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COMPLETE CONVERGENCE THEOREMS FOR NORMED ROW SUMS FROM AN ARRAY OF ROWWISE PAIRWISE NEGATIVE QUADRANT DEPENDENT RANDOM VARIABLES WITH APPLICATION TO THE DEPENDENT BOOTSTRAP

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Abstract. Let $\{X_{n,j},\, 1 \leqslant j \leqslant m(n),\, n \geqslant 1\}$ be an array of rowwise pairwise negative quadrant dependent mean 0 random variables and let $0 < b_n \to \infty$. Conditions are given for $\sum_{j=1}^{m(n)} X_{n,j}/b_n \to 0$ completely and for $\max_{1\leqslant k\leqslant m(n)} \Bigl|\sum_{j=1}^{k} X_{n,j}\Bigr|/b_n \to 0$ completely. As an application of these results, we obtain a complete convergence theorem for the row sums $\sum_{j=1}^{m(n)} X^*_{n,j}$ of the dependent bootstrap samples $\{\{X^*_{n,j},\, 1 \leqslant j \leqslant m(n)\},\, n \geqslant 1\}$ arising from a sequence of i.i.d. random variables $\{X_n, n \geqslant 1\}$.

Keywords: array of rowwise pairwise negative quadrant dependent random variables; complete convergence; dependent bootstrap; sequence of i.i.d. random variables

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1. Introduction and preliminaries

The concept of the complete convergence was introduced by Hsu and Robbins [8]. A sequence of random variables $\{U_n, n \geqslant 1\}$ is said to *converge completely* to a real number c if $\sum\limits_{n=1}^{\infty} P(|U_n - c| > \varepsilon) < \infty$ for all $\varepsilon > 0$. This of course implies by the Borel-Cantelli lemma that $U_n \to c$ almost surely (a.s.). The converse is true if the random variables U_n, $n \geqslant 1$, are independent. Hsu and Robbins [8] proved that the sequence of arithmetic means $\Bigl\{n^{-1} \sum\limits_{i=1}^{n} X_i,\, n \geqslant 1\Bigr\}$ of independent and identically

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distributed (i.i.d.) random variables $\{X_n,\ n \geqslant 1\}$ converges completely to EX_1 if $\operatorname{Var} X_1 < \infty$. Erdős [6] proved the converse. The Hsu-Robbins-Erdős result is precisely stated as follows.

Theorem 1.1 (Hsu and Robbins [8], Erdős [6])**.** *For a sequence of i.i.d. random variables* $\{X_n,\ n \geqslant 1\}$, $\sum\limits_{i=1}^{n} X_i/n$ *converges completely to* 0 *if and only if* $EX_1 = 0$ *and* $EX_1^2 < \infty$.

In Theorems 2.1 and 2.2, the main results of the current work, we establish complete convergence theorems for normed row sums from an array of rowwise pairwise negative quadrant dependent random variables. This concept of dependence was introduced by Lehmann [12] and will be defined below.

In many stochastic models, an independence assumption among the random variables in the model is not a reasonable one, since the random variables in the model may be "repelling" in the sense that small values of any of the random variables increase the probability that the other random variables are large. Thus an assumption of some type of negative dependence is often more suitable than the classical assumption of independence.

Definition 1.1. A sequence of random variables $\{X_n,\ n \geqslant 1\}$ is said to be *pairwise negative quadrant dependent* (PNQD) if for all $i, j \geqslant 1$ $(i \neq j)$ and all $x, y \in \mathbb{R}$,

$$P(X_i \leqslant x, X_j \leqslant y) \leqslant P(X_i \leqslant x)P(X_j \leqslant y). \tag{1.1}$$

The choice of the adjective "negative" in this definition stems from the fact that (1.1) implies that

$$P(X_j > y \mid X_i \leqslant x) \geqslant P(X_j > y).$$

It is well known and easy to prove that the sequence $\{X_n, n \geqslant 1\}$ is PNQD if and only if for all $i, j \geqslant 1$ $(i \neq j)$ and all $x, y \in \mathbb{R}$,

$$P(X_i \geqslant x, X_j \geqslant y) \leqslant P(X_i \geqslant x)P(X_j \geqslant y).$$

It is of course immediate that if $\{X_n, n \geqslant 1\}$ is a sequence of pairwise independent (*a fortiori*, independent) random variables, then $\{X_n, n \geqslant 1\}$ is PNQD.

