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THE BASIC CONSTRUCTION FROM THE CONDITIONAL EXPECTATION ON THE QUANTUM DOUBLE OF A FINITE GROUP

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Abstract. Let G be a finite group and H a subgroup. Denote by $D(G;H)$ (or $D(G)$) the crossed product of $C(G)$ and $\mathbb{C}H$ (or $\mathbb{C}G$) with respect to the adjoint action of the latter on the former. Consider the algebra $\langle D(G), e\rangle$ generated by $D(G)$ and e, where we regard E as an idempotent operator e on $D(G)$ for a certain conditional expectation E of $D(G)$ onto $D(G;H)$. Let us call $\langle D(G), e\rangle$ the basic construction from the conditional expectation $E\colon D(G) \to D(G;H)$. The paper constructs a crossed product algebra $C(G/H \times G) \rtimes \mathbb{C}G$, and proves that there is an algebra isomorphism between $\langle D(G), e\rangle$ and $C(G/H \times G) \rtimes \mathbb{C}G$.

Keywords: conditional expectation; basic construction; quantum double; quasi-basis

MSC 2010: 16S99, 16W22

1. Introduction

Index theory for subfactors started with the breakthrough achieved by Jones ([8]). This theory was developed first for subfactors of type II_1, i.e., those possessing a faithful normal tracial state. Jones' index theory has found important applications in conformal field theory and quantum field theory ([10], [12]) and in the study of the tensor categories arising from compact groups and quantum groups. For a nontechnical but broad overview of the subject including a lot of important connections with other areas, the reader can refer to [7].

Let M be a factor of type II_1 on the Hilbert space $\mathcal{H}$, and let N (contained in M) be a subfactor. By $\dim_M(\mathcal{H})$ we denote the coupling constant of M on $\mathcal{H}$. The Jones

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index $[M:N]$, as defined by Jones, is

$$[M:N]=\frac{\dim_N(\mathcal{H})}{\dim_M(\mathcal{H})}$$

if N', the set of all bounded operators on $\mathcal{H}$ commuting with every operator in $\mathcal{H}$, is finite. In particular, if G is an infinite conjugacy class group, namely, a discrete group all of whose conjugacy classes are infinite except that of the identity, and $H \leqslant G$, then the Jones index is exactly $[G:H]$, the index of H in G. Surprisingly, Jones answers this question: what are the possible values of the index $[M:N]$ for subfactors? The answer is $\{4\cos^2 \pi/n\colon\ n \geqslant 3\} \cup [4,\infty]$. The values $[4,\infty]$ can be easily obtained. However, the situation between 1 and 4 is more difficult. In this case, what is called the basic construction plays an important role in constructing subfactors. More precisely, one can identify M with the algebra of left multiplication operators on $L^2(M,\mathrm{tr})$ and consider the extension e_N to $L^2(M,\mathrm{tr})$ of the conditional expectation onto N, where "tr" is the faithful normal normalized trace and $L^2(M,\mathrm{tr})$ denotes the Hilbert space completion of M with respect to the inner product $\langle a,b\rangle = \mathrm{tr}(b^*a)$ coming from the Gelfand-Naimark-Segal (G.N.S.) construction. One defines $M_1 \triangleq \langle M, e_N\rangle$ to be the von Neumann algebra generated by M and e_N, the Jones projection, on $L^2(M,\mathrm{tr})$ (this is called the basic construction). Subsequently, Jones used the basic construction to obtain an increasing sequence of type II_1 factors, $N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$, which is called the Jones tower, iteratively, by adding the Jones projections $\{e_n\colon\ n \geqslant 1, e_1 = e_N\}$, which satisfy the Temperley-Lieb relations. Eventually Jones used this structure to construct examples such that the values of the index can exhaust the set $\{4\cos^2 \pi/n\colon\ n \geqslant 3\}$. Let J denote the $*$-operation in $L^2(M,\mathrm{tr})$, regarded as an isometric conjugate linear operator, i.e. $Jx = x^*$ for $x \in L^2(M,\mathrm{tr})$; then $\langle M, e_N\rangle = JN'J$, which is a very elegant result.

