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# CONGRUENCES FOR WOLSTENHOLME PRIMES

Romeo Meštrović, Kotor

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*Abstract.* A prime $p$ is said to be a Wolstenholme prime if it satisfies the congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}$. For such a prime $p$, we establish an expression for $\binom{2p-1}{p-1} \pmod{p^8}$ given in terms of the sums $R_i := \sum\limits_{k=1}^{p-1} 1/k^i$ $(i = 1, 2, 3, 4, 5, 6)$. Further, the expression in this congruence is reduced in terms of the sums $R_i$ $(i = 1, 3, 4, 5)$. Using this congruence, we prove that for any Wolstenholme prime $p$ we have

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} - 2p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^7}.$$

Moreover, using a recent result of the author, we prove that a prime $p$ satisfying the above congruence must necessarily be a Wolstenholme prime.

Furthermore, applying a technique of Helou and Terjanian, the above congruence is given as an expression involving the Bernoulli numbers.

*Keywords*: congruence; prime power; Wolstenholme prime; Wolstenholme's theorem; Bernoulli number

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## 1. Introduction and statements of results

*Wolstenholme's theorem* (see, e.g., [23], [7]) asserts that if $p$ is a prime greater than 3, then the binomial coefficient $\binom{2p-1}{p-1}$ satisfies the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

It is well known (see, e.g., [8]) that this theorem is equivalent to the assertion that for any prime $p \geqslant 5$ the numerator of the fraction

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1},$$

written in reduced form, is divisible by $p^2$. A. Granwille [7] established broader generalizations of Wolstenholme's theorem. As an application, it is obtained in [7] that for a prime $p \geqslant 5$ we have

$$\binom{2p-1}{p-1} \Big/ \binom{2p}{p}^3 \equiv \binom{3}{2} \Big/ \binom{2}{1}^3 \pmod{p^5}.$$

Notice that C. Helou and G. Terjanian [9] established many Wolstenholme type congruences modulo $p^k$ with a prime $p$ and $k \in \mathbb{N}$ such that $k \leqslant 6$. One of their main results ([9], Proposition 2, pages 488–489) is a congruence of the form $\binom{np}{mp} \equiv f(n,m,p)\binom{n}{m} \pmod{p}$, where $p \geqslant 3$ is a prime number, $m,n \in \mathbb{N}$ with $0 \leqslant m \leqslant n$, and $f$ is a function on $m,n$ and $p$ involving the Bernoulli numbers $B_k$. As an application, ([9], Corollary 2 (2), page 493; also see Corollary 6 (2), page 495), for any prime $p \geqslant 5$ we have

$$\binom{2p-1}{p-1} \equiv 1 - p^3 B_{p^3-p^2-2} + \frac{1}{3}p^5 B_{p-3} - \frac{6}{5}p^5 B_{p-5} \pmod{p^6}.$$

A similar congruence modulo $p^7$ (Corollary 1.2) is obtained in this paper for Wolstenholme primes.

A prime $p$ is said to be a *Wolstenholme prime* if it satisfies the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}.$$

The two known such primes are 16843 and 2124679, and in 2007 R. J. McIntosh and E. L. Roettger [17] reported that these primes are the only two Wolstenholme primes less than $10^9$. However, using an argument based on the prime number theorem, McIntosh [16], page 387, conjectured that there are infinitely many Wolstenholme primes, and that no prime satisfies the congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^5}$.

Wolstenholme primes form a subset of irregular primes. Indeed, Wolstenholme primes are those irregular primes $p$ which divide the numerator of $B_{p-3}$ (see, e.g., [16] or [19]). Recall that the irregular primes as well as Wieferich and related primes are connected with the first case of Fermat's last theorem; see [21], Lecture I, pages 9–12, and [21], Lecture VIII, pages 151–154, [2], [3], [12], [13], [22].

The following result is basic in our investigations.

**Proposition 1.1.** *Let $p$ be a Wolstenholme prime. Then*

$$\binom{2p-1}{p-1} \equiv 1 + p\sum_{k=1}^{p-1}\frac{1}{k} - \frac{p^2}{2}\sum_{k=1}^{p-1}\frac{1}{k^2} + \frac{p^3}{3}\sum_{k=1}^{p-1}\frac{1}{k^3} - \frac{p^4}{4}\sum_{k=1}^{p-1}\frac{1}{k^4} + \frac{p^5}{5}\sum_{k=1}^{p-1}\frac{1}{k^5} - \frac{p^6}{6}\sum_{k=1}^{p-1}\frac{1}{k^6} \pmod{p^8}.$$

The above congruence can be simplified as follows.

**Proposition 1.2.** *Let $p$ be a Wolstenholme prime. Then*

$$\binom{2p-1}{p-1} \equiv 1 + \frac{3p}{2}\sum_{k=1}^{p-1}\frac{1}{k} - \frac{p^2}{4}\sum_{k=1}^{p-1}\frac{1}{k^2} + \frac{7p^3}{12}\sum_{k=1}^{p-1}\frac{1}{k^3} + \frac{5p^5}{12}\sum_{k=1}^{p-1}\frac{1}{k^5} \pmod{p^8}.$$

Reducing the modulus in the previous congruence, we can obtain the following simpler congruences.