The concept of a finite set of random variables being PNQD is defined in a manner completely analogous to the definition provided by Definition 1.1. An array of random variables $\{X_{n,j},\ 1 \leqslant j \leqslant m(n),\ n \geqslant 1\}$ is said to be *rowwise* PNQD if for each $n \geqslant 1$, the finite set of random variables $\{X_{n,j}, 1 \leqslant j \leqslant m(n)\}$ is PNQD.

A collection of m PNQD random variables arises by sampling without replacement from a set of m real numbers (see, e.g., Bozorgnia, Patterson, and Taylor [2]). Li, Rosalsky, and Volodin [13] showed that for every sequence of continuous distribution functions $\{F_n,\, n \geqslant 1\}$, there exists a PNQD sequence of random variables $\{X_n,\, n \geqslant 1\}$ such that the distribution function of X_n is F_n, $n \geqslant 1$ and such that for all $k \geqslant 1$, $\{X_n, n \geqslant k\}$ is not a sequence of independent random variables.

Theorems 2.1 and 2.2 extend, improve, or relate to various other complete convergence results in the literature; see Hu, Móricz, and Taylor [9], Bozorgnia, Patterson, and Taylor [1], Bozorgnia, Patterson, and Taylor [3], Hu and Taylor [11], Taylor, Patterson, and Bozorgnia [20], Gan and Chen [7], Wu and Zhu [24], and Wu and Wang [23]. We gratefully acknowledge that the statement of Theorems 2.1 and 2.2 and their proofs owe much to some of these earlier results.

The following two lemmas are used in the proof of Theorem 2.1; the first lemma is also used in the proofs of Corollary 2.1, Theorem 2.2, and Corollary 2.2.

Lemma 1.1 follows from Lemma 1 of Lehmann [12]; a more direct proof of it was provided by Matuła [14].

Lemma 1.1 (Lehmann [12], Matuła [14]). *Let $\{X_n,\, n \geqslant 1\}$ be a sequence of PNQD random variables and let $\{f_n,\, n \geqslant 1\}$ be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$. If the sequence $\{f_n, n \geqslant 1\}$ consists of only nondecreasing functions or only nonincreasing functions, then $\{f_n(X_n),\, n \geqslant 1\}$ is a sequence of PNQD random variables.*

The next lemma is well known (see, e.g., Patterson and Taylor [16]) but we are not able to track down its origin. In any event, its proof is immediate from Lemmas 1 and 3 of Lehmann [12].

Lemma 1.2. *If $\{X_1, \dots, X_n\}$ is a set of $n \geqslant 2$ PNQD integrable random variables, then*

$$\operatorname{Var}\left(\sum_{j=1}^{n} X_j\right) \leqslant \sum_{j=1}^{n} \operatorname{Var} X_j.$$

The following lemma provides a maximal inequality for a sequence of PNQD mean 0 random variables and is used in the proof of Theorem 2.2. Throughout this paper, for $x \geqslant 1$ we let $\log x$ denote $\log_{\mathrm{e}}(\max\{\mathrm{e}, x\})$.

Lemma 1.3 (Wu [22]). *There exists a universal constant $C < \infty$ such that for every sequence $\{X_n, n \geqslant 1\}$ of PNQD mean 0 random variables,*

$$E\left(\max_{1 \leqslant k \leqslant n}\left(\sum_{j=1}^{k} X_j\right)^2\right) \leqslant C(\log n)^2 \sum_{j=1}^{n} EX_j^2 \quad \forall\, n \geqslant 1.$$

Theorems 2.1 and 2.2 and their corollaries will be stated and proved in Section 2. In Section 3, we will apply the results in Section 2 to obtain a complete convergence theorem (Theorem 3.1) for the row sums of the dependent bootstrap samples arising from a sequence of i.i.d. random variables. The notion of dependent bootstrap samples will be reviewed in Section 3.