Jones' index theory for subfactors of type II_1 has been generalized to general inclusions of factors by Kosaki ([11]) and Longo ([13]). Based on the work of Jones, Kosaki and Longo, Watatani ([16]) investigated Index Γ for a conditional expectation $\Gamma\colon\ B \to A$ of a C^*-algebra B onto a C^*-subalgebra A, and introduced the C^*-algebra basic construction $C^*\langle B, \gamma_A\rangle$ as follows: one can view B as a pre-Hilbert module over A by defining the A-valued inner product $\langle x,y\rangle = \Gamma(x^*y)$. Denote by $\overline{B}$ the completion of B with respect to $\|\cdot\|_B$, where $\|x\|_B = \langle x,x\rangle^{1/2}$ for $x \in B$. Then $\overline{B}$ is a Hilbert C^*-module over A. Denote by $\mathrm{End}^*_A(\overline{B})$ the set of all adjointable operators from $\overline{B}$ to $\overline{B}$. It is a C^*-algebra. In such a case, one can identify B with the algebra of left multiplication operators on $\overline{B}$. The conditional expectation Γ can be extended to a projection γ_{A} on $\overline{B}$ via $\gamma_{A}(x) = \Gamma(x)$ for $x \in B$. The C^*-algebra $C^*\langle B, \gamma_{A}\rangle$ is generated by $\{x\gamma_{A} y\colon\ x, y \in B\}$.

However, in contrast to the basic construction for type II_1 factors, the C^*-algebra $C^*\langle B, \gamma_{_A}\rangle$ does not have the concrete form, such as $\langle M, e_N\rangle = JN'J$, where M is a factor of type II_1 and $N \subseteq M$ is a subfactor. The reason for this phenomenon is that any factor of type II_1 admits a faithful trace which is a state, for which the G.N.S. construction may be performed, while for general algebras, the existence of such a functional is uncertain.

In this paper, we think about the basic construction for special algebras. Let G be a finite group and H a subgroup of G, denoted by $H \leqslant G$. Then $D(G;H)$ is defined as the crossed product of $C(G)$, the algebra of complex functions on G, and the group algebra $\mathbb{C}H$ with respect to the adjoint action of the latter on the former. In particular, if $H = G$, then $D(G;H) \triangleq D(G)$ is the quantum double of G. Then we consider the conditional expectation $E\colon D(G) \to D(G;H)$ defined in a natural way and give an appropriate form of the basic construction. In detail, considering the algebra $\langle D(G), e\rangle$ generated by $\{(g,h)\colon g,h \in G\}$ and e, where we regard E as an idempotent operator e on $D(G)$, one finds that the algebra $\langle D(G), e\rangle$ is isomorphic to the crossed product algebra $C(G/H \times G) \rtimes \mathbb{C}G$. Moreover, by constructing a conditional expectation $\widetilde{E}\colon \langle D(G), e\rangle \to D(G)$ of index-finite type, which is called the dual conditional expectation of $E\colon D(G) \to D(G;H)$, one finds that the index of $\widetilde{E}$ is exactly the index of E.

2. The basic construction for the quantum double of a finite group

2.1. Conditional expectations between algebras. At the beginning of this section, we first briefly review some known results we will need later. Suppose that B is an algebra over $\mathbb{C}$ and A a subalgebra with a common identity element 1. These definitions can be found in [16].

Definition 2.1. A linear map $\Gamma\colon B \to A$ is called a conditional expectation if it satisfies the following conditions: for all $a \in A$, $b \in B$,

(1) $\Gamma(1) = 1$;

(2) (bimodular property)

$$\Gamma(ab) = a\Gamma(b), \quad \Gamma(ba) = \Gamma(b)a.$$

Moreover, we say Γ is non-degenerate if $E(Bb) = 0$ or $E(bB) = 0$ implies $b = 0$ for $b \in B$.

Remark 2.2. If B is a C^*-algebra with a C^*-subalgebra A of B with a common identity element, we always assume that Γ is positive, i.e., $\Gamma(b^*b)$ is a positive element in A for any $b \in B$. Actually, Γ is a projection of norm one ([2]).

Definition 2.3. A finite family $\{(u_i, v_i)\colon\ i = 1, 2, \ldots, n\} \subseteq B \times B$ is called a quasi-basis for Γ if for all $x \in B$,

$$\sum_{i=1}^{n} u_i \Gamma(v_i x) = x = \sum_{i=1}^{n} \Gamma(x u_i) v_i.$$

Furthermore, if there exists a quasi-basis for Γ, we say Γ is of index-finite type. In this case we define the index of Γ by

$$\text{Index } \Gamma = \sum_{i=1}^{n} u_i v_i.$$

Remark 2.4. (1) If Γ is of index-finite type, then Index Γ is in the center of B and does not depend on the choice of the quasi-basis. In fact, for any $b \in B$,

$$\begin{aligned}(\text{Index } \Gamma) b &= \sum_{i=1}^{n} u_i v_i b = \sum_{i=1}^{n} u_i \left(\sum_{j=1}^{n} \Gamma(v_i b u_j) v_j \right) \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} u_i \Gamma(v_i b u_j) \right) v_j = \sum_{j=1}^{n} b u_j v_j = b(\text{Index } \Gamma).\end{aligned}$$

This means Index Γ is in the center of B. Let $\{(s_j, t_j)\colon\ j = 1, 2, \ldots, n\}$ be another quasi-basis for Γ, then

$$\sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} s_j \Gamma(t_j u_i) \right) v_i = \sum_{j=1}^{n} s_j \left(\sum_{i=1}^{n} \Gamma(t_j u_i) v_i \right) = \sum_{j=1}^{n} s_j t_j,$$

which shows that the value Index Γ does not depend on the choice of the quasi-basis.