**Corollary 1.1.** *Let $p$ be a Wolstenholme prime. Then*

$$\begin{aligned}\binom{2p-1}{p-1} &\equiv 1 - 2p\sum_{k=1}^{p-1}\frac{1}{k} - 2p^2\sum_{k=1}^{p-1}\frac{1}{k^2}\\ &\equiv 1 + 2p\sum_{k=1}^{p-1}\frac{1}{k} + \frac{2p^3}{3}\sum_{k=1}^{p-1}\frac{1}{k^3} \pmod{p^7}.\end{aligned}$$

The *Bernoulli numbers* $B_k$ $(k \in \mathbb{N})$ are defined by the generating function

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{\mathrm{e}^x - 1}.$$

It is easy to find the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_n = 0$ for odd $n \geqslant 3$. Furthermore, $(-1)^{n-1}B_{2n} > 0$ for all $n \geqslant 1$. These and many other properties can be found, for instance, in [10] or [4].

The second congruence from Corollary 1.1 can be given in terms of the Bernoulli numbers by the following result.

**Corollary 1.2.** *Let $p$ be a Wolstenholme prime. Then*

$$\binom{2p-1}{p-1} \equiv 1 - p^3 B_{p^4-p^3-2} - \frac{3}{2}p^5 B_{p^2-p-4} + \frac{3}{10}p^6 B_{p-5} \pmod{p^7}.$$

The above congruence can be given by the following expression involving lower order Bernoulli numbers.

**Corollary 1.3.** *Let $p$ be a Wolstenholme prime. Then*

$$\begin{aligned}\binom{2p-1}{p-1} \equiv 1 &- p^3\Big(\frac{8}{3}B_{p-3} - 3B_{2p-4} + \frac{8}{5}B_{3p-5} - \frac{1}{3}B_{4p-6}\Big)\\ &- p^4\Big(\frac{8}{9}B_{p-3} - \frac{3}{2}B_{2p-4} + \frac{24}{25}B_{3p-5} - \frac{2}{9}B_{4p-6}\Big)\\ &- p^5\Big(\frac{8}{27}B_{p-3} - \frac{3}{4}B_{2p-4} + \frac{72}{125}B_{3p-5} - \frac{4}{27}B_{4p-6} + \frac{12}{5}B_{p-5} - B_{2p-6}\Big)\\ &- \frac{2}{25}p^6 B_{p-5} \pmod{p^7}.\end{aligned}$$

Combining the first congruence in Corollary 1.1 and a recent result of the author in [18], Theorem 1.1, we obtain a new characterization of Wolstenholme primes as follows.

**Corollary 1.4** ([18], Remark 1.6)**.** *A prime $p$ is a Wolstenholme prime if and only if*

$$\binom{2p-1}{p-1} \equiv 1 - 2p\sum_{k=1}^{p-1}\frac{1}{k} - 2p^2\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^7}.$$

**Remark 1.1.** A computation shows that no prime $p < 10^5$ satisfies the second congruence in Corollary 1.1, except the Wolstenholme prime 16843. Accordingly, an interesting question is as follows: *Is it true that the second congruence in Corollary* 1.1 *implies that a prime $p$ is necessarily a Wolstenholme prime*? We conjecture that this is true.

A proof of Proposition 1.1 is given in the next section. Proofs of Proposition 1.2 and Corollaries 1.1–1.3 are presented in Section 3.

## 2. Proof of Proposition 1.1

For the proof of Proposition 1.1, we will need some auxiliary results.

**Lemma 2.1.** *For any prime $p \geqslant 7$, we have*

$$2\sum_{k=1}^{p-1}\frac{1}{k} \equiv -p\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^4}. \tag{2.1}$$

Proof. The above congruence is in fact the congruence (14) in ([25], Proof of Theorem 3.2). □

**Lemma 2.2.** *For any prime $p \geqslant 7$, we have*

$$\binom{2p-1}{p-1} \equiv 1 + 2p\sum_{k=1}^{p-1}\frac{1}{k} \pmod{p^5}, \tag{2.2}$$

*and*

$$\binom{2p-1}{p-1} \equiv 1 - p^2\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^5}. \tag{2.3}$$

P r o o f. Let $R_1(p) = \sum\limits_{k=1}^{p-1} 1/k$. Following ([25], Definition 3.1) we define $w_p < p^2$ to be the unique nonnegative integer such that $w_p \equiv R_1(p)/p^2 \pmod{p^2}$. Then by ([25], Theorem 3.2), for all nonnegative integers $n$ and $r$ with $n \geqslant r$,

$$\binom{np}{rp}\Big/\binom{n}{r} \equiv 1 + w_p nr(n-r)p^3 \pmod{p^5}. \tag{2.4}$$

Since $\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1}$, taking $n = 2$ and $r = 1$, (2.4) becomes

$$\binom{2p-1}{p-1} \equiv 1 + 2w_p p^3 \pmod{p^5},$$

which is actually (2.2). Now the congruence (2.3) follows immediately from (2.2) and (2.1) of Lemma 2.1. □

**Lemma 2.3.** *The following statements about a prime $p \geqslant 7$ are equivalent:*

(i) *$p$ is a Wolstenholme prime;*

(ii) $\sum\limits_{k=1}^{p-1} 1/k \equiv 0 \pmod{p^3}$;

(iii) $\sum\limits_{k=1}^{p-1} 1/k^2 \equiv 0 \pmod{p^2}$;

(iv) *$p$ divides the numerator of the Bernoulli number $B_{p-3}$.*

P r o o f. The equivalences (i)⇔(ii)⇔(iii) are immediate from Lemma 2.2 if we consider the congruences (2.2) and (2.3) modulo $p^4$. Further, by a special case of Glaisher's congruence ([5], page 21, [6], page 323; also see [16], Theorem 2), we have

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3}p^3 B_{p-3} \pmod{p^4},$$

which implies the equivalence (i)⇔(iv). This concludes the proof. □

For the proof of Proposition 1.1, we use the congruences (2.2) and (2.3) of Lemma 2.2 with $\pmod{p^4}$ instead of $\pmod{p^5}$. By a classical result of E. Lehmer [15]; (also see [24], Theorem 2.8), $\sum_{k=1}^{p-1} 1/k \equiv -\frac{1}{3}B_{p-3} \pmod{p^3}$. Substituting this into Glaisher's congruence given above, we obtain immediately (2.2) of Lemma 2.2, with $\pmod{p^4}$ instead of $\pmod{p^5}$.

