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FIBER PRODUCT PRESERVING BUNDLE FUNCTORS AS MODIFIED VERTICAL WEIL FUNCTORS

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Dedicated my wife Ula Mikulska on her birthday

Abstract. We introduce the concept of modified vertical Weil functors on the category $\mathcal{FM}_m$ of fibred manifolds with m-dimensional bases and their fibred maps with embeddings as base maps. Then we describe all fiber product preserving bundle functors on $\mathcal{FM}_m$ in terms of modified vertical Weil functors. The construction of modified vertical Weil functors is an (almost direct) generalization of the usual vertical Weil functor. Namely, in the construction of the usual vertical Weil functors, we replace the usual Weil functors T^A corresponding to Weil algebras A by the so called modified Weil functors $T^{\mathcal{A}}$ corresponding to Weil algebra bundle functors $\mathcal{A}$ on the category $\mathcal{M}_m$ of m-dimensional manifolds and their embeddings.

Keywords: Weil algebra; Weil functor; vertical Weil functor; Weil algebra bundle functor; modified Weil functor; modified vertical Weil functor; bundle functor; fiber product preserving bundle functor; natural transformation

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1. Introduction

We assume that any manifold considered in the paper is Hausdorff, second countable, finite dimensional, without boundary and smooth (i.e. of class $\mathcal{C}^\infty$). All maps between manifolds are assumed to be smooth (of class $\mathcal{C}^\infty$).

Let $\mathcal{M}f$ be the category of all manifolds and their maps, $\mathcal{M}f_m$ the category of m-dimensional manifolds and their embeddings, $\mathcal{FM}$ the category of fibred manifolds (surjective submersions between manifolds) and their fibred maps and $\mathcal{FM}_m$ the category of fibred manifolds with m-dimensional bases and their fibred maps with embeddings as base maps.

A bundle functor F on $\mathcal{FM}_m$ in the sense of [5] is a functor $F\colon \mathcal{FM}_m \to \mathcal{FM}$ such that the value FY of Y is a fibred manifold $\pi_Y\colon FY \to Y$ for any $\mathcal{FM}_m$-object $p\colon Y \to M$, the value $Ff\colon FY \to FY^1$ of $f\colon Y \to Y^1$ is a fiber map covering f for any $\mathcal{FM}_m$-map $f\colon Y \to Y^1$, and $Fi_U\colon FU \to \pi_Y^{-1}U$ is a diffeomorphism for the inclusion map $i_U\colon U \to Y$ of an open subset U of Y. The definitions of bundle functors on $\mathcal{M}f$ or $\mathcal{M}f_m$ are quite similar (we replace $\mathcal{FM}_m$ by $\mathcal{M}f$ or by $\mathcal{M}f_m$). A bundle functor F on $\mathcal{FM}_m$ is fiber product preserving if $F(Y \times_M Y^1) \cong FY \times_M FY^1$ for any $\mathcal{FM}_m$-objects Y and Y^1 with the same base M.

The usual vertical functor $V\colon \mathcal{FM}_m \to \mathcal{FM}$ sends any $\mathcal{FM}_m$-object $p\colon Y \to M$ into

$$VY = \ker_{0_M} Tp = \bigcup_{x\in M} (Tp)^{-1}(0_M(x))$$

and any $\mathcal{FM}_m$-map $f\colon Y \to Y_1$ into $Vf := Tf_{|VY}\colon VY \to VY_1$. One can easily see that the vertical functor $V\colon \mathcal{FM}_m \to \mathcal{FM}$ is a fiber product preserving bundle functor.

A natural transformation $\eta\colon F \to F^1$ between bundle functors on $\mathcal{FM}_m$ is a family of maps $\eta_Y\colon FY \to F^1Y$ for any $\mathcal{FM}_m$-manifold Y such that $F^1f \circ \eta_Y = \eta_{Y^1} \circ Ff$ for any $\mathcal{FM}_m$-map $f\colon Y \to Y^1$. One can show that then $\eta_Y\colon FY \to F^1Y$ is a fibred map covering the identity map id_Y for any $\mathcal{FM}_m$-manifold Y ([5]).

A Weil algebra is a finite dimensional real local commutative algebra A with unity (i.e. $A = \mathbb{R}\cdot 1 \oplus N_A$, where N_A is the ideal of nilpotent elements).