(2) The existence of a quasi-basis guarantees that Γ is non-degenerate ([16]).

2.2. The quantum double and the basic construction. Suppose that G is a finite group with a unit u, and that H is a subgroup of G. Let $t_1 = u, t_2, \ldots, t_k$ be a left coset representation of H in G, namely $G = \bigcup\limits_{i=1}^{k} t_i H$ and $i \neq j$ induces that $t_i H \cap t_j H = \emptyset$. Let us begin with the following definitions.

Definition 2.5 ([5], [4]). $D(G; H)$ is the crossed product of $C(G)$ and the group algebra $\mathbb{C}H$, where $C(G)$ denotes the set of complex functions on G, with respect to the adjoint action of the latter on the former.

In particular, if $H = G$, then $D(G;H) \triangleq D(G)$ is the quantum double of G. For more information about $D(G)$, one can refer to [1], [3], [9]. The main difference between $D(G)$ and $D(G;H)$ is that the former is a quasi-triangular Hopf algebra while the latter is not ([14]).

It is easy to see that $\sum\limits_{g\in G}(g,u) \triangleq I$ is the unit of $D(G;H)$, where u is the unit of G. There is a unique element $z_H = |H|^{-1}\sum\limits_{h\in H}(e,h)$, called a cointegral, satisfying

$$az_H = z_H a = \varepsilon(a)z_H, \quad a \in D(G;H), \quad \text{and} \quad \varepsilon(z_H) = 1.$$

As a result, $D(G;H)$ is a semisimple finite dimensional algebra with a natural $*$-structure ([15]). Consequently, it can be a C^*-algebra.

In [6], the authors investigate the index of conditional expectation of index-finite type from $D(G)$ onto the subalgebra $D(G;H)$. Consider

$$\begin{aligned} E\colon\ D(G) &\to D(G;H)\\ \sum_{g,h\in G}\lambda_{g,h}(g,h) &\mapsto \sum_{g\in G,h\in H}\lambda_{g,h}(g,h)\end{aligned}$$

where $\lambda_{g,h} \in \mathbb{C}$, and set $u_i = \sum\limits_{\alpha\in G}(\alpha,t_i)$ and $v_i = \sum\limits_{\beta\in G}(\beta,t_i^{-1})$. Then $\{(u_i,v_i)\colon\ i = 1,2,\ldots,k\}$ is a quasi-basis for E, and therefore Index $E = kI$.

In the sequel we will consider the basic construction from the conditional expectation $E\colon\ D(G) \to D(G;H)$.

Recall that a homomorphic map $T\colon\ D(G) \to D(G)$ is said to be a right $D(G;H)$-module homomorphism if $T(xy) = T(x)y$ for $x \in D(G), y \in D(G;H)$. Denote by $\mathrm{End}_{D(G;H)}D(G)$ the set of all right $D(G;H)$-module homomorphisms from $D(G)$ to $D(G)$. Then it is a nonzero algebra. In such a case, one can regard $D(G)$ as the algebra of left multiplication operators on $D(G)$, while the conditional expectation E as an idempotent operator on $D(G)$, denoted by e.

Lemma 2.6. *As operators on $D(G)$, e and $(g,h) \in D(G)$ satisfy the following covariant relations:*
(1) $e(g,h)e = E(g,h)e$;
(2) *$h \in H$ if and only if $e(g,h) = (g,h)e$.*

P r o o f. To prove the first part, just observe that for $(m,n) \in D(G)$,

$$(e(g,h)e)(m,n) = e((g,h)E(m,n)) = E(g,h)E(m,n) = (E(g,h)e)(m,n).$$

For the second part, assume that $h \in H$, then

$$(e(g,h))(m,n) = E((g,h)(m,n)) = (g,h)E(m,n) = ((g,h)e)(m,n)$$

for $(m,n)\in D(G)$. Thus (g,h) commutes with e. Conversely, if $(g,h)\in D(G)$ and $e(g,h)=(g,h)e$, then

$$(g,h)=(g,h)e\sum_{\alpha\in G}(\alpha,u)=e(g,h)\sum_{\alpha\in G}(\alpha,u)=E(g,h),$$

which shows that $h\in H$. □

Definition 2.7. Let $\langle D(G),e\rangle$ be the subalgebra of $\mathrm{End}_{D(G;H)}D(G)$ generated by $\{(g,h)\colon\ g,h\in G\}$ and e. We call it the basic construction from the conditional expectation $E\colon\ D(G)\to D(G;H)$.