We close this section by remarking that a major survey article concerning a general "theory of negative dependence" was prepared by Pematle [17]. That article discusses the relationship between various definitions of "negative dependence", outlines some possible directions that the theory can take, and provides some interesting conjectures.

2. The main results

With the preliminaries accounted for, the main results may now be stated and proved. We note that the condition (2.2) is in the spririt of a condition of Chung [4] for a sequence of independent random variables to obey a general strong law of large numbers (SLLN).

Theorem 2.1. *Let $\{m(n), n \geqslant 1\}$ be a sequence of positive integers, let $\{X_{n,j}, 1 \leqslant j \leqslant m(n), n \geqslant 1\}$ be an array of rowwise PNQD mean 0 random variables, and let $\{b_n, n \geqslant 1\}$ be a sequence of positive real numbers with $\lim\limits_{n\to\infty} b_n = \infty$. Let $\{\Psi_{n,j}, 1 \leqslant j \leqslant m(n), n \geqslant 1\}$ be an array of functions defined on $[0,\infty)$ such that for all $n \geqslant 1$ and all $1 \leqslant j \leqslant m(n)$,*

$$\Psi_{n,j}(0) = 0 < \frac{\Psi_{n,j}(t)}{t} \uparrow \quad \text{and} \quad \frac{\Psi_{n,j}(t)}{t^2} \downarrow \quad \text{as } 0 < t \uparrow . \tag{2.1}$$

If

$$\sum_{n=1}^{\infty} \sum_{j=1}^{m(n)} \frac{E\Psi_{n,j}(|X_{n,j}|)}{\Psi_{n,j}(b_n)} < \infty, \tag{2.2}$$

then

$$\sum_{j=1}^{m(n)} X_{n,j}/b_n \longrightarrow 0 \quad \text{completely.} \tag{2.3}$$

Proof. For $n \geqslant 1$ and $1 \leqslant j \leqslant m(n)$, let

$$Y_{n,j} = -b_n I(X_{n,j} < -b_n) + X_{n,j} I(|X_{n,j}| \leqslant b_n) + b_n I(X_{n,j} > b_n)$$

and

$$Z_{n,j} = (X_{n,j} + b_n)I(X_{n,j} < -b_n) + (X_{n,j} - b_n)I(X_{n,j} > b_n).$$

Then

$$Y_{n,j} + Z_{n,j} = X_{n,j}, \quad 1 \leqslant j \leqslant m(n),\ n \geqslant 1 \tag{2.4}$$

and so to prove (2.3) it suffices to show

$$\sum_{j=1}^{m(n)} Z_{n,j}/b_n \longrightarrow 0 \quad \text{completely,} \tag{2.5}$$

$$\sum_{j=1}^{m(n)} (Y_{n,j} - EY_{n,j})/b_n \longrightarrow 0 \quad \text{completely,} \tag{2.6}$$

and

$$\sum_{j=1}^{m(n)} EY_{n,j}/b_n \longrightarrow 0. \tag{2.7}$$

We first prove (2.5). Let $n \geqslant 1$ and $1 \leqslant j \leqslant m(n)$. If $X_{n,j} > b_n$, then