In [9], A. Weil introduced the concept of near A-point on a manifold M as an algebra homomorphism of the algebra $C^\infty(M,\mathbb{R})$ of smooth functions on M into a Weil algebra A. Nowadays, the space T^AM of all near A-points on M is called a Weil bundle. About 1985, D. Eck (see [2]), O. O. Luciano (see [8]) and G. Kainz and P. W. Michor (see [3]) proved independently that the product preserving bundle functors $G\colon \mathcal{M}f \to \mathcal{FM}$ (i.e. satisfying $G(M \times M_1) = GM \times GM_1$ for any $\mathcal{M}f$-objects M and M_1) are the Weil functors $T^A\colon \mathcal{M}f \to \mathcal{FM}$ for Weil algebras $A = G\mathbb{R}$ and that the natural transformations $\eta\colon G \to G_1$ between product preserving bundle functors on $\mathcal{M}f$ are in bijection with the algebra homomorphisms $\eta_{\mathbb{R}}\colon G\mathbb{R} \to G_1\mathbb{R}$ between the corresponding Weil algebras.

Replacing (in the construction of V) the tangent functor T by the Weil functor T^A corresponding to a Weil algebra A and 0_M by the canonical section e_M of T^AM, one can define (in the same way) the vertical Weil functor $V^A\colon \mathcal{FM}_m \to \mathcal{FM}$. The functor $V^A\colon \mathcal{FM}_m \to \mathcal{FM}$ is a fiber product preserving bundle functor on $\mathcal{FM}_m$, too.

A Weil algebra bundle functor on $\mathcal{M}f_m$ is a bundle functor $A\colon \mathcal{M}f_m \to \mathcal{FM}$ such that A_xM is a Weil algebra and $A_xg\colon A_xM \to A_{g(x)}M_1$ is an algebra isomorphism for any $\mathcal{M}f_m$-map $f\colon M \to M_1$ and any point $x \in M$ (or shortly and more precisely

A is a bundle functor from $\mathcal{M}f_m$ into the category of all Weil algebra bundles and their algebra bundle maps).

We have the following Weil algebra bundle functors (see [7]).

- ▷ The trivial Weil algebra bundle functor A on $\mathcal{M}f_m$ given by $AM = M \times A$ and $Ag = g \times \mathrm{id}_A$, where A is a fixed Weil algebra.
- ▷ The Weil algebra bundle functor A on $\mathcal{M}f_m$ given by $AM = (\bigwedge TM)^0$ and $Ag = \bigwedge Tg_{|(\bigwedge TM)^0}$, where $\bigwedge TM = (\bigwedge TM)^0 \oplus (\bigwedge TM)^1 = \bigcup_{x\in M} \bigwedge T_xM$ is the Grassmann algebra bundle of the tangent bundle TM and $(\bigwedge TM)^0$ is the even degree subalgebra bundle.
- ▷ In the previous example we can replace the tangent functor T by an arbitrary vector bundle functor G on $\mathcal{M}f_m$.
- ▷ The Weil algebra bundle functor A on $\mathcal{M}f_m$ given by $AM = J^r(M, \mathbb{R})$ and $Ag = J^r(g, \mathrm{id}_{\mathbb{R}})$.
- ▷ In the previous example we can replace the holonomic r-jet functor J^r by the semi-holonomic r-jet functor $\overline{J}^r$ or by the non-holonomic r-jet functor $\widetilde{J}^r$.
- ▷ We can apply fibre-wise tensor product to the above examples of Weil algebra bundle functors on $\mathcal{M}f_m$.

A natural transformation (homomorphism) between Weil algebra bundle functors A and A^1 on $\mathcal{M}f_m$ is an $\mathcal{M}f_m$-natural transformation $\nu\colon A \to A^1$ of bundle functors such that $(\nu_M)_x\colon A_xM \to A^1_xM$ is an algebra homomorphism for any m-manifold M and any point $x \in M$.

In the present paper, we modify the above concept of vertical Weil functor $V^A\colon \mathcal{FM}_m \to \mathcal{FM}$ as follows.

First, given a Weil algebra bundle functor A on $\mathcal{M}f_m$, we define the so called modified Weil functor $T^A\colon \mathcal{FM}_m \to \mathcal{FM}$ such that

$$T^AY = \bigcup_{y\in Y} T_y^{A_{p(y)}M} Y$$

for any $\mathcal{FM}_m$-object $p\colon Y \to M$ (the details can be found in Section 2). By restriction, we define the modified Weil functor $T^A_{|\mathcal{M}f_m}$ on $\mathcal{M}f_m$. Next, given an $\mathcal{M}f_m$-canonical section σ of $T^A_{|\mathcal{M}f_m}\colon \mathcal{M}f_m \to \mathcal{FM}$ (i.e. a system of sections $\sigma_M\colon M \to T^AM$ for any m-manifold such that $T^Ag \circ \sigma_M = \sigma_{M_1} \circ g$ for any $\mathcal{M}f_m$-map $g\colon M \to M_1$) we define the so called modified vertical Weil functor $V^{A,\sigma}\colon \mathcal{FM}_m \to \mathcal{FM}$ by

$$V^{A,\sigma}Y = \ker_{\sigma_M} T^Ap = \bigcup_{x\in M} (T^Ap)^{-1}(\sigma_M(x))$$

for any $\mathcal{FM}_m$-object $p\colon Y \to M$ (the details can be found in Section 3). Then $V^{A,\sigma}\colon \mathcal{FM}_m \to \mathcal{FM}$ is a fiber product preserving bundle functor.