Remark 2.8. The set $\{(g_1,h_1)e(g_2,h_2)\colon\ g_i,h_i\in G,\ i=1,2\}$ can generate the algebra $\langle D(G),e\rangle$. Clearly, $e=\sum\limits_{\alpha,\beta\in G}(\alpha,u)e(\beta,u)$. And for $(g,h)\in D(G)$, we have that

$$\begin{aligned}\sum_{i=1}^{k}(g,t_i)e(t_i^{-1}gt_i,t_i^{-1}h)(m,n)&=\delta_{gh,hm}\sum_{i=1}^{k}(g,t_i)E(t_i^{-1}gt_i,t_i^{-1}hn)\\&=\delta_{gh,hm}(g,t_{i_0})(t_{i_0}^{-1}gt_{i_0},t_{i_0}^{-1}hn)\\&=\delta_{gh,hm}(g,hn)\\&=(g,h)(m,n),\end{aligned}$$

where in the second equation we use the fact that for $hn\in G$, there exists t_{i_0} such that $hn\in t_{i_0}H$. This implies that $(g,h)=\sum\limits_{i=1}^{k}(g,t_i)e(t_i^{-1}gt_i,t_i^{-1}h)$, and that the element of $D(G)$ can be linearly expressed by $\{(g_1,h_1)e(g_2,h_2)\colon\ g_i,h_i\in G,\ i=1,2\}$.

3. The construction of the crossed product algebra and the isomorphism theorem

Let us continue to assume that G is a finite group with a subgroup H. Denote by G/H the set of all left cosets of H, i.e. $G/H=\{[t_1],[t_2],\dots,[t_k]\}$. Let $C(G/H\times G)$ and $\mathbb{C}G$ be the algebra of complex functions on $G/H\times G$ and the group algebra over the $\mathbb{C}$, respectively.

The set $\{\chi_{[t_i]}\colon\ i=1,2,\dots,k\}$ is a linear basis of $C(G/H)$ where

$$\chi_{[t_i]}[t_j]=\begin{cases}1 & \text{if } j=i,\\ 0 & \text{if } j\neq i,\end{cases}$$

while the set $\{F_g\colon\ g \in G\}$ is a linear basis of $C(G)$ where

$$F_g(h) = \begin{cases} 1 & \text{if } h = g, \\ 0 & \text{if } h \neq g. \end{cases}$$

Since G is a finite group and $C(G/H \times G) \cong C(G/H) \otimes C(G)$, hence $\{(t_i, g)\colon\ i = 1, 2, \ldots, k;\ g \in G\}$ can be viewed as a linear basis of $C(G/H \times G)$, where we use (t_i, g) in place of $(\chi_{[t_i]}, F_g)$ for notational convenience. The map $\sigma\colon\ \mathbb{C}G \times C(G/H \times G) \to C(G/H \times G)$ given on the basis elements of $C(G/H \times G)$ as

$$\sigma(h \times (t_i, g)) = (ht_i, hgh^{-1})$$

for $h, g \in G$, can be linearly extended both to $\mathbb{C}G$ and $C(G/H \times G)$. One can show that σ defines an automorphic action of $\mathbb{C}G$ on $C(G/H \times G)$. Here and from now on, by $h(t_i, g)$ we always denote $\sigma(h \times (t_i, g))$.

Definition 3.1. The associative algebra generated by $\{(t_i, g), \alpha\colon\ g, \alpha \in G;\ i = 1, 2, \ldots, k\}$ is called the crossed product of $C(G/H \times G)$ by $\mathbb{C}G$ with respect to σ, and we denote it by $C(G/H \times G) \rtimes_\sigma \mathbb{C}G$ or $C(G/H \times G) \rtimes \mathbb{C}G$ for convenience.