Notice that the congruence (2.3) is also given in [16], page 385, but its proof is there omitted.

For a prime $p \geqslant 3$ and a positive integer $n \leqslant p-2$ we denote

$$R_n(p) := \sum_{i=1}^{p-1} \frac{1}{k^n} \quad \text{and} \quad H_n(p) := \sum_{1 \leqslant i_1 < i_2 < \ldots < i_n \leqslant p-1} \frac{1}{i_1 i_2 \ldots i_n}.$$

In the sequel we shall often write $R_n$ and $H_n$ instead of $R_n(p)$ and $H_n(p)$, respectively.

**Lemma 2.4** ([1], Theorem 3; also see [24], Remark 2.3)**.** *For any prime $p \geqslant 3$ and a positive integer $n \leqslant p-3$, we have*

$$R_n(p) \equiv 0 \pmod{p^2} \quad \text{if } n \text{ is odd}, \quad \text{and} \quad R_n(p) \equiv 0 \pmod{p} \quad \text{if } n \text{ is even.}$$

**Lemma 2.5** (Newton's formula, see, e.g., [11])**.** *Let $m$ and $s$ be positive integers such that $m \leqslant s$. Define the symmetric polynomials*

$$P_m(s) = P_m(s; x_1, x_2, \ldots, x_s) = x_1^m + x_2^m + \ldots + x_s^m,$$

*and*

$$A_m(s) = A_m(s; x_1, x_2, \ldots, x_s) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant s} x_{i_1} x_{i_2} \ldots x_{i_m}.$$

*Then for $n = 1, 2, \ldots, s$, we have*

$$\begin{aligned} &P_n(s) - A_1(s)P_{n-1}(s) + A_2(s)P_{n-2}(s) \\ &\qquad + \ldots + (-1)^{n-1}A_{n-1}(s)P_1(s) + (-1)^n n A_n(s) = 0. \end{aligned}$$

**Lemma 2.6** (see [20], Lemma 2.2, the case $l = 1$)**.** *For any prime $p \geqslant 5$ and a positive integer $n \leqslant p-3$, we have*

$$H_n(p) \equiv 0 \pmod{p^2} \quad \text{if } n \text{ is odd} \quad \text{and} \quad H_n(p) \equiv 0 \pmod{p} \quad \text{if } n \text{ is even.}$$

Lemma 2.6 is an immediate consequence of a result of X. Zhou and T. Cai [26], Lemma 2; (also see [24], Theorem 2.14).

**Lemma 2.7.** *For any Wolstenholme prime $p$, we have*

$$R_2(p) \equiv -2H_2(p) \pmod{p^6}, \quad R_3(p) \equiv 3H_3(p) \pmod{p^5},$$
$$R_4(p) \equiv -4H_4(p) \pmod{p^4}, \quad R_5(p) \equiv 5H_5(p) \pmod{p^4}$$
$$\text{and} \qquad R_6(p) \equiv -6H_6(p) \pmod{p^3}.$$

P r o o f. By Newton's formula (see Lemma 2.5), for $n = 2, 3, 4, 5, 6$ we have

$$R_n + (-1)^n nH_n = H_1R_{n-1} - H_2R_{n-2} + \ldots + (-1)^n H_{n-1}R_1. \tag{2.5}$$

First note that by Lemma 2.3, $R_1 = H_1 \equiv 0 \pmod{p^3}$ and $R_2 \equiv 0 \pmod{p^2}$. Therefore, (2.5) implies $R_2 + 2H_2 = H_1R_1 \equiv 0 \pmod{p^6}$, so that $R_2 \equiv -2H_2 \pmod{p^6}$. From this and Lemma 2.3 we conclude that $H_2 \equiv R_2 \equiv 0 \pmod{p^2}$.

Further, by Lemma 2.4 and Lemma 2.6, $R_3 \equiv H_3 \equiv R_5 \equiv H_5 \equiv 0 \pmod{p^2}$ and $R_4 \equiv H_4 \equiv 0 \pmod{p}$. Substituting the previous congruences for $H_i$ and $R_i$ ($i = 1, 2, 3, 4, 5$) into (2.5) with $n = 3, 4, 5, 6$, we get

$$\begin{aligned}
&R_3 - 3H_3 = H_1R_2 - H_2R_1 \equiv 0 \pmod{p^5},\\
&R_4 + 4H_4 = H_1R_3 - H_2R_2 + H_3R_1 \equiv 0 \pmod{p^4},\\
&R_5 - 5H_5 = H_1R_4 - H_2R_3 + H_3R_2 - H_4R_1 \equiv 0 \pmod{p^4},\\
&R_6 + 6H_6 = H_1R_5 - H_2R_4 + H_3R_3 - H_4R_2 + H_5R_1 \equiv 0 \pmod{p^3}.
\end{aligned}$$