So, we have the category $\mathcal{MVWF}_m$ of modified vertical Weil functors on $\mathcal{FM}_m$ and their natural transformations, and the obvious (forgetting, inclusion) functor

$$\mathcal{I}\colon \mathcal{MVWF}_m \to \mathcal{FPPBF}_m,$$

where $\mathcal{FPPBF}_m$ denotes the category of fiber product preserving bundle functors on $\mathcal{FM}_m$ and their natural transformations.

In [7], given a fiber product preserving bundle functor F on $\mathcal{FM}_m$ we constructed canonically a Weil algebra bundle functor A^F on $\mathcal{M}f_m$ by

$$A^F M := F(M \times \mathbb{R}),$$

where $M \times \mathbb{R}$ is the trivial bundle with fiber $\mathbb{R}$ and base M. (For the reader's convenience we will review this construction in Section 4.) In Section 4, we also define a canonical section σ^F of $T^{A^F}_{|\mathcal{M}f_m}\colon \mathcal{M}f_m \to \mathcal{FM}$. So, we get the modified vertical Weil functor V^{A^F,σ^F} on $\mathcal{FM}_m$.

In this way, we obtain the functor

$$\mathcal{V}\colon \mathcal{FPPBF}_m \to \mathcal{MVWF}_m, \quad \mathcal{V}(F) := V^{A^F,\sigma^F}.$$

The main result of the present note is the following.

Theorem 1.1. *There is the equivalence*

$$\mathcal{MVWF}_m \cong \mathcal{FPPBF}_m$$

of categories. More precisely, $\mathcal{I} \circ \mathcal{V} \cong \mathrm{id}_{\mathcal{FPPBF}_m}$ *and* $\mathcal{V} \circ \mathcal{I} \cong \mathrm{id}_{\mathcal{MVWF}_m}$.

There is another purely theoretical description of all fiber product preserving bundle functors on $\mathcal{FM}_m$ by means of triples (A, H, t), see [4] or [6]. We will not use this description.

2. The modified Weil functors

We modify the concept of Weil functors as follows.

Example 2.1. Let $A\colon \mathcal{M}f_m \to \mathcal{FM}$ be a Weil algebra bundle functor. We have a functor $T^A\colon \mathcal{FM}_m \to \mathcal{FM}$ sending any $\mathcal{FM}_m$-object $p\colon Y \to M$ into fibred manifold

$$T^A Y = \bigcup_{y \in Y} T_y^{A_{p(y)}M} Y$$

(where $T_y^{A_{p(y)}M}Y$ is the fiber over $y \in Y$ of the usual Weil bundle $T^{A_{p(y)}M}Y$ of Y corresponding to the Weil algebra $A_{p(y)}M$) with the obvious projection onto Y, and any $\mathcal{FM}_m$-map $f\colon Y \to Y_1$ covering $\underline{f}\colon M \to M_1$ into fibred map $T^A f\colon T^A Y \to T^A Y_1$ covering f given by the composition

$$T_y^{A_{p(y)}M}Y \to T_{f(y)}^{A_{p(y)}M}Y_1 \to T_{f(y)}^{A_{p(f(y))}M_1}Y_1$$

for any $y \in Y$, where $T_y^{A_{p(y)}M}Y \to T_{f(y)}^{A_{p(y)}M}Y_1$ is the restriction of the prolongation $T^{A_{p(y)}M}f\colon T^{A_{p(y)}M}Y \to T^{A_{p(y)}M}Y_1$ of $f\colon Y \to Y_1$ with respect to the (usual) Weil functor corresponding to the Weil algebra $A_{p(y)}M$, and where $T_{f(y)}^{A_{p(y)}M}Y_1 \to T_{f(y)}^{A_{p(f(y))}M_1}Y_1$ is the restriction of the extension (natural transformation) $T^{A_{p(y)}M}Y_1 \to T^{A_{p(f(y))}M_1}Y_1$ of the Weil algebra homomorphism $A_{p(y)}\underline{f}\colon A_{p(y)}M \to A_{p(f(y))}M_1$. (Every fibred chart (U, φ) on $Y \to M$ induces a fibred chart $(T^A U, T^A \varphi)$ on $T^A Y \to M$ provided we use the obvious "translation" identification $T^A(\mathbb{R}^m \times \mathbb{R}^n) = (A_0\mathbb{R}^m)^{m+n}$.) Clearly, $T^A\colon \mathcal{FM}_m \to \mathcal{FM}$ is a bundle functor.