Using the linear basis elements $(t_i, g; \alpha)$ of $C(G/H \times G) \rtimes \mathbb{C}G$, the product is defined as

$$(t_i, g; \alpha)(t_j, h; \beta) = \delta_{[t_i],[\alpha t_j]}\delta_{g\alpha,\alpha h}(t_i, g; \alpha\beta),$$

where $\delta_{\zeta,\eta} = \begin{cases} 1 & \text{if } \zeta = \eta, \\ 0 & \text{if } \zeta \neq \eta. \end{cases}$

It is clear that the element $\sum\limits_{g\in G}\sum\limits_{i=1}^{k}(t_i, g; u)$ is the unit of $C(G/H \times G) \rtimes \mathbb{C}G$, where u is a unit of G. Moreover, if H is a normal subgroup of G, then $C(G/H \times G) \rtimes \mathbb{C}G$ is a coalgebra. In fact, the coproduct $\triangle$ and counit ε are defined on the basis elements as

$$\triangle(t_i, g; \alpha) = \sum_{j=1}^{k}\sum_{h\in G}(t_j, h; \alpha) \otimes (t_j^{-1}t_i, h^{-1}g; \alpha),$$
$$\varepsilon(t_i, g; \alpha) = \delta_{[t_i],[t_1]}\delta_{g,u}$$

and are linearly extended to $C(G/H \times G) \rtimes \mathbb{C}G$. Since ε is not an algebra homomorphism, $C(G/H \times G) \rtimes \mathbb{C}G$ is not a bialgebra. Naturally, a question may emerge: in what case $C(G/H \times G) \rtimes \mathbb{C}G$ can be a bialgebra. However, this is unknown.

Lemma 3.2. *The elements* $\sum\limits_{i=1}^{k}(t_i,g;\alpha)$ *and* $\sum\limits_{h\in G}(t_1,h;u)$ *satisfy the following covariant relations:*

(1) $\sum\limits_{h\in G}(t_1,h;u)\sum\limits_{i=1}^{k}(t_i,g;\alpha)\sum\limits_{h\in G}(t_1,h;u)=\delta_{[u],[\alpha]}\sum\limits_{i=1}^{k}(t_i,g;\alpha)\sum\limits_{h\in G}(t_1,h;u)$;

(2) $\sum\limits_{h\in G}(t_1,h;u)$ *is an idempotent element in* $C(G/H\times G)\rtimes\mathbb{C}G$, *that is*

$$\Big(\sum_{h\in G}(t_1,h;u)\Big)^2=\sum_{h\in G}(t_1,h;u).$$

Proof. (1) Notice that

$$\begin{aligned}
&\sum_{h\in G}(t_1,h;u)\sum_{i=1}^{k}(t_i,g;\alpha)\sum_{h\in G}(t_1,h;u)\\
&\qquad=\sum_{h\in G}(t_1,h;u)\sum_{i=1}^{k}\sum_{h\in G}\delta_{[t_i],[\alpha]}\delta_{g\alpha,\alpha h}(t_i,g;\alpha)\\
&\qquad=\sum_{h\in G}(t_1,h;u)(\alpha,g;\alpha)=\sum_{h\in G}\delta_{[u],[\alpha]}\delta_{h,g}(t_1,g;\alpha)\\
&\qquad=\delta_{[u],[\alpha]}(t_1,g;\alpha),
\end{aligned}$$

which together with $\sum\limits_{i=1}^{k}(t_i,g;\alpha)\sum\limits_{h\in G}(t_1,h;u)=(\alpha,g;\alpha)$ completes the proof.

(2) $\Big(\sum\limits_{h\in G}(t_1,h;u)\Big)^2=\sum\limits_{h\in G}\sum\limits_{g\in G}(t_1,h;u)(t_1,g;u)=\sum\limits_{h\in G}\sum\limits_{g\in G}\delta_{g,h}(t_1,h;u)=\sum\limits_{h\in G}(t_1,h;u)$. □

Now we give the main theorem of this paper.

Theorem 3.3. *There is an isomorphism of algebras*

$$\langle D(G),e\rangle\cong C(G/H\times G)\rtimes\mathbb{C}G.$$

Proof. Let $\pi\colon\ \langle D(G),e\rangle\to C(G/H\times G)\rtimes\mathbb{C}G$ be a map with

$$\begin{aligned}
(g,\alpha)&\mapsto\sum_{i=1}^{k}(t_i,g;\alpha)\\
e&\mapsto\sum_{g\in G}(t_1,g;u).
\end{aligned}$$

First, π is a well-defined map and can be linearly extended to $\langle D(G), e\rangle$ preserving algebraic structure. Indeed, by Lemma 2.6 and Lemma 3.2 we obtain that

$$\pi(e(g,\alpha)e) = \pi(e)\pi(g,\alpha)\pi(e) = \pi(E(g,\alpha)e).$$

And we have

$$\begin{aligned}\pi(g,\alpha)\pi(h,\beta) &= \sum_{i=1}^{k}(t_i, g; \alpha)\sum_{j=1}^{k}(t_j, h; \beta) = \sum_{i=1}^{k}\sum_{j=1}^{k}\delta_{[t_i],[\alpha t_j]}\delta_{g\alpha,\alpha h}(t_i, g; \alpha\beta)\\ &= \delta_{g\alpha,\alpha h}\sum_{i=1}^{k}(t_i, g, \alpha\beta) = \pi(\delta_{g\alpha,\alpha h}(g,\alpha\beta))\\ &= \pi((g,\alpha)(h,\beta)).\end{aligned}$$

Thus, π is an algebra homomorphism.