$$0 < Z_{n,j} = X_{n,j} - b_n < X_{n,j}.$$

If $X_{n,j} < -b_n$, then

$$X_{n,j} < X_{n,j} + b_n = Z_{n,j} < 0.$$

If $|X_{n,j}| \leqslant b_n$, then

$$|Z_{n,j}| = 0 \leqslant |X_{n,j}|.$$

Thus for all $n \geqslant 1$ and $1 \leqslant j \leqslant m(n)$,

$$|Z_{n,j}| \leqslant |X_{n,j}|I(|X_{n,j}| > b_n). \tag{2.8}$$

Hence, for arbitrary $\varepsilon > 0$,

$$
\begin{aligned}
(2.9)\qquad \sum_{n=1}^{\infty} P\left(\left|\sum_{j=1}^{m(n)} Z_{n,j}\right| > b_n\varepsilon\right) &\leqslant \sum_{n=1}^{\infty} P\left(\sum_{j=1}^{m(n)} |Z_{n,j}| > b_n\varepsilon\right) \\
&\leqslant \frac{1}{\varepsilon}\sum_{n=1}^{\infty}\sum_{j=1}^{m(n)} E|Z_{n,j}|/b_n \quad \text{(by the Markov inequality)} \\
&\leqslant \frac{1}{\varepsilon}\sum_{n=1}^{\infty}\sum_{j=1}^{m(n)} \frac{E(|X_{n,j}|I(|X_{n,j}| > b_n))}{b_n} \quad \text{(by (2.8))} \\
&\leqslant \frac{1}{\varepsilon}\sum_{n=1}^{\infty}\sum_{j=1}^{m(n)} \frac{E\Psi_{n,j}(|X_{n,j}|)}{\Psi_{n,j}(b_n)} \quad \text{(by the first half of (2.1))} \\
&< \infty \quad \text{(by (2.2))}
\end{aligned}
$$

thereby proving (2.5).

Next, we prove (2.6). It follows from Lemma 1.1 that $\{Y_{n,j} - EY_{n,j}, 1 \leqslant j \leqslant m(n), n \geqslant 1\}$ is an array of rowwise PNQD random variables. Hence, for arbitrary $\varepsilon > 0$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\sum_{j=1}^{m(n)} (Y_{n,j} - EY_{n,j})\right| > b_n\varepsilon\right) \\
&\qquad\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty} b_n^{-2}\operatorname{Var}\left(\sum_{j=1}^{m(n)} Y_{n,j}\right) \quad \text{(by Chebyshev's inequality)} \\
&\qquad\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty} b_n^{-2}\sum_{j=1}^{m(n)} \operatorname{Var} Y_{n,j} \quad \text{(by Lemma 1.2)} \\
&\qquad\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\sum_{j=1}^{m(n)} \frac{EY_{n,j}^2}{b_n^2} \\
&\qquad\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\sum_{j=1}^{m(n)} \frac{E\Psi_{n,j}(|Y_{n,j}|)}{\Psi_{n,j}(b_n)} \quad \text{(by the second half of (2.1))} \\
&\qquad\leqslant \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\sum_{j=1}^{m(n)} \frac{E\Psi_{n,j}(|X_{n,j}|)}{\Psi_{n,j}(b_n)} \quad \text{(since } |Y_{n,j}| \leqslant |X_{n,j}|\text{)} \\
&\qquad< \infty \quad \text{(by (2.2))}
\end{aligned}
$$

thereby proving (2.6).

Finally, we prove (2.7). Note that for all $n \geqslant 1$ and $1 \leqslant j \leqslant m(n)$, it follows from (2.4) and the $EX_{n,j} = 0$ assumption that $EY_{n,j} = -EZ_{n,j}$ and hence

$$\left|\sum_{j=1}^{m(n)} EY_{n,j}\right| \Big/ b_n = \left|\sum_{j=1}^{m(n)} EZ_{n,j}\right| \Big/ b_n \leqslant \sum_{j=1}^{m(n)} E|Z_{n,j}|/b_n \longrightarrow 0,$$

since

$$\sum_{n=1}^{\infty} \sum_{j=1}^{m(n)} E|Z_{n,j}|/b_n < \infty$$

by (2.9) thereby proving (2.7) and completing the proof of Theorem 2.1. □

Remark 2.1. In Theorem 2.1, if the array $\{X_{n,j},\ 1 \leqslant j \leqslant m(n),\ n \geqslant 1\}$ is comprised of symmetric random variables, then the theorem is true with (2.1) replaced by the following weaker condition. For some $0 < q \leqslant 1$ and all $n \geqslant 1$ and all $1 \leqslant j \leqslant m(n)$,

$$\Psi_{n,j}(0) = 0 < \frac{\Psi_{n,j}(t)}{t^q} \uparrow \quad \text{and} \quad \frac{\Psi_{n,j}(t)}{t^2} \downarrow \quad \text{as } 0 < t \uparrow .$$

The reader can easily verify (2.5), and verification of (2.6) is the same as before. Symmetry immediately gives (2.7).