By restriction, we have the bundle functor $T^A\colon \mathcal{M}f_m \to \mathcal{FM}$ sending any m-manifold M into $T^A M$, where we treat M as the $\mathcal{FM}_m$-object $\mathrm{id}_M\colon M \to M$, and any $\mathcal{M}f_m$ map $g\colon M \to M_1$ into $T^A g\colon T^A M \to T^A M_1$, where we treat $g\colon M \to M_1$ as the $\mathcal{FM}_m$-map $g\colon M \to M_1$ covering g.

The functor $T^A\colon \mathcal{FM}_m \to \mathcal{FM}$ or $T^A\colon \mathcal{M}f_m \to \mathcal{FM}$ from Example 2.1 is called the modified Weil functor on $\mathcal{FM}_m$ or on $\mathcal{M}f_m$, respectively, corresponding to Weil algebra bundle functor A.

Clearly, if A is the trivial Weil algebra bundle functor (i.e. $AM = M \times A$ and $Ag = g \times \mathrm{id}_A$ for a Weil algebra A), then T^A is (equivalent to) the usual Weil functor T^A on $\mathcal{FM}_m$ or $\mathcal{M}f_m$.

If $\nu\colon A \to A^1$ is a natural homomorphism of Weil algebra bundle functors on $\mathcal{M}f_m$ we have the corresponding natural transformation $\nu\colon T^A \to T^{A^1}$ given by

$$\nu_Y = \bigcup_{y \in Y} (((\nu_M)_{p(y)})_Y)_y\colon T^A Y \to T^{A^1} Y$$

for any $\mathcal{FM}_m$-object $p\colon Y \to M$, where $(\nu_M)_{p(y)}\colon T^{A_{p(y)}M} \to T^{A^1_{p(y)}M}$ is the natural transformation (of usual Weil functors) corresponding to algebra homomorphism $(\nu_M)_{p(y)}\colon A_{p(y)}M \to A^1_{p(y)}M$.

3. The modified vertical Weil functors

We are in position to modify the concept of vertical Weil functors.

Example 3.1. Let $A\colon \mathcal{M}f_m \to \mathcal{FM}$ be a Weil algebra bundle functor. Suppose we have a canonical section σ of $T^A\colon \mathcal{M}f_m \to \mathcal{FM}$, i.e., an $\mathcal{M}f_m$-invariant family of sections $\sigma_M\colon M \to T^A M$ for any $\mathcal{M}f_m$-object M. (The invariance means that $T^A g \circ \sigma_M = \sigma_{M_1} \circ g$ for any $\mathcal{M}f_m$-map $g\colon M \to M_1$.) We define a functor $V^{A,\sigma}\colon \mathcal{FM}_m \to \mathcal{FM}$ as follows. For any $\mathcal{FM}_m$-object $p\colon Y \to M$ we put

$$V^{A,\sigma} Y := \ker_{\sigma_M} T^A p = \bigcup_{x\in M} (T^A p)^{-1}(\sigma_M(x))$$

(we treat $p\colon Y \to M$ as the $\mathcal{FM}_m$-map covering id_M). Then $V^{A,\sigma}Y$ is a fibred manifold with the obvious projection onto Y. Clearly, for any $\mathcal{FM}_m$-map $f\colon Y \to Y_1$ covering $\underline{f}\colon M \to M_1$, we have

$$T^A f(V^{A,\sigma} Y) \subset V^{A,\sigma} Y_1.$$

We put

$$V^{A,\sigma} f := T^A f_{|V^{A,\sigma}Y}\colon V^{A,\sigma} Y \to V^{A,\sigma} Y_1.$$

Then $V^{A,\sigma} f$ is a fibred map covering f. Clearly, $V^{A,\sigma}\colon \mathcal{FM}_m \to \mathcal{FM}$ is a fiber product preserving bundle functor.

In the case of the trivial Weil algebra bundle functor $A\colon \mathcal{M}f_m \to \mathcal{FM}$ (i.e., $AM = M \times A$, $Ag = g \times \mathrm{id}_A$ for a Weil algebra A), T^A is the usual Weil functor, and for $\sigma_M = e_M$ we reobtain the usual vertical Weil functor V^A corresponding to the Weil algebra A.