In order to complete the proof of the theorem we need to show that π is bijective. Indeed, for any $(t_i, g; \alpha) \in C(G/H \times G) \rtimes \mathbb{C}G$, choose $(g, t_i)e(t_i^{-1}gt_i, t_i^{-1}\alpha) \in \langle D(G), e\rangle$ such that

$$\pi((g,t_i)e(t_i^{-1}gt_i, t_i^{-1}\alpha)) = (t_i, g; \alpha),$$

which implies that π is surjective, and then $\dim\langle D(G), e\rangle \geqslant k|G|^2$. From the following relations, we know that every element $(g,h)e(\alpha,\beta)$ in $\langle D(G), e\rangle$ can be linearly expressed by $\{(g,t_i)e(t_i^{-1}gt_i, t_i^{-1}h)\colon\ g,h \in G;\ i = 1,2,\ldots,k\}$. For $(m,n) \in D(G)$,

$$\begin{aligned}(g,h)e(\alpha,\beta)(m,n) &= (g,h)e(\delta_{\alpha\beta,\beta m}(\alpha,\beta n))\\ &= \delta_{[\beta n],[u]}\delta_{\alpha\beta,\beta m}(g,h)(\alpha,\beta n)\\ &= \delta_{[\beta n],[u]}\delta_{gh,h\alpha}\delta_{\alpha\beta,\beta m}(g, h\beta n),\end{aligned}$$

which yields induce that

$$(g,h)e(h^{-1}gh,\beta)(m,n) = \delta_{[\beta n],[u]}\delta_{h^{-1}gh,\beta m\beta^{-1}}(g, h\beta n).$$

And for $h \in H$, there exists t_i such that $h \in t_iH$, that is $t_i^{-1}h \in H$. Then

$$\begin{aligned}(g,t_i)e(t_i^{-1}gt_i, t_i^{-1}h\beta)(m,n) &= \delta_{t_i^{-1}gh\beta, t_i^{-1}h\beta m}(g,t_i)e(t_i^{-1}gt_i, t_i^{-1}h\beta n)\\ &= \delta_{h^{-1}gh,\beta m\beta^{-1}}(g,t_i)E(t_i^{-1}gt_i, t_i^{-1}h\beta n)\\ &= \delta_{[\beta n],[u]}\delta_{h^{-1}gh,\beta m\beta^{-1}}(g,t_i)(t_i^{-1}gt_i, t_i^{-1}h\beta n)\\ &= \delta_{[\beta n],[u]}\delta_{h^{-1}gh,\beta m\beta^{-1}}(g, h\beta n).\end{aligned}$$

From the above, π is surjective and $\{(g,t_i)e(t_i^{-1}gt_i, t_i^{-1}h)\colon\ g,h \in G;\ i = 1, 2,\ldots,k\}$ is a linear basis of $\langle D(G), e\rangle$, which mean π is bijective. □

Remark 3.4. The basic constructions do not depend on the choice of conditional expectations (see Proposition 2.10.11 in [16]), which, together with Theorem 3.3, implies that the crossed product does not depend on the choice of a conditional expectation.

Example 3.5. (1) Consider the basic construction from the conditional expectation $\Phi\colon \mathbb{C}G \to \mathbb{C}H$ defined by

$$\Phi\biggl(\sum_{g\in G}\lambda_g g\biggr) = \sum_{h\in H}\lambda_h h$$

where $\lambda_g \in \mathbb{C}$ for $g \in G$. Let $\tau\colon \mathbb{C}G \to \operatorname{Aut} C(G/H)$ be the action induced by translation from the left, where $\operatorname{Aut} C(G/H)$ stands for the collection of all automorphisms on $C(G/H)$. There is an inclusion $i\colon \mathbb{C}G \to D(G)$ given by $i(g) = \sum\limits_{f\in G}(f,g)$, which together with Theorem 3.3 yields that $\langle \mathbb{C}G, \varphi\rangle$ is algebra isomorphic to $C(G/H)\rtimes_\tau \mathbb{C}G$, where we view Φ as an idempotent operator on $\mathbb{C}G$, denoted by φ.