This completes the proof. $\square$

P r o o f of Proposition 1.1. For any prime $p \geqslant 7$, we have

$$\begin{aligned}
\binom{2p-1}{p-1} &= \frac{(p+1)(p+2)\ldots(p+k)\ldots(p+(p-1))}{1\cdot 2\ldots k\ldots p-1}\\
&= \Big(\frac{p}{1}+1\Big)\Big(\frac{p}{2}+1\Big)\ldots\Big(\frac{p}{k}+1\Big)\ldots\Big(\frac{p}{p-1}+1\Big)\\
&= 1 + \sum_{i=1}^{p-1}\frac{p}{i} + \sum_{1\leqslant i_1<i_2\leqslant p-1}\frac{p^2}{i_1i_2} + \ldots + \sum_{1\leqslant i_1<i_2<\ldots<i_k\leqslant p-1}\frac{p^k}{i_1i_2\ldots i_k}\\
&\quad + \ldots + \frac{p^{p-1}}{(p-1)!} = 1 + \sum_{k=1}^{p-1}p^kH_k = 1 + \sum_{k=1}^{6}p^kH_k + \sum_{k=7}^{p-1}p^kH_k.
\end{aligned}$$

Since by Lemma 2.6, $p^9 \,\Big|\, \sum\limits_{k=7}^{p-1} p^kH_k$ for any prime $p \geqslant 11$, the above identity yields

$$\binom{2p-1}{p-1} \equiv 1 + pH_1 + p^2H_2 + p^3H_3 + p^4H_4 + p^5H_5 + p^6H_6 \pmod{p^8}.$$

Now by Lemma 2.7, for $n = 2, 3, 4, 5, 6$, we have

$$H_n \equiv (-1)^{n-1}\frac{R_n}{n} \pmod{p^{e_n}} \quad \text{for } e_2 = 6,\ e_3 = 5,\ e_4 = 4,\ e_5 = 4 \text{ and } e_6 = 3.$$

Substituting the above congruences into the previous one, and setting $H_1 = R_1$, we obtain

$$\binom{2p-1}{p-1} \equiv 1 + pR_1 - \frac{p^2}{2}R_2 + \frac{p^3}{3}R_3 - \frac{p^4}{4}R_4 + \frac{p^5}{5}R_5 - \frac{p^6}{6}R_6 \pmod{p^8}.$$

This is the desired congruence from Proposition 1.1. □

## 3. Proofs of Proposition 1.2 and Corollaries 1.1–1.3

In order to prove Proposition 1.2 and Corollaries 1.1–1.3, we need some auxiliary results.

**Lemma 3.1.** *Let $p$ be a prime, and let $m$ be any even positive integer. Then the denominator $d_m$ of the Bernoulli number $B_m$, written in reduced form, is given by*

$$d_m = \prod_{p-1\mid m} p,$$

*where the product is taken over all primes $p$ such that $p-1$ divides $m$.*

Proof. The assertion is an immediate consequence of the von Staudt-Clausen theorem (see, e.g., [10], page 233, Theorem 3) which asserts that $B_m + \sum\limits_{p-1\mid m} 1/p$ is an integer for all even $m$, where the summation is over all primes $p$ such that $p-1$ divides $m$. □

Recall that for a prime $p$ and a positive integer $n$, we denote

$$R_n(p) = R_n = \sum_{k=1}^{p-1}\frac{1}{k^n} \quad \text{and} \quad P_n(p) = \sum_{k=1}^{p-1} k^n.$$

**Lemma 3.2** ([9], page 8)**.** *Let $p$ be a prime greater than 5, and let $n, r$ be positive integers. Then*

$$P_n(p) \equiv \sum_{s-\mathrm{ord}_p(s)\leqslant r} \frac{1}{s}\binom{n}{s-1}p^s B_{n+1-s} \pmod{p^r}, \tag{3.1}$$

*where $\mathrm{ord}_p(s)$ is the largest power of $p$ dividing $s$, and the summation is taken over all integers $1 \leqslant s \leqslant n+1$ such that $s - \mathrm{ord}_p(s) \leqslant r$.*

The following result is well known as the Kummer congruences.

**Lemma 3.3** ([10], page 239). *Suppose that $p \geqslant 3$ is a prime and $m$, $n$, $r$ are positive integers such that $m$ and $n$ are even, $r \leqslant n-1 \leqslant m-1$, and $m \not\equiv 0 \pmod{p-1}$. If $n \equiv m \pmod{\varphi(p^r)}$, where $\varphi(p^r) = p^{r-1}(p-1)$ is Euler's totient function, then*

$$\frac{B_m}{m} \equiv \frac{B_n}{n} \pmod{p^r}. \tag{3.2}$$

The following congruences are also due to Kummer.

**Lemma 3.4** ([14]; also see [9], page 20). *Let $p \geqslant 3$ be a prime and let $m$, $r$ be positive integers such that $m$ is even, $r \leqslant m-1$ and $m \not\equiv 0 \pmod{p-1}$. Then*

$$\sum_{k=0}^{r}(-1)^k\binom{m}{k}\frac{B_{m+k(p-1)}}{m+k(p-1)} \equiv 0 \pmod{p^r}. \tag{3.3}$$

**Lemma 3.5.** *For any prime $p \geqslant 11$, we have*

(i) $R_1(p) \equiv -\frac{1}{2}p^2B_{p^4-p^3-2} - \frac{1}{4}p^4B_{p^2-p-4} + \frac{1}{6}p^5B_{p-3} + \frac{1}{20}p^5B_{p-5} \pmod{p^6}$,
(ii) $R_3(p) \equiv -\frac{3}{2}p^2B_{p^4-p^3-4} \pmod{p^4}$,
(iii) $R_4(p) \equiv pB_{p^4-p^3-4} \pmod{p^3}$,
(iv) $pR_6(p) \equiv -\frac{2}{5}R_5(p) \pmod{p^4}$.