The fiber product preserving bundle functor $V^{A,\sigma}\colon \mathcal{FM}_m \to \mathcal{FM}$ from Example 3.1 is called the modified vertical Weil functor on $\mathcal{FM}_m$ corresponding to the pair (A, σ) consisting of a Weil algebra bundle functor A on $\mathcal{M}f_m$ and a canonical section σ of $T^A\colon \mathcal{M}f_m \to \mathcal{FM}$.

Let A^1 be an another Weil algebra bundle functor on $\mathcal{M}f_m$ and σ^1 an $\mathcal{M}f_m$-canonical section of T^{A^1}. Thus we have $V^{A^1,\sigma^1}\colon \mathcal{FM}_m \to \mathcal{FM}$. Suppose we have a natural homomorphism $\nu\colon A \to A^1$ of Weil algebra bundle functors on $\mathcal{M}f_m$ such that the corresponding $\mathcal{M}f_m$-natural transformation $\nu\colon T^A \to T^{A^1}$ sends σ into σ^1, i.e.,

$$\nu_M \circ \sigma_M = \sigma^1_M$$

for any m-manifold M. The natural transformation $\nu_Y\colon T^A Y \to T^{A^1} Y$ corresponding to ν sends $V^{A,\sigma}Y$ into $V^{A^1,\sigma^1}Y$, i.e.,

$$\nu_Y(V^{A,\sigma} Y) \subset V^{A^1,\sigma^1} Y$$

for any $\mathcal{FM}_m$-object $p\colon Y \to M$. So, we have $\mathcal{FM}_m$-natural transformation $\eta_\nu\colon V^{A,\sigma} \to V^{A^1,\sigma^1}$ given by

$$(\eta_\nu)_Y = (\nu_Y)_{|V^{A,\sigma}Y}\colon V^{A,\sigma}Y \to V^{A^1,\sigma^1}Y.$$

Let $\mathcal{FPPBF}_m$ denote the category of all fiber product preserving bundle functors on $\mathcal{FM}_m$ and their natural transformations. By Example 3.1, we have the category $\mathcal{MVWF}_m$ of all modified vertical Weil functors on $\mathcal{FM}_m$ and their natural transformations. The obvious (forgetting, inclusion) functor will be denoted by

$$\mathcal{I}\colon \mathcal{MVWF}_m \to \mathcal{FPPBF}_m.$$

4. Modified vertical Weil functors from fiber product preserving bundle functors

We are going to describe all fiber product preserving bundle functors on $\mathcal{FM}_m$ in terms of modified vertical Weil functors. We start from the following example.

Example 4.1. Let $F\colon \mathcal{FM}_m \to \mathcal{FM}$ be a fiber product preserving bundle functor. In [7], we defined bundle functor $A^F\colon \mathcal{M}f_m \to \mathcal{FM}$ by

$$A^F M := F(M \times \mathbb{R})$$

(the projection $A^F M \to M$ is the composition of the bundle functor projection $F(M \times \mathbb{R}) \to M \times \mathbb{R}$ with the canonical projection $M \times \mathbb{R} \to M$ onto the first factor) for any m-manifold M, where $M \times \mathbb{R}$ is the trivial bundle over M with fiber $\mathbb{R}$ (it is an $\mathcal{FM}_m$-object), and

$$A^F g := F(g \times \mathrm{id}_{\mathbb{R}})\colon A^F M \to A^F M_1$$

for any $\mathcal{M}f_m$-map $g\colon M \to M_1$, where $g \times \mathrm{id}_{\mathbb{R}}\colon M \times \mathbb{R} \to M_1 \times \mathbb{R}$ is treated as an $\mathcal{FM}_m$-map. Then using the standard fibre-wise product preserving argument (i.e. using the mentioned result on description of product preserving bundle functors in terms of Weil algebras) we observed that $A^F\colon \mathcal{M}f_m \to \mathcal{FM}$ is a Weil algebra bundle functor.

More precisely, given a point $x \in M$, we have the bundle functor $A^{F,M,x}\colon \mathcal{M}f \to \mathcal{FM}$ by $A^{F,M,x}N := F_x(M \times N)$ for any manifold N and $A^{F,M,x}h := F_x(\mathrm{id}_M \times h)\colon A^{F,M,x}N \to A^{F,M,x}N^1$ for any map $h\colon N \to N^1$. Since $F\colon \mathcal{FM}_m \to \mathcal{FM}$ is fiber product preserving, $A^{F,M,x}\colon \mathcal{M}f \to \mathcal{FM}$ is product preserving. So, $A^F_x M = A^{F,M,x}\mathbb{R}$ is a Weil algebra, the Weil algebra of $A^{F,M,x}$.