(2) If G is an Abelian finite group, then $D(G)$ reduces to a symmetry group $\widehat{G}\times G$, where $\widehat{G}$ denotes the Pontryagin dual of G (the group of characters of G). Let $E\colon \widehat{G}\times G \to \widehat{G}\times H$ be a conditional expectation such that

$$E\biggl(\sum_{g,h\in G}\lambda_{g,h}(g,h)\biggr) = \sum_{g\in G,h\in H}\lambda_{g,h}(g,h).$$

Set $\varsigma\colon \mathbb{C}G\times C(G/H\times G) \to C(G/H\times G)$ given on the basis elements of $C(G/H\times G)$ as

$$\sigma(h\times(t_i,g)) = (ht_i,g) \quad \text{for } h,g\in G.$$

One can prove that ς defines an automorphic action of $\mathbb{C}G$ on $C(G/H\times G)$. Then, $\langle \widehat{G}\times G, e\rangle$ is algebra isomorphic to $C(G/H\times G)\rtimes_\varsigma \mathbb{C}G$. Moreover, there is an algebra isomorphism between $\langle \widehat{G}\times G, e\rangle$ and $\widehat{G}\times (C(G/H)\rtimes_\tau \mathbb{C}G)$, where τ is defined as above.

Finally, we consider the conditional expectation from $\langle D(G), e\rangle$ onto $D(G)$.

Theorem 3.6. *The map $\widetilde{E}\colon \langle D(G), e\rangle \to D(G)$ defined on the basis elements of $\langle D(G), e\rangle$ by*

$$\widetilde{E}((g,t_i)e(t_i^{-1}gt_i, t_i^{-1}h)) = \frac{1}{k}(g,h)$$

for $g,h\in G$ and $i = 1,2,\dots,k$, linearly extended to $\langle D(G), e\rangle$, is a conditional expectation of index-finite type, and we call it the dual conditional expectation of $E\colon D(G)\to D(G;H)$. More precisely, let $w_i = \sqrt{k}\sum\limits_{\alpha,\beta\in G}(\alpha,t_i)e(\beta,u)$ and $z_i =$

$\sqrt{k}\sum_{\eta,\xi\in G}(\eta,u)e(\xi,t_i^{-1})$, *then* $\{(w_i,z_i)\colon\ i=1,2,\ldots,k\}$ *is a quasi-basis for* $\widetilde{E}$ *and* Index $\widetilde{E}=\text{Index }E$.

Proof. Since $\sum_{g\in G}\sum_{i=1}^{k}(g,t_i)e(t_i^{-1}gt_i,t_i^{-1})$ and I are the units of $\langle D(G),e\rangle$ and $D(G)$, respectively, we have

$$\widetilde{E}\biggl(\sum_{g\in G}\sum_{i=1}^{k}(g,t_i)e(t_i^{-1}gt_i,t_i^{-1})\biggr)=\frac{1}{k}\sum_{g\in G}\sum_{i=1}^{k}(g,u)=\sum_{g\in G}(g,u)=I.$$

Since Index $E=kI$ is in the center of $D(G)$, $\widetilde{E}$ is a $D(G)$-bimodule homomorphism.

For $g,h\in G$ and t_j, $j=1,2,\ldots,k$, we have that

$$\begin{aligned}
&\sum_{i=1}^{k}w_i\widetilde{E}(z_i(g,t_j)e(t_j^{-1}gt_j,t_jh))\\
&\qquad=\sqrt{k}\sum_{i=1}^{k}w_i\widetilde{E}\biggl(\sum_{\eta,\xi\in G}(\eta,u)e(\xi,t_i^{-1})(g,t_j)e(t_j^{-1}gt_j,t_jh)\biggr)\\
&\qquad=\sqrt{k}\sum_{i=1}^{k}w_i\widetilde{E}\biggl(\sum_{\eta\in G}(\eta,u)E(t_i^{-1}gt_i,t_i^{-1}t_j)e(t_j^{-1}gt_j,t_jh)\biggr)\\
&\qquad=\frac{1}{\sqrt{k}}\sum_{i=1}^{k}w_iE(t_i^{-1}gt_i,t_i^{-1}t_j)(t_j^{-1}gt_j,t_jh)\\
&\qquad=\frac{1}{\sqrt{k}}w_j(t_j^{-1}gt_j,u)(t_j^{-1}gt_j,t_jh)\\
&\qquad=\sum_{\alpha\in G}(\alpha,t_j)e(t_j^{-1}gt_j,u)(t_j^{-1}gt_j,t_jh)\\
&\qquad=\sum_{\alpha\in G}(\alpha,t_j)(t_j^{-1}gt_j,u)e(t_j^{-1}gt_j,t_jh)\\
&\qquad=(g,t_j)e(t_j^{-1}gt_j,t_jh)
\end{aligned}$$

and similarly

$$\sum_{i=1}^{k}\widetilde{E}((g,t_j)e(t_j^{-1}gt_j,t_jh)w_i)z_i=(g,t_j)e(t_j^{-1}gt_j,t_jh).$$

