractional differential equation (1.2)

In this section, we will extend Theorem 2.2 and prove that fractional differential equation (1.2) is Hyers-Ulam stable.

Theorem 3.1. *Let $\lambda \in \mathbb{R}$, $m, n \in \mathbb{N}$, $m \leqslant n$, $n-1 < \alpha \leqslant n$, $m-1 < \beta \leqslant m$, $0 < \beta < \alpha$, and let $g(x)$ be a given real function defined on $\mathbb{R}_+$. If a function $y\colon (0,\infty) \to \mathbb{R}$ satisfies the inequality*

$$|({}^C D^{\alpha}_{0+} y)(x) - \lambda({}^C D^{\beta}_{0+} y)(x) - g(x)| \leqslant \varepsilon \tag{3.1}$$

for all $x > 0$ and some $\varepsilon > 0$, then there exists a solution $y_a\colon (0,\infty) \to \mathbb{R}$ of the fractional differential equation

$$({}^C D^{\alpha}_{0+} y)(x) - \lambda({}^C D^{\beta}_{0+} y)(x) = g(x) \tag{3.2}$$

such that

$$|y - y_a| \leqslant \varepsilon x^{\alpha} E_{\alpha-\beta,\ \alpha+1}(|\lambda| x^{\alpha-\beta}). \tag{3.3}$$

Proof. Putting $y^{(k)}(0) = b_k \in \mathbb{R}$ for $k = 0, 1, \ldots, n-1$, and $Y(x) = ({}^C D^{\alpha}_{0+} y)(x) - \lambda({}^C D^{\beta}_{0+} y)(x) - g(x)$ for each $x > 0$, by Lemma 1.1 we have

$$\begin{aligned}
\mathcal{L}\{Y(x)\} &= \mathcal{L}\{({}^C D^{\alpha}_{0+} y)(x) - \lambda({}^C D^{\beta}_{0+} y)(x) - g(x)\} \\
&= \mathcal{L}\{({}^C D^{\alpha}_{0+} y)(x)\} - \lambda\mathcal{L}\{({}^C D^{\beta}_{0+} y)(x)\} - \mathcal{L}\{g(x)\} \\
&= s^{\alpha}\mathcal{L}\{y(x)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} b_k - \lambda\Big\{s^{\beta}\mathcal{L}\{y(x)\} - \sum_{k=0}^{m-1} s^{\beta-k-1} b_k\Big\} \\
&\quad - \mathcal{L}\{g(x)\} \\
&= (s^{\alpha} - \lambda s^{\beta})\mathcal{L}\{y(x)\} - \sum_{k=0}^{n-1} s^{\alpha-k-1} b_k + \lambda \sum_{k=0}^{m-1} s^{\beta-k-1} b_k - \mathcal{L}\{g(x)\}.
\end{aligned} \tag{3.4}$$

By (3.4), it follows that

$$\mathcal{L}\{y(x)\} = \frac{\sum_{k=0}^{n-1} s^{\alpha-k-1}b_k - \lambda\sum_{k=0}^{m-1} s^{\beta-k-1}b_k + \mathcal{L}\{g(x)\}}{s^\alpha - \lambda s^\beta} + \frac{\mathcal{L}\{Y(x)\}}{s^\alpha - \lambda s^\beta}. \tag{3.5}$$

Put

$$y_a(x) = \sum_{k=0}^{n-1} b_k y_k(x) + \int_0^x (x-t)^{\alpha-1} E_{\alpha-\beta,\alpha}[\lambda(x-t)^{\alpha-\beta}]g(t)\,\mathrm{d}t, \tag{3.6}$$

where

$$y_k(x) = x^k E_{\alpha-\beta,k+1}(\lambda x^{\alpha-\beta}) - \lambda x^{\alpha-\beta+k} E_{\alpha-\beta,\alpha-\beta+k+1}(\lambda x^{\alpha-\beta}), \quad k = 0,1,\dots,m-1, \tag{3.7}$$

$$y_k(x) = x^k E_{\alpha-\beta,k+1}(\lambda x^{\alpha-\beta}), \quad k = m,\dots,n-1. \tag{3.8}$$