We see that there is a section $\sigma^F_M\colon M \to T^{A^F}M$ such that

$$\sigma_M(x)(h) := F_x(\mathrm{id}_M, h)(\theta^F_x) \in A^F_x M,$$

$h \in C^\infty_x(M)$= the algebra of germs at $x \in M$ of smooth maps $h\colon M \to \mathbb{R}$, where θ^F_x is the unique point of F_xM (we treated the system of maps $(\mathrm{id}_M, h)\colon M \to M\times\mathbb{R}$ as an $\mathcal{FM}_m$-map covering id_M (defined near x)). Here and later for the simplicity of notations we denote by the same symbol a map and the germ of a map if the source of the germ is clear.

More precisely, recalling the definitions of the algebra operations of $A^F_x M$ we see that $\sigma_M(x) \in \mathrm{Hom}(C^\infty_x(M), A^F_x M)$, i.e. $\sigma_M(x) \in T^{A^F_x M}_x M = T^{A^F}_x M$. Because of the canonical character of the construction of σ^F_M, $\sigma^F = \{\sigma^F_M\}$ is a canonical section of T^{A^F}.

The modified vertical Weil functor $V^{A^F,\sigma^F}\colon \mathcal{FM}_m \to \mathcal{FM}$ is called the modified vertical Weil functor corresponding to the fiber product preserving bundle functor $F\colon \mathcal{FM}_m \to \mathcal{FM}$.

Suppose $\eta\colon F \to F^1$ is an $\mathcal{FM}_m$-natural transformation of fiber product preserving bundle functors on $\mathcal{FM}_m$. We define an $\mathcal{M}f_m$-natural transformation $\nu^{\langle\eta\rangle}\colon A^F \to A^{F^1}$ of bundle functors by

$$(\nu^{\langle\eta\rangle})_M := \eta_{M\times\mathbb{R}|A^F M}\colon A^F M \to A^{F^1}M.$$

By the standard fiber-wise product preserving argument, $\nu^{\langle\eta\rangle}$ is a natural transformation of Weil algebra bundle functors. Let $\nu^{\langle\eta\rangle}\colon T^{A^F} \to T^{A^{F^1}}$ be the (induced) $\mathcal{M}f_m$-natural transformation corresponding to $\nu^{\langle\eta\rangle}\colon A^F \to A^{F^1}$. We have

$$\nu^{\langle\eta\rangle} \circ \sigma^F = \sigma^{F^1}$$

as $\nu^{\langle\eta\rangle}_M(\sigma^F_M(x))(h) = \eta_{M\times\mathbb{R}}(F_x(\mathrm{id}_M, h)(\theta^F_x)) = F^1_x((\mathrm{id}_M, h)(\theta^{F^1}_x)) = \sigma^{F^1}(x)(h)$ for any m-manifold M, any point $x \in M$ and any $h \in C^\infty_x(M)$. Thus we have the $\mathcal{FM}_m$-natural transformation $\widetilde{\eta}\colon V^{A^F,\sigma^F} \to V^{A^{F^1},\sigma^{F^1}}$ defined by

$$\widetilde{\eta} := \eta_{\nu^{\langle\eta\rangle}}\colon V^{A^F,\sigma^F} \to V^{A^{F^1},\sigma^{F^1}},$$

where $\eta_{\nu^{\langle\eta\rangle}}$ is the natural transformation corresponding to natural homomorphism $\nu^{\langle\eta\rangle}\colon A^F \to A^{F^1}$ of Weil algebra bundle functors with $\nu^{\langle\eta\rangle} \circ \sigma^F = \sigma^{F^1}$.

Consequently, we have defined the functor

$$\mathcal{V}\colon \mathcal{FPPBF}_m \to \mathcal{MVWF}_m.$$

More precisely, we put $\mathcal{V}(F) = V^{A^F,\sigma^F}$ and $\mathcal{V}(\eta) := \widetilde{\eta}$.

5. The main result

The main result is the following theorem (corresponding to Theorem 1.1).

Theorem 5.1. *There is the equivalence*

$$\mathcal{MVWF}_m \cong \mathcal{FPPBF}_m$$

of categories. More precisely, we have isomorphisms

$$\Theta\colon \mathcal{I} \circ \mathcal{V} \to \mathrm{id}_{\mathcal{FPPBF}_m} \quad \text{and} \quad \mathcal{T}\colon \mathcal{V} \circ \mathcal{I} \to \mathrm{id}_{\mathcal{MVWF}_m}.$$

We start with the following proposition.

Proposition 5.2.