This shows that $\{(w_i, z_i)\colon\ i = 1, 2, \ldots, k\}$ is a quasi-basis for $\widetilde{E}$. Hence $\widetilde{E}$ is of index-finite type and

$$\begin{aligned}\text{Index } \widetilde{E} &= \sum_{i=1}^{k} w_i z_i = k \sum_{i=1}^{k} \sum_{\alpha,\beta,\xi,\eta\in G} (\alpha, t_i) e(\beta, u)(\eta, u) e(\xi, t_i^{-1}) \\ &= k \sum_{i=1}^{k} \sum_{\alpha,\beta,\xi\in G} (\alpha, t_i) E(\beta, u) e(\xi, t_i^{-1}) \\ &= k \sum_{i=1}^{k} \sum_{\alpha,\xi\in G} (\alpha, t_i) e(\xi, t_i^{-1}) = k \sum_{\gamma\in G} (\gamma, u) \\ &= \text{Index } E.\end{aligned}$$

Here we use the properties that $\Big\{\Big(\sum\limits_{\alpha\in G}(\alpha, t_i), \sum\limits_{\xi\in G}(\xi, t_i^{-1})\Big)\colon\ i = 1, 2, \ldots, k\Big\}$ is a quasi-basis for E, and the unit $I = \sum\limits_{\gamma\in G}(\gamma, u)$ of $D(G)$ can be expressed as $\sum\limits_{i=1}^{k} \sum\limits_{\alpha,\xi\in G} (\alpha, t_i) e(\xi, t_i^{-1})$. $\square$

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References

[1] *P. Bántay*: Orbifolds and Hopf algebras. Phys. Lett., B *245* (1990), 477–479.

[2] *O. Bratteli, D. W. Robinson*: Operator Algebras and Quantum Statistical Mechanics. 1. C^*- and W^*-Algebras, Symmetry Groups, Decomposition of States. Texts and Monographs in Physics, Springer, New York, 1987.

[3] *K. A. Dancer, P. S. Isac, J. Links*: Representations of the quantum doubles of finite group algebras and spectral parameter dependent solutions of the Yang-Baxter equation. J. Math. Phys. *47* (2006), 103511, 18 pages.

[4] *L. Jiang*: Towards a quantum Galois theory for quantum double algebras of finite groups. Proc. Am. Math. Soc. *138* (2010), 2793–2801.

[5] *L. Jiang*: C^*-index of observable algebras in G-spin model. Sci. China, Ser. A *48* (2005), 57–66.

[6] *L. Jiang, G. Zhu*: C^*-index in double algebra of finite group. Trans. Beijing Inst. Technol. *23* (2003), 147–148. (In Chinese.)

[7] *V. F. R. Jones*: Subfactors and Knots. Expository lectures from the CBMS regional conference, Annapolis, USA, 1988. Regional Conference Series in Mathematics 80, AMS, Providence, 1991.

[8] *V. F. R. Jones*: Index for subfactors. Invent. Math. *72* (1983), 1–25.

[9] *C. Kassel*: Quantum Groups. Graduate Texts in Mathematics 155, Springer, Berlin, 1995.

[10] *Y. Kawahigashi, R. Longo*: Classification of local conformal nets. Case $c < 1$. Ann. Math. *160* (2004), 493–522.

[11] *H. Kosaki*: Extension of Jones' theory on index to arbitrary factors. J. Funct. Anal. *66* (1986), 123–140.
[12] *R. Longo*: Index of subfactors and statistics of quantum fields. II. Correspondences, braid group statistics and Jones polynomial. Commun. Math. Phys. *130* (1990), 285–309.
[13] *R. Longo*: Index of subfactors and statistics of quantum fields. I. Commun. Math. Phys. *126* (1989), 217–247.
[14] *D. E. Radford*: Minimal quasitriangular Hopf algebras. J. Algebra *157* (1993), 285–315.
[15] *M. E. Sweedler*: Hopf Algebras. Mathematics Lecture Note Series, W. A. Benjamin, New York, 1969.
[16] *Y. Watatani*: Index for C^*-subalgebras. Mem. Am. Math. Soc. *83* (1990).

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