In the following corollary, note that there is a trade-off involving p in the conditions (2.10) and (2.11); the larger p, the stronger is the condition (2.10) but the weaker is the condition (2.11).

Corollary 2.1. *Let $\{m(n),\ n \geqslant 1\}$ be a sequence of positive integers and let $\{X_{n,j},\ 1 \leqslant j \leqslant m(n),\ n \geqslant 1\}$ be an array of rowwise identically distributed PNQD random variables with*

$$E|X_{n,1} - EX_{n,1}|^p = O(1) \tag{2.10}$$

for some $1 \leqslant p \leqslant 2$. Let $\{b_n,\ n \geqslant 1\}$ be a sequence of positive real numbers with $\lim\limits_{n\to\infty} b_n = \infty$. If

$$\sum_{n=1}^{\infty} \frac{m(n)}{b_n^p} < \infty, \tag{2.11}$$

then

$$b_n^{-1} \sum_{j=1}^{m(n)} (X_{n,j} - EX_{n,j}) \longrightarrow 0 \quad \textit{completely.} \tag{2.12}$$

P r o o f. For all $n \geqslant 1$ and all $1 \leqslant j \leqslant m(n)$, let

$$\Psi_{n,j}(t) = t^p, \quad t \geqslant 0.$$

Then $\{\Psi_{n,j}, 1 \leqslant j \leqslant m(n), n \geqslant 1\}$ satisfies (2.1) and by Lemma 1.1, $\{X_{n,j} - EX_{n,j}, 1 \leqslant j \leqslant m(n), n \geqslant 1\}$ is an array of rowwise PNQD mean 0 random variables. Since

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{j=1}^{m(n)} \frac{E\Psi_{n,j}(|X_{n,j} - EX_{n,j}|)}{\Psi_{n,j}(b_n)} &= \sum_{n=1}^{\infty} \frac{m(n)E|X_{n,1} - EX_{n,1}|^p}{b_n^p} \\ &< \infty \quad \text{(by (2.10) and (2.11))}, \end{aligned}$$

the conclusion (2.12) follows immediately from Theorem 2.1. □

Theorem 2.2. *Let $\{m(n), n \geqslant 1\}$ be a sequence of positive integers, let $\{X_{n,j}, 1 \leqslant j \leqslant m(n), n \geqslant 1\}$ be an array of rowwise PNQD mean 0 random variables, and let $\{b_n, n \geqslant 1\}$ be a sequence of positive real numbers with $\lim_{n\to\infty} b_n = \infty$. Let $\{\Psi_{n,j}, 1 \leqslant j \leqslant m(n), n \geqslant 1\}$ be an array of functions defined on $[0,\infty)$ satisfying (2.1) for all $n \geqslant 1$ and all $1 \leqslant j \leqslant m(n)$. If*

$$\sum_{n=1}^{\infty} \bigl(\log m(n)\bigr)^2 \sum_{j=1}^{m(n)} \frac{E\Psi_{n,j}(|X_{n,j}|)}{\Psi_{n,j}(b_n)} < \infty, \tag{2.13}$$

then

$$\frac{1}{b_n} \max_{1 \leqslant k \leqslant m(n)} \Bigl| \sum_{j=1}^{k} X_{n,j} \Bigr| \longrightarrow 0 \quad \textit{completely.} \tag{2.14}$$