(i) *Given a fiber product preserving bundle functor F on $\mathcal{FM}_m$ we have $F = V^{A^F,\sigma^F}$ modulo a (canonical in F) $\mathcal{FM}_m$-natural isomorphism $\Theta^F\colon V^{A^F,\sigma^F} \to F$.*

(ii) *Given a modified vertical Weil functor $V^{A,\sigma}$ on $\mathcal{FM}_m$, we have $V^{A,\sigma} = V^{A^{V^{A,\sigma}},\sigma^{V^{A,\sigma}}}$ modulo a canonical (in $V^{A,\sigma}$) $\mathcal{FM}_m$-natural isomorphism $\mathcal{T}^{V^{A,\sigma}}$.*

P r o o f. (i) Let $p\colon Y \to M$ be an $\mathcal{FM}_m$-object. Let $v \in F_yY$, $p(y) = x$. Define $\widetilde{v}\colon C^\infty_y(Y) \to A^F_x M = F_x(M,\mathbb{R})$ by

$$\widetilde{v}(h) := F(p,h)(v),$$

where the system $(p,h)\colon Y \to M \times \mathbb{R}$ is treated as an $\mathcal{FM}_m$-map covering id_M. Recalling the definitions of the operations in the Weil algebra $A^F_x M$ we see that $\widetilde{v} \in \mathrm{Hom}(C^\infty_y(M), A^F_x M) = T^{A^F_x M}_y Y = T^{A^F}_y Y$. Moreover, $T^{A^F} p(\widetilde{v}) = \sigma^F(x)$ as

$$\begin{aligned} T^{A^F} p(\widetilde{v})(h_1) &= T^{A^F_x M} p(\widetilde{v})(h_1) = \widetilde{v}(h_1 \circ p) = F(p, h_1 \circ p)(v) \\ &= F(\mathrm{id}_M, h_1)(Fp(v)) = F(\mathrm{id}_M, h_1)(\theta^F_x) = \sigma^F(x)(h_1) \end{aligned}$$

for any $h_1 \in C^\infty_x(M)$. So, we have defined an $\mathcal{FM}_m$-natural transformation $\Theta^F\colon F \to V^{A^F,\sigma^F}$ by

$$\Theta^F_Y(v) := \widetilde{v}$$

for any $\mathcal{FM}_m$-object $p\colon Y \to M$ and any $v \in F_yY$, $p(y) = x$.

Since F and V^{A^F,σ^F} are fiber product preserving, to prove that Θ^F is a natural equivalence it suffices to show that $(\Theta^F_{\mathbb{R}^m\times\mathbb{R}})_{(0,0)}\colon F_{(0,0)}(\mathbb{R}^m \times \mathbb{R}) \to V^{A^F,\sigma^F}_{(0,0)}(\mathbb{R}^m \times \mathbb{R})$ is a diffeomorphism. But the inverse diffeomorphism is the restriction of

$$\mathrm{Hom}(C^\infty_{(0,0)}(\mathbb{R}^m \times \mathbb{R}), A^F_0 \mathbb{R}^m) \to A^F_0 \mathbb{R}^m$$

given by $w \mapsto w(p_2)$, where $p_2\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is the projection onto the second factor.

(ii) We define an $\mathcal{M}f_m$-natural isomorphism $\tau\colon A \to A^{V^{A,\sigma}}$ as follows.

Let M be an m-manifold and $p\colon M \times \mathbb{R} \to M$ be the usual projection. For any $x \in M$ and $t \in \mathbb{R}$, the map

$$T^{A_xM}_{(x,t)}p\colon T^{A_xM}_{(x,t)}(M \times \mathbb{R}) = T^{A_xM}_x M \times (A_xM)_t \to T^{A_xM}_x M$$

is the usual projection. Then we have the obvious identification

$$(A_xM)_t \cong (T^{A_xM}_{(x,t)}p)^{-1}(\sigma_M(x)) = \{\sigma_M(x)\} \times (A_xM)_t.$$

We put $(\tau_M)_x\colon A_xM \to A^{V^{A,\sigma}}_x M$ to be the composition

$$A_xM \cong \bigcup_{y \in \{x\} \times \mathbb{R}} (T^{A_xM}_y p)^{-1}(\sigma_M(x)) = V^{A,\sigma}_x(M \times \mathbb{R}) = A^{V^{A,\sigma}}_x M,$$

where the equalities are by Examples 3.1 and 4.1 (respectively), and the identification $\cong$ is described above.

One can easily see that $\tau\colon A \to A^{V^{A,\sigma}}$ is a natural isomorphism of Weil algebra bundle functors and that

$$\tau_M \circ \sigma_M = \sigma^{V^{A,\sigma}}_M$$

for any $\mathcal{M}f_m$-manifold M. So, we have the corresponding $\mathcal{FM}_m$-natural isomorphism

$$\mathcal{T}^{V^{A,\sigma}} := \eta_\tau\colon V^{A,\sigma} \to V^{A^{V^{A,\sigma}}, \sigma^{V^{A,\sigma}}}$$

(see the last part of Section 3 for τ playing the role of ν).