By Lemma 1.2 and (1.5), we get

$$\begin{aligned}
\mathcal{L}\{y_a(x)\} &= \mathcal{L}\Big\{\sum_{k=0}^{m-1} b_k y_k(x)\Big\} + \mathcal{L}\Big\{\sum_{k=m}^{n-1} b_k y_k(x)\Big\} \\
&\quad + \mathcal{L}\Big\{\int_0^x (x-t)^{\alpha-1} E_{\alpha-\beta,\alpha}[\lambda(x-t)^{\alpha-\beta}]g(t)\,\mathrm{d}t\Big\} \\
&= \sum_{k=0}^{m-1} b_k \mathcal{L}\{x^k E_{\alpha-\beta,k+1}(\lambda x^{\alpha-\beta}) - \lambda x^{\alpha-\beta+k} E_{\alpha-\beta,\alpha-\beta+k+1}(\lambda x^{\alpha-\beta})\} \\
&\quad + \sum_{k=m}^{n-1} b_k \mathcal{L}\{x^k E_{\alpha-\beta,k+1}(\lambda x^{\alpha-\beta})\} \\
&\quad + \mathcal{L}\{x^{\alpha-1} E_{\alpha-\beta,\alpha}(\lambda x^{\alpha-\beta})\}\mathcal{L}\{g(x)\} \\
&= \frac{\sum_{k=0}^{n-1} b_k s^{\alpha-k-1} - \lambda\sum_{k=0}^{m-1} b_k s^{\beta-k-1} + \mathcal{L}\{g(x)\}}{s^\alpha - \lambda s^\beta}.
\end{aligned} \tag{3.9}$$

By (3.9), one can get

$$\mathcal{L}\{({}^C D_{0+}^\alpha y_a)(x) - \lambda({}^C D_{0+}^\beta y_a)(x)\} = \mathcal{L}\{g(x)\}, \tag{3.10}$$

so $y_a(x)$ is a solution of (3.2).

Using (3.5) and (3.9), we obtain

$$\mathcal{L}\{y(x) - y_a(x)\} = \frac{\mathcal{L}\{Y(x)\}}{s^\alpha - \lambda s^\beta}. \tag{3.11}$$

By (1.5) and Lemma 1.2, we have

$$\begin{aligned}(3.12)\qquad \mathcal{L}\{[x^{\alpha-1}E_{\alpha-\beta,\alpha}(\lambda x^{\alpha-\beta})]*Y(x)\} &= \mathcal{L}[x^{\alpha-1}E_{\alpha-\beta,\alpha}(\lambda x^{\alpha-\beta})]\mathcal{L}\{Y(x)\}\\ &= \frac{\mathcal{L}\{Y(x)\}}{s^{\alpha}-\lambda s^{\beta}}.\end{aligned}$$

Using (3.11) and (3.12), we see that

$$(3.13)\qquad y(x)-y_a(x)=[x^{\alpha-1}E_{\alpha-\beta,\alpha}(\lambda x^{\alpha-\beta})]*Y(x).$$

Therefore, from (3.1), it follows that

$$\begin{aligned}(3.14)\qquad |y(x)-y_a(x)| &= |[x^{\alpha-1}E_{\alpha-\beta,\alpha}(\lambda x^{\alpha-\beta})]*Y(x)|\\ &= \left|\int_0^x (x-t)^{\alpha-1}E_{\alpha-\beta,\alpha}[\lambda(x-t)^{\alpha-\beta}]Y(t)\,\mathrm{d}t\right|\\ &= \left|\int_0^x \sum_{k=0}^{\infty}\frac{\lambda^k(x-t)^{\alpha k-\beta k+\alpha-1}}{\Gamma[(\alpha-\beta)k+\alpha]}Y(t)\,\mathrm{d}t\right|\\ &\leqslant \sum_{k=0}^{\infty}\frac{|\lambda|^k}{\Gamma[(\alpha-\beta)k+\alpha]}\left|\int_0^x (x-t)^{\alpha k-\beta k+\alpha-1}Y(t)\,\mathrm{d}t\right|\\ &\leqslant \sum_{k=0}^{\infty}\frac{|\lambda|^k\varepsilon}{\Gamma[(\alpha-\beta)k+\alpha]}\int_0^x (x-t)^{\alpha k-\beta k+\alpha-1}\,\mathrm{d}t\\ &= \varepsilon x^{\alpha}\sum_{k=0}^{\infty}\frac{(|\lambda|x^{\alpha-\beta})^k}{\Gamma[(\alpha-\beta)k+\alpha+1]}\\ &= \varepsilon x^{\alpha}E_{\alpha-\beta,\alpha+1}(|\lambda|x^{\alpha-\beta}),\end{aligned}$$