P r o o f. For $n \geqslant 1$ and $1 \leqslant j \leqslant m(n)$, define $Y_{n,j}$ and $Z_{n,j}$ as in the proof of Theorem 2.1. Then (2.4) holds and to prove (2.14) it suffices to show

$$\max_{1 \leqslant k \leqslant m(n)} \Bigl| \sum_{j=1}^{k} Z_{n,j} \Bigr| \Big/ b_n \longrightarrow 0 \quad \text{completely,}$$

$$\max_{1 \leqslant k \leqslant m(n)} \Bigl| \sum_{j=1}^{k} (Y_{n,j} - EY_{n,j}) \Bigr| \Big/ b_n \longrightarrow 0 \quad \text{completely,}$$

and

$$\max_{1 \leqslant k \leqslant m(n)} \Bigl| \sum_{j=1}^{k} EY_{n,j} \Bigr| \Big/ b_n \longrightarrow 0.$$

These are proved in the same way as (2.5), (2.6), and (2.7) were proved in the proof of Theorem 2.1, *mutatis mutandis*. The argument involves using (2.13) and Lemma 1.3 in the same manner as (2.2) and Lemma 1.2 were used in the proof of Theorem 2.1. The details are left to the reader. □

The following corollary will only be stated as its proof is virtually identical to that of Corollary 2.1. Theorem 2.2 is applied instead of Theorem 2.1.

Corollary 2.2. *Let* $\{m(n),\, n \geqslant 1\}$ *be a sequence of positive integers and let* $\{X_{n,j},\, 1 \leqslant j \leqslant m(n),\, n \geqslant 1\}$ *be an array of rowwise identically distributed PNQD random variables satisfying* (2.10) *for some* $1 \leqslant p \leqslant 2$. *Let* $\{b_n,\, n \geqslant 1\}$ *be a sequence of positive real numbers with* $\lim_{n\to\infty} b_n = \infty$. *If*

$$\sum_{n=1}^{\infty} \frac{m(n)(\log m(n))^2}{b_n^2} < \infty,$$

then

$$b_n^{-1} \max_{1\leqslant k\leqslant m(n)} \left| \sum_{j=1}^{k} (X_{n,j} - EX_{n,j}) \right| \longrightarrow 0 \quad \textit{completely.}$$

3. The dependent bootstrap

The dependent bootstrap was introduced by Smith and Taylor [18] and [19] for a sequence of i.i.d. random variables as follows. Let $\{X_n,\, n \geqslant 1\}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{m(n), n \geqslant 1\}$ and $\{k(n), n \geqslant 1\}$ be two sequences of integers such that $1 \leqslant m(n) \leqslant nk(n), n \geqslant 1$. For $\omega \in \Omega$ and $n \geqslant 1$, the *dependent bootstrap* is defined to be the sample of size $m(n)$, denoted $\{X_{n,j}^{*(\omega)}, 1 \leqslant j \leqslant m(n)\}$, drawn at random *without replacement* from the collection of $nk(n)$ items comprised of $k(n)$ of each of the n sample observations $X_1(\omega), \dots, X_n(\omega)$. In other words, $\{X_{n,j}^{*(\omega)}, 1 \leqslant j \leqslant m(n)\}$ consists of $m(n)$ selections taken at random *without replacement* from

$$\underbrace{X_1(\omega), \dots, X_1(\omega)}_{k(n)\ \text{times}}, \ \dots, \ \underbrace{X_n(\omega), \dots, X_n(\omega)}_{k(n)\ \text{times}}.$$

Thus for each of the $m(n)$ selections, each $X_i(\omega)$ has probability $1/n$ of being chosen (without conditioning on the other selections). Hence, for $\omega \in \Omega$ and $n \geqslant 1$, $\{X_{n,j}^{*(\omega)}, 1 \leqslant j \leqslant m(n)\}$ is a set of $m(n)$ identically distributed dependent random variables with

$$P^*(X_{n,1}^{*(\omega)} = X_i(\omega)) = \frac{1}{n}, \quad 1 \leqslant i \leqslant n,$$

where P^* is the (conditional) probability measure (given $\{X_n,\, n \geqslant 1\}$) carrying for each $n \geqslant 1$, the (uniform) distribution on $X_1(\omega), \dots, X_n(\omega)$ of each resampled