Proof. If $s$ is a positive integer such that $\mathrm{ord}_p(s) = e \geqslant 1$, then for $p \geqslant 11$ we have $s - e \geqslant p^e - e \geqslant 10$. This shows that the condition $s - \mathrm{ord}_p(s) \leqslant 6$ implies that $\mathrm{ord}_p(s) = 0$, and thus, $s \leqslant 6$ must hold for such an $s$. Therefore, by Lemma 3.2,

$$P_n(p) \equiv \sum_{s=1}^{6}\frac{1}{s}\binom{n}{s-1}p^sB_{n+1-s} \pmod{p^6} \quad \text{for } n = 1, 2, \ldots. \tag{3.4}$$

By Euler's theorem, for $1 \leqslant k \leqslant p-1$ and positive integers $n$, $e$ we have $1/k^{\varphi(p^e)-n} \equiv k^n \pmod{p^e}$, where $\varphi(p^e) = p^{e-1}(p-1)$ is Euler's totient function. Hence, $R_{\varphi(p^e)-n}(p) \equiv P_n(p) \pmod{p^e}$. In particular, if $n = \varphi(p^6) - 1 = p^5(p-1) - 1$, then by Lemma 3.1, $p^6 \mid p^6B_{p^5(p-1)-6}$ for each prime $p \geqslant 11$. Therefore, using the fact that $B_{p^5(p-1)-1} = B_{p^5(p-1)-3} = B_{p^5(p-1)-5} = 0$, (3.4) yields

$$\begin{aligned} R_1(p) \equiv P_{p^5(p-1)-1}(p) &\equiv \frac{1}{2}(p^5(p-1)-1)p^2B_{p^5(p-1)-2} \\ &+ \frac{1}{4}\frac{(p^5(p-1)-1)(p^5(p-1)-2)(p^5(p-1)-3)}{6}p^4B_{p^5(p-1)-4} \pmod{p^6}, \end{aligned}$$

whence we have

$$R_1(p) \equiv -\frac{p^2}{2} B_{p^6-p^5-2} - \frac{p^4}{4} B_{p^6-p^5-4} \pmod{p^6}. \tag{3.5}$$

By the Kummer congruences (3.2) from Lemma 3.3, we have

$$B_{p^6-p^5-2} \equiv \frac{p^6-p^5-2}{p^4-p^3-2} B_{p^4-p^3-2} \equiv \frac{2B_{p^4-p^3-2}}{p^3+2} \equiv \Big(1-\frac{p^3}{2}\Big) B_{p^4-p^3-2} \pmod{p^4}.$$

Substituting this into (3.5), we obtain

$$R_1(p) \equiv -\frac{p^2}{2} B_{p^4-p^3-2} + \frac{p^5}{4} B_{p^4-p^3-2} - \frac{p^4}{4} B_{p^6-p^5-4} \pmod{p^6}. \tag{3.6}$$

Similarly, we have

$$B_{p^4-p^3-2} \equiv \frac{p^4-p^3-2}{p-3} B_{p-3} \equiv \frac{2}{3} B_{p-3} \pmod{p}$$

and

$$B_{p^6-p^5-4} \equiv \frac{p^6-p^5-4}{p^2-p-4} B_{p^2-p-4} \equiv \frac{4B_{p^2-p-4}}{p+4} \equiv \Big(1-\frac{p}{4}\Big) B_{p^2-p-4} \pmod{p^2}.$$

Substituting the above two congruences into (3.6), we get

$$R_1(p) \equiv -\frac{p^2}{2} B_{p^4-p^3-2} + \frac{p^5}{6} B_{p-3} - \frac{p^4}{4} B_{p^2-p-4} + \frac{p^5}{16} B_{p^2-p-4} \pmod{p^6}. \tag{3.7}$$

Finally, since

$$B_{p^2-p-4} \equiv \frac{p^2-p-4}{p-5} B_{p-5} \equiv \frac{4}{5} B_{p-5} \pmod{p},$$

the substitution of the above congruence into (3.7) immediately gives the congruence (i).

To prove the congruences (ii) and (iii), note that if $n-3 \not\equiv 0 \pmod{p-1}$, then by Lemma 3.1, $p^4 \mid p^4 B_{n-3}$ for odd $n \geqslant 5$, while $B_{n-3} = 0$ for even $n \geqslant 6$. Therefore, reducing the modulus in (3.4) to $p^4$, for all odd $n \geqslant 3$ with $n-3 \not\equiv 0 \pmod{p-1}$ and for all even $n \geqslant 2$ we have

$$P_n(p) \equiv pB_n + \frac{p^2}{2} nB_{n-1} + \frac{p^3}{6} n(n-1) B_{n-2} \pmod{p^4}. \tag{3.8}$$

In particular, for $n = p^4 - p^3 - 3$ we have $B_{p^4-p^3-3} = B_{p^4-p^3-5} = 0$, and thus (3.8) yields

$$R_3(p) \equiv P_{p^4-p^3-3}(p) \equiv \frac{p^2(p^4-p^3-3)}{2} B_{p^4-p^3-4} \equiv -\frac{3p^2}{2} B_{p^4-p^3-4} \pmod{p^4}.$$