The proof of the proposition is complete. □

P r o o f of Theorem 5.1. Theorem 5.1 is a simple consequence of Proposition 5.2. □

6. A final remark

The vertical functor V on $\mathcal{FM}_m$ can be also defined (in an another way) by

$$VY = \bigcup_{x \in M} TY_x, \quad Vf = \bigcup_{x \in M} Tf_x\colon VY \to VY_1$$

for any $\mathcal{FM}_m$-object $p\colon Y \to M$ and any $\mathcal{FM}_m$-map $f\colon Y \to Y_1$ covering $\underline{f}\colon M \to M_1$. If we replace the tangent functor T by the Weil functor T^A corresponding to a Weil algebra A, we construct the vertical Weil functor V^A on $\mathcal{FM}_m$ corresponding to a Weil algebra A.

In [7], we generalized the last construction as follows. Let $A\colon \mathcal{M}f_m \to \mathcal{FM}$ be a Weil algebra bundle functor. We put

$$V^A Y = \bigcup_{x\in M} T^{A_x M} Y_x$$

for any $\mathcal{FM}_m$-object $p\colon Y \to M$. If $f\colon Y \to Y^1$ is an $\mathcal{FM}_m$-map covering $\underline{f}\colon M \to M^1$ we define $V^A f\colon V^A Y \to V^A Y_1$ such that for any $x \in M$ the restriction $V^A_x f\colon V^A_x Y = T^{A_x M} Y_x \to V^A_{\underline{f}(x)} Y^1 = T^{A_{\underline{f}(x)} M^1} Y^1_{\underline{f}(x)}$ of $V^A f$ to respective fibres is the composition

$$T^{A_x M} Y_x \to T^{A_x M} Y^1_{\underline{f}(x)} \to T^{A_{\underline{f}(x)} M^1} Y^1_{\underline{f}(x)}$$

of the map $T^{A_x M} f_x\colon T^{A_x M} Y_x \to T^{A_x M} Y^1_{\underline{f}(x)}$ (the extension of $f_x\colon Y_x \to Y^1_{\underline{f}(x)}$ by means of $T^{A_x M}$) with $(A_x \underline{f})_{Y^1_{\underline{f}(x)}}\colon T^{A_x M} Y^1_{\underline{f}(x)} \to T^{A_{\underline{f}(x)} M^1} Y^1_{\underline{f}(x)}$ (the extension of the homomorphism $A_x \underline{f}\colon A_x M \to A_{\underline{f}(x)} M^1$ of Weil algebras). The functor V^A as above is called the generalized vertical Weil functor corresponding to a Weil algebra bundle functor A.

In [7], we described all fiber product preserving bundle functors F on $\mathcal{FM}_m$ of vertical type in terms of the generalized vertical Weil functors V^A corresponding to Weil algebra bundle functors A.

We inform that a bundle functor F on $\mathcal{FM}_m$ is of vertical type if for any $\mathcal{FM}_m$-map $f\colon Y \to Y^1$ covering $\underline{f}\colon M \to M^1$ and any $x \in M$, the restriction $F_x f\colon F_x Y \to F_{\underline{f}(x)} Y^1$ of $Ff\colon FY \to FY^1$ to respective fibres depends on $f_x\colon Y_x \to Y^1_{\underline{f}(x)}$ only.

In the present paper we described all (not necessarily of vertical type) fiber product preserving bundle functors on $\mathcal{FM}_m$.

One can show that fiber product preserving bundle functors on $\mathcal{FM}_m$ of vertical type commute. Indeed, for any Weil algebra bundle functors A and B on $\mathcal{M}f_m$, we have

$$V^A \circ V^B = V^{B\otimes A}$$

modulo the obvious and standard identifications

$$\begin{aligned}(V^A \circ V^B)_x Y &= V^A_x (V^B Y) = T^{A_x M}(V^B_x Y) = T^{A_x M}(T^{B_x M} Y_x)\\ &= T^{B_x M \otimes A_x M} Y_x = T^{(B\otimes A)_x M} Y_x = V^{B\otimes A}_x Y.\end{aligned}$$

Then the exchange isomorphism $A\otimes B \cong B\otimes A$ induces the identification $V^A \circ V^B = V^B \circ V^A$. In this way we have solved the problem from Remark 2 in [7].

In [1], the authors showed that fiber product preserving bundle functors on $\mathcal{FM}_m$ cannot commute.

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