which completes the proof. □

Remark 3.2. If $\beta=0$ and $f(x)=g(x)$, then the equation $({}^CD^{\alpha}_{0+}y)(x)-\lambda({}^CD^{\beta}_{0+}y)(x)=g(x)$ coincides with $({}^CD^{\alpha}_{0+}y)(x)-\lambda y(x)=f(x)$, and $\varepsilon x^{\alpha}E_{\alpha-\beta,\alpha+1}(|\lambda|x^{\alpha-\beta})$ coincides with $\varepsilon x^{\alpha}E_{\alpha,\alpha+1}(|\lambda|x^{\alpha})$, so Theorem 3.1 generalizes Theorem 2.2.

Corollary 3.3. *Let $\lambda\in\mathbb{R}$, $m,n\in\mathbb{N}$, $m\leqslant n$, $n-1<\alpha\leqslant n$, $m-1<\beta\leqslant m$, $0<\beta<\alpha$, and let $g(x)$ be a given real function defined on $\mathbb{R}_+$. If a function $y\colon(0,\infty)\to\mathbb{R}$ satisfies the inequality*

$$(3.15)\qquad |({}^CD^{\alpha}_{0+}y)(x)-\lambda({}^CD^{\beta}_{0+}y)(x)-g(x)|\leqslant F(x)$$

for all $x>0$ and some function $F(x)>0$, then there exists a solution $y_a\colon(0,\infty)\to\mathbb{R}$ of the fractional differential equation (3.2) *such that*

$$(3.16)\qquad |y-y_a|\leqslant F(x)x^{\alpha}E_{\alpha-\beta,\ \alpha+1}(|\lambda|x^{\alpha-\beta}).$$

4. An example

Consider the fractional differential equation

$$({}^{C}D_{0+}^{2}y)(x)-\frac{1}{2}({}^{C}D_{0+}^{\frac{3}{2}}y)(x)=\frac{7}{3}-2\sqrt{\frac{x}{\pi}}, \tag{4.1}$$

where $\alpha=2$, $\beta=\frac{3}{2}$, $\lambda=\frac{1}{2}$, $g(x)=\frac{7}{3}-2\sqrt{x/\pi}$.

For $\varepsilon=\frac{1}{2}$, it is easy to check that the function $y_1(x)=x^2$ satisfies

$$\Big|({}^{C}D_{0+}^{2}y_1)(x)-\frac{1}{2}({}^{C}D_{0+}^{\frac{3}{2}}y_1)(x)-\frac{7}{3}+2\sqrt{\frac{x}{\pi}}\Big|<\frac{1}{2}, \tag{4.2}$$

and initial values of $y_1(x)$ are $y_1(0)=0$, $y_1'(0)=0$.

From (3.6) and the initial values of $y_1(x)$, we get an exact solution of equation (4.1)

$$y_a(x)=\int_0^x (x-t)E_{\frac{1}{2},2}\Big[\frac{1}{2}(x-t)^{\frac{1}{2}}\Big]\Big(\frac{7}{3}-2\sqrt{\frac{t}{\pi}}\Big)\,\mathrm{d}t. \tag{4.3}$$

By Theorem 3.1, the control function of $y_1(x)$ is $\frac{1}{2}x^2E_{1/2,3}(\frac{1}{2}x^{1/2})$, thus

$$|y_1(x)-y_a(x)|<\frac{1}{2}x^2E_{1/2,3}\Big(\frac{1}{2}x^{1/2}\Big), \tag{4.4}$$

and the error of the approximate solution $y_1(x)$ can be estimated.

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