Similarly, if $n = p^4 - p^3 - 4$, then since $p^4 - p^3 - 6 \not\equiv 0 \pmod{p-1}$, by Lemma 3.1 we have $p^3 \mid p^3 B_{p^4-p^3-6}$ for each prime $p \geqslant 11$. Using this and the fact that $B_{p^4-p^3-5} = 0$, from (3.8) modulo $p^3$ we find that

$$R_4(p) \equiv P_{p^4-p^3-4}(p) \equiv pB_{p^4-p^3-4} \pmod{p^3}.$$

It remains to show (iv). If $n$ is odd such that $n-3 \not\equiv 0 \pmod{p-1}$, then by (3.8) and Lemma 3.1, $P_n(p) \equiv (n/2)p^2 B_{n-1} \pmod{p^4}$ and $P_{n-1}(p) \equiv pB_{n-1} \pmod{p^3}$. Thus, for such an $n$ we have

$$P_n(p) \equiv \frac{n}{2} p P_{n-1} \pmod{p^4}.$$

In particular, for $n = p^4 - p^3 - 5$, from the above we get

$$\begin{aligned} R_5(p) \equiv P_{p^4-p^3-5}(p) &\equiv \frac{(p^4-p^3-5)p}{2} P_{p^4-p^3-6}(p) \\ &\equiv -\frac{5}{2} p P_{p^4-p^3-6}(p) \equiv -\frac{5}{2} p R_6(p) \pmod{p^4}. \end{aligned}$$

This implies (iv) and the proof is complete. □

**Lemma 3.6.** *For any prime $p$ and any positive integer $r$, we have*

$$2R_1 \equiv -\sum_{i=1}^{r} p^i R_{i+1} \pmod{p^{r+1}}. \tag{3.9}$$

Proof. Multiplying the identity

$$1 + \frac{p}{i} + \ldots + \frac{p^{r-1}}{i^{r-1}} = \frac{p^r - i^r}{i^{r-1}(p-i)}$$

by $-p/i^2$, $1 \leqslant i \leqslant p-1$, we obtain

$$-\frac{p}{i^2}\Big(1 + \frac{p}{i} + \ldots + \frac{p^{r-1}}{i^{r-1}}\Big) = \frac{-p^{r+1} + pi^r}{i^{r+1}(p-i)} \equiv \frac{p}{i(p-i)} \pmod{p^{r+1}}.$$

Therefore,

$$\Big(\frac{1}{i} + \frac{1}{p-i}\Big) \equiv -\Big(\frac{p}{i^2} + \frac{p^2}{i^3} + \ldots + \frac{p^r}{i^{r+1}}\Big) \pmod{p^{r+1}},$$

from which we immediately obtain (3.9) after summing over all $i$ from 1 to $p-1$. □

P r o o f of Proposition 1.2. We begin with the congruence from Proposition 1.1:

$$\binom{2p-1}{p-1} \equiv 1 + pR_1 - \frac{p^2}{2}R_2 + \frac{p^3}{3}R_3 - \frac{p^4}{4}R_4 + \frac{p^5}{5}R_5 - \frac{p^6}{6}R_6 \pmod{p^8}. \tag{3.10}$$

As by Lemma 2.4 we have $p^2 \mid R_7$, Lemma 3.6 with $r = 7$ yields

$$2R_1 \equiv -pR_2 - p^2R_3 - p^3R_4 - p^4R_5 - p^5R_6 \pmod{p^8}, \tag{3.11}$$

and after multiplying by $p/4$ it follows that

$$-\frac{p^4}{4}R_4 \equiv \frac{p}{2}R_1 + \frac{1}{4}(p^2R_2 + p^3R_3 + p^5R_5 + p^6R_6) \pmod{p^8}.$$

Substituting this into the congruence (3.10), we obtain

$$\binom{2p-1}{p-1} \equiv 1 + \frac{3p}{2}R_1 - \frac{p^2}{4}R_2 + \frac{7p^3}{12}R_3 + \frac{9p^5}{20}R_5 + \frac{p^6}{12}R_6 \pmod{p^8}.$$

Further, from (iv) of Lemma 3.5 we see that

$$p^6R_6 \equiv -\frac{2}{5}p^5R_5 \pmod{p^8}.$$

The substitution of this into the previous congruence immediately gives

$$\binom{2p-1}{p-1} \equiv 1 + \frac{3p}{2}R_1 - \frac{p^2}{4}R_2 + \frac{7p^3}{12}R_3 + \frac{5p^5}{12}R_5 \pmod{p^8},$$

as desired. □

**Remark 3.1.** Proceeding in the same way as in the previous proof and using (3.11), we can eliminate $R_2$ to obtain

$$\binom{2p-1}{p-1} \equiv 1 + 2pR_1 + \frac{5p^3}{6}R_3 + \frac{p^4}{4}R_4 + \frac{17p^5}{30}R_5 \pmod{p^8}.$$

**Remark 3.2.** If we suppose that there exists a prime $p$ such that $\binom{2p-1}{p-1} \equiv 1 \pmod{p^5}$, then by Lemma 2.2, for such a $p$ we must have $R_1 \equiv 0 \pmod{p^4}$ and $R_2 \equiv 0 \pmod{p^3}$. Starting with these two congruences, in the same manner as in the proof of Lemma 2.7, it can be deduced that for $n = 2, 3, 4, 5, 6, 7, 8$,

$$H_n \equiv (-1)^{n-1}\frac{R_n}{n} \pmod{p^{e_n}},$$

where $e_2 = 8$, $e_3 = 7$, $e_4 = 6$, $e_5 = 5$, $e_6 = 4$, $e_7 = 3$ and $e_8 = 2$. Since as in the proof of Proposition 1.1 we have

$$\binom{2p-1}{p-1} \equiv 1+pH_1+p^2H_2+p^3H_3+p^4H_4+p^5H_5+p^6H_6+p^7H_7+p^8H_8 \pmod{p^{10}},$$

then substituting the previous congruences into the right hand side of the above congruence and setting $H_1 = R_1$, we obtain

$$\binom{2p-1}{p-1} \equiv 1+pR_1-\frac{p^2}{2}R_2+\frac{p^3}{3}R_3-\frac{p^4}{4}R_4+\frac{p^5}{5}R_5-\frac{p^6}{6}R_6+\frac{p^7}{7}R_7-\frac{p^8}{8}R_8 \pmod{p^{10}}.$$