$X_{n,j}^{*(\omega)}, 1 \leqslant j \leqslant m(n)$. We refer to n as the *sample size* and to $m(n)$ as the *dependent bootstrap sample size.*

Alternatively, for $n \geqslant 1$ and $1 \leqslant i \leqslant n$, let

$$Y_j = X_i \quad \text{for } (i-1)k(n) + 1 \leqslant j \leqslant ik(n)$$

and

$$X_{n,j}^* = Y_{Z(n,j)}, \quad 1 \leqslant j \leqslant m(n),$$

where the $Z(n,j), 1 \leqslant j \leqslant m(n)$ are taken at random without replacement from the finite set $\{1, \dots, nk(n)\}$ and such that the families of random variables

$$\{Z(n,j), 1 \leqslant j \leqslant m(n)\}, \quad n \geqslant 1, \quad \{X_n, n \geqslant 1\}$$

are independent; we may and do assume without loss of generality that the underlying probability space $(\Omega, \mathcal{F}, P)$ is rich enough to accommodate all of these random variables. (Of course, for each $n \geqslant 1$, the $m(n)$ random variables $Z(n,j)$, $1 \leqslant j \leqslant m(n)$ are not independent.) Then for each $n \geqslant 1$, $X_{n,1}^*, \dots, X_{n,m(n)}^*$ are conditionally identically distributed (but not conditionally independent) given $(X_1, \dots, X_n)$ with

$$P(X_{n,1}^* = X_i \mid X_1, \dots, X_n) = \frac{1}{n} \quad \text{a.s.}, \quad 1 \leqslant i \leqslant n.$$

Smith and Taylor [18] and [19] proposed the dependent bootstrap as a procedure for reducing variation of estimators and for obtaining better confidence intervals than those obtained using the traditional Efron [5] nonparametric (or resampling *with* replacement) bootstrap. The reader may refer to Smith and Taylor [19] for details and where simulated confidence intervals are obtained to examine possible gains in coverage probabilities and reductions in the interval lengths.

Smith and Taylor [18] and [19] established the following important properties of the dependent bootstrap.

(i) For all $n \geqslant 1$, the set of random variables $\{X_{n,1}^{*(\omega)}, \dots, X_{n,m(n)}^{*(\omega)}\}$ comprising a dependent bootstrap sample are PNQD.

(ii) Letting E^* and Var^* denote, respectively, the operation of taking the expected value and variance (with respect to P^*) of real-valued functions of the dependent bootstrap samples $\{X_{n,j}^{*(\omega)}, 1 \leqslant j \leqslant m(n)\}$, $n \geqslant 1$, we have that

$$E^* X_{n,1}^{*(\omega)} = \overline{X}_n(\omega) \quad \text{and} \quad \mathrm{Var}^* X_{n,1}^{*(\omega)} = S_n^2(\omega), \quad n \geqslant 1, \tag{3.1}$$

where $\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ and $S_n^2 = n^{-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$, $n \geqslant 1$ are the sample mean and the biased version of the sample variance, respectively.

Smith and Taylor [18] established the conditional consistency of the dependent bootstrap mean

$$\overline{X}^*_{n,m(n)} = \sum_{j=1}^{m(n)} X^*_{n,j} \Big/ m(n),$$

whereas Patterson, Smith, Taylor, and Bozorgnia [15] established the conditional asymptotic normality of the dependent bootstrap mean. Hu, Ordóñez Cabrera, and Volodin [10] found an upper bound for the exact convergence rate (i.e., a law of the iterated logarithm type result) for dependent bootstrap means. In the following theorem, we establish a complete convergence theorem for the row sums $\sum_{j=1}^{m(n)} X^*_{n,j}$ of the dependent bootstrap samples $\{\{X^*_{n,j}, 1 \leqslant j \leqslant m(n)\}, n \geqslant 1\}$. The only other results that we are aware of concerning complete convergence of the row sums of dependent bootstrap samples are those of Volodin, Ordóñez Cabrera, and Hu [21] but that work and the current work do not entail each other.

Theorem 3.1. *Let $\{m(n), n \geqslant 1\}$ and $\{k(n), n \geqslant 1\}$ be integer sequences such that $1 \leqslant m(n) \leqslant nk(n)$, $n \geqslant 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of i.i.d. square integrable random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and for $\omega \in \Omega$, let $\{\{X^{*(\omega)}_{n,j}, 1 \leqslant j \leqslant m(n)\}, n \geqslant 1\}$ be the corresponding sequence of dependent bootstrap samples. Let $\{b_n, n \geqslant 1\}$ be a sequence of positive real numbers with $\lim_{n\to\infty} b_n = \infty$. Then letting $\overline{X}_n = n^{-1} \sum_{i=1}^{n} X_i$, $n \geqslant 1$, the dependent bootstrap samples satisfy the following:*

(i) *If $\sum_{n=1}^{\infty} m(n) b_n^{-2} < \infty$, then for almost every $\omega \in \Omega$,*

$$b_n^{-1} \sum_{j=1}^{m(n)} (X^{*(\omega)}_{n,j} - \overline{X}_n(\omega)) \longrightarrow 0 \quad \textit{completely};$$

that is, for almost every $\omega \in \Omega$,

$$\sum_{n=1}^{\infty} P^*\left(\left|\sum_{j=1}^{m(n)} (X^{*(\omega)}_{n,j} - \overline{X}_n(\omega))\right| > b_n \varepsilon\right) < \infty \quad \forall \varepsilon > 0.$$

(ii) *If $\sum_{n=1}^{\infty} m(n)(\log m(n))^2 b_n^{-2} < \infty$, then for almost every $\omega \in \Omega$,*

$$b_n^{-1} \max_{1 \leqslant k \leqslant m(n)} \left|\sum_{j=1}^{k} (X^{*(\omega)}_{n,j} - \overline{X}_n(\omega))\right| \longrightarrow 0 \quad \textit{completely};$$

that is, for almost every $\omega \in \Omega$,

$$\sum_{n=1}^{\infty} P^*\Bigg(\max_{1\leqslant k\leqslant m(n)}\Bigg|\sum_{j=1}^{k}(X_{n,j}^{*(\omega)} - \overline{X}_n(\omega))\Bigg| > b_n\varepsilon\Bigg) < \infty \quad \forall\,\varepsilon > 0.$$

P r o o f. (i) We will verify that for almost every $\omega \in \Omega$, the array $\{X_{n,j}^{*(\omega)}, 1 \leqslant j \leqslant m(n), n \geqslant 1\}$ of rowwise identically distributed PNQD random variables satisfies the hypotheses of Corollary 2.1 with $p = 2$. Now recalling (3.1), for almost every $\omega \in \Omega$,

$$E^*(X_{n,1}^{*(\omega)} - E^* X_{n,1}^{*(\omega)})^2 = \operatorname{Var}^* X_{n,1}^{*(\omega)} = S_n^2(\omega) = O(1),$$

since

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \Bigg(\frac{1}{n}\sum_{i=1}^{n} X_i\Bigg)^2 \longrightarrow EX_1^2 - (EX_1)^2 < \infty \quad \text{a.s.}$$

by the Kolmogorov SLLN. Thus by (3.1) and Corollary 2.1, for almost every $\omega \in \Omega$,

$$b_n^{-1}\sum_{j=1}^{m(n)}(X_{n,j}^{*(\omega)} - \overline{X}_n(\omega)) = b_n^{-1}\sum_{j=1}^{m(n)}(X_{n,j}^{*(\omega)} - E^* X_{n,j}^{*(\omega)}) \longrightarrow 0 \quad \text{completely.}$$

(ii) The proof is identical to that of part (i) except that Corollary 2.2 is used instead of Corollary 2.1. $\square$

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