Since by Lemma 2.4, $p^2 \mid R_7$ and $p \mid R_8$, from the above we get

$$\binom{2p-1}{p-1} \equiv 1+pR_1-\frac{p^2}{2}R_2+\frac{p^3}{3}R_3-\frac{p^4}{4}R_4+\frac{p^5}{5}R_5-\frac{p^6}{6}R_6 \pmod{p^9}.$$

Then as in the above proof, using (3.11) and the fact that by (iv) of Lemma 3.5, $p^6R_6(p) \equiv -(2/5)p^5R_5(p) \pmod{p^9}$, we can find that

$$\binom{2p-1}{p-1} \equiv 1+\frac{3p}{2}R_1-\frac{p^2}{4}R_2+\frac{7p^3}{12}R_3+\frac{5p^5}{12}R_5 \pmod{p^9}.$$

P r o o f of Corollary 1.1. In view of the fact that by Lemma 2.4, $p^2 \mid R_5$, the congruence from Proposition 1.2 immediately yields

$$\binom{2p-1}{p-1} \equiv 1+\frac{3p}{2}R_1-\frac{p^2}{4}R_2+\frac{7p^3}{12}R_3 \pmod{p^7}. \tag{3.12}$$

Lemma 3.6 with $r = 5$ and the fact that by Lemma 2.4, $p^2 \mid R_5$ and $p \mid R_6$ imply

$$2R_1 \equiv -pR_2 - p^2R_3 - p^3R_4 \pmod{p^6}.$$

From (ii) and (iii) of Lemma 3.5 we see that $pR_4 \equiv -\frac{2}{3}R_3 \pmod{p^4}$, so that $p^3R_4 \equiv -\frac{2}{3}p^2R_3 \pmod{p^6}$. Substituting this into the previous congruence, we obtain

$$2R_1 + pR_2 \equiv -\frac{1}{3}p^2R_3 \pmod{p^6},$$

whence we have

$$p^3R_3 \equiv -6pR_1 - 3p^2R_2 \pmod{p^7}. \tag{3.13}$$

Substituting this into (3.12), we get

$$\binom{2p-1}{p-1} \equiv 1 - 2pR_1 - 2p^2R_2 \pmod{p^7},$$

which is actually the first congruence from Corollary 1.1. Finally, from (3.13) we have

$$p^2R_2 \equiv -2pR_1 - \frac{1}{3}p^3R_3 \pmod{p^7},$$

and substituting this into (3.12) gives

$$\binom{2p-1}{p-1} \equiv 1 + 2pR_1 + \frac{2}{3}p^3R_3 \pmod{p^7}. \tag{3.14}$$

This completes the proof. □

P r o o f of Corollary 1.2. By (ii) of Lemma 3.5, we have

$$p^3R_3(p) \equiv -(3/2)p^5B_{p^4-p^3-4} \pmod{p^7}.$$

Substituting this into (3.14), we obtain

$$\binom{2p-1}{p-1} \equiv 1 + 2pR_1 - p^5B_{p^4-p^3-4} \pmod{p^7}. \tag{3.15}$$

By Lemma 2.3, $p \mid B_{p-3}$ so that $p^6 \mid (p^5/6)B_{p-3}$, and hence from (i) of Lemma 3.5 we obtain

$$2pR_1(p) \equiv -p^3B_{p^4-p^3-2} - \frac{p^5}{2}B_{p^2-p-4} + \frac{p^6}{10}B_{p-5} \pmod{p^7}.$$

Furthermore, by the Kummer congruences (3.2), since $p^4 - p^3 - 2 \not\equiv 0 \pmod{p-1}$ and $p^4 - p^3 - 2 \equiv p^2 - p - 2 \pmod{\varphi(p^2)}$, we have

$$B_{p^4-p^3-4} \equiv \frac{p^4-p^3-4}{p^2-p-4}B_{p^2-p-4} \equiv \frac{4}{p+4}B_{p^2-p-4} \equiv \left(1 - \frac{p}{4}\right)B_{p^2-p-4} \pmod{p^2}.$$

The substitution of the above two congruences into (3.15) immediately gives

$$\binom{2p-1}{p-1} \equiv 1 - p^3B_{p^4-p^3-2} - \frac{3p^5}{2}B_{p^2-p-4} + \frac{p^6}{10}B_{p-5} + \frac{p^6}{4}B_{p^2-p-4} \pmod{p^7}. \tag{3.16}$$

Finally, since by the Kummer congruences (3.2),

$$B_{p^2-p-4} \equiv \frac{p^2-p-4}{p-5}B_{p-5} \equiv \frac{4}{5}B_{p-5} \pmod{p},$$

after substitution of this into (3.16) we obtain

$$\binom{2p-1}{p-1} \equiv 1 - p^3B_{p^4-p^3-2} - \frac{3p^5}{2}B_{p^2-p-4} + \frac{3p^6}{10}B_{p-5} \pmod{p^7}. \tag{3.17}$$

This is the required congruence. □

P r o o f of Corollary 1.3. As noticed in [9], congruence (3) on page 494, combining the Kummer congruences (3.2) and (3.3) for $m = \varphi(p^n) - s$, $n, s \in \mathbb{N}$ with $s \not\equiv 0 \pmod{p-1}$, we obtain

$$\frac{B_{p^n - p^{n-1} - s}}{p^n - p^{n-1} - s} \equiv \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{B_{k(p-1)-s}}{k(p-1)-s} \pmod{p^n}. \tag{3.18}$$

Now (3.18) with $n = 2$ and $s = 4$ gives

$$\frac{B_{p^2-p-4}}{p^2-p-4} \equiv \frac{2B_{p-5}}{p-5} - \frac{B_{2p-6}}{2p-6} \pmod{p^2},$$

or equivalently,

$$B_{p^2-p-4} \equiv -\frac{2(p+4)}{p-5} B_{p-5} + \frac{p+4}{2(p-3)} B_{2p-6} \pmod{p^2}.$$

Substituting $1/(p-5) \equiv -(5+p)/25 \pmod{p^2}$ and $1/(p-3) \equiv -(3+p)/9 \pmod{p^2}$, the above congruence becomes

$$B_{p^2-p-4} \equiv \frac{18p+40}{25} B_{p-5} - \frac{7p+12}{18} B_{2p-6} \pmod{p^2}. \tag{3.19}$$

Similarly, (3.18) with $n = 4$ and $s = 2$ yields

$$\frac{B_{p^4-p^3-2}}{p^4-p^3-2} \equiv \sum_{k=1}^{4} (-1)^{k+1} \binom{4}{k} \frac{B_{k(p-1)-2}}{k(p-1)-2} \pmod{p^4},$$

whence, multiplying by $p^3 + 2$, we get

$$\begin{aligned} -B_{p^4-p^3-2} &\equiv (p^3+2)\Big(\frac{4B_{p-3}}{p-3} - \frac{6B_{2p-4}}{2p-4} + \frac{4B_{3p-5}}{3p-5} - \frac{B_{4p-6}}{4p-6}\Big) \\ &\equiv p^3\Big(\frac{4B_{p-3}}{-3} - \frac{6B_{2p-4}}{-4} + \frac{4B_{3p-5}}{-5} - \frac{B_{4p-6}}{-6}\Big) \\ &\quad + 2\Big(\frac{4B_{p-3}}{p-3} - \frac{6B_{2p-4}}{2p-4} + \frac{4B_{3p-5}}{3p-5} - \frac{B_{4p-6}}{4p-6}\Big) \pmod{p^4}. \end{aligned} \tag{3.20}$$

As by the Kummer congruences (3.2),

$$\frac{B_{4p-6}}{4p-6} \equiv \frac{B_{3p-5}}{3p-5} \equiv \frac{B_{2p-4}}{2p-4} \equiv \frac{B_{p-3}}{p-3} \pmod{p},$$

we have

$$B_{4p-6} \equiv 2B_{p-3} \pmod{p}, \quad B_{3p-5} \equiv \frac{5}{3} B_{p-3} \pmod{p}, \quad B_{2p-4} \equiv \frac{4}{3} B_{p-3} \pmod{p}.$$

Substituting this into the first term on the right-hand side in the congruence (3.20), we obtain

$$p^3\Big(\frac{4B_{p-3}}{-3}-\frac{6B_{2p-4}}{-4}+\frac{4B_{3p-5}}{-5}-\frac{B_{4p-6}}{-6}\Big)\equiv-\frac{p^3}{3}B_{p-3}\equiv 0 \pmod{p^4},$$

where we have used the fact that by Lemma 2.3, $p$ divides the numerator of $B_{p-3}$.

Further, as for all integers $a, b, n$ such that $b\not\equiv 0 \pmod p$ we have

$$\frac{1}{ap-b}\equiv-\frac{1}{b}\sum_{k=0}^{3}\frac{a^kp^k}{b^k} \pmod{p^4},$$

applying this to $1/(p-3)$, $1/(2p-4)$ and $1/(3p-5)$, the second term on the right-hand side in the congruence (3.20) becomes

$$\begin{aligned}-B_{p^4-p^3-2}\equiv 2\Big(&-\frac{4}{3}\Big(1+\frac{p}{3}+\frac{p^2}{9}\Big)B_{p-3}+\frac{3}{2}\Big(1+\frac{p}{2}+\frac{p^2}{4}\Big)B_{2p-4}\\&-\frac{4}{5}\Big(1+\frac{3p}{5}+\frac{9p^2}{25}\Big)B_{3p-5}+\frac{1}{6}\Big(1+\frac{2p}{3}+\frac{4p^2}{9}\Big)B_{4p-6}\Big) \pmod{p^4}.\end{aligned}$$

Muptiplying by $p^3$, the above congruence becomes

$$\begin{aligned}-p^3B_{p^4-p^3-2}\equiv&-\frac{8}{3}\Big(p^3+\frac{p^4}{3}+\frac{p^5}{9}\Big)B_{p-3}+3\Big(p^3+\frac{p^4}{2}+\frac{p^5}{4}\Big)B_{2p-4}\\&-\frac{8}{5}\Big(p^3+\frac{3p^4}{5}+\frac{9p^5}{25}\Big)B_{3p-5}+\frac{1}{3}\Big(p^3+\frac{2p^4}{3}+\frac{4p^5}{9}\Big)B_{4p-6} \pmod{p^7}.\end{aligned}$$

Finally, substituting this and the congruence (3.19) into (3.17), we obtain the congruence from Corollary 1.3. □

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*Author's address*: Romeo Meštrović, Maritime Faculty, University of Montenegro, Dobrota 36, 85330 Kotor, Montenegro, e-mail: `romeo@ac.me`.