Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 65 (2015), No. 1, 191–205

Persistent URL: `http://dml.cz/dmlcz/144221`

Czechoslovak Mathematical Journal, 65 (140) (2015), 191–205

GRADUAL DOUBLING PROPERTY OF HUTCHINSON ORBITS

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(Received November 26, 2013)

Abstract. The classical self-similar fractals can be obtained as fixed points of the iteration technique introduced by Hutchinson. The well known results of Mosco show that typically the limit fractal equipped with the invariant measure is a (normal) space of homogeneous type. But the doubling property along this iteration is generally not preserved even when the starting point, and of course the limit point, both have the doubling property. We prove that the elements of Hutchinson orbits possess the doubling property except perhaps for radii which decrease to zero as the step of the iteration grows, and in this sense, we say that the doubling property of the limit is achieved gradually. We use this result to prove the uniform upper doubling property of the orbits.

Keywords: metric space; doubling measure; Hausdorff-Kantorovich metric; iterated function system

MSC 2010: 28A78, 28A75

1. Introduction

Since the earlier works by Coifman and Guzmán in [5] and Coifman and Weiss in [6], the harmonic analysis has a natural environment of the (quasi) metric spaces with doubling measures. There the expression *space of homogeneous type* for this setting seems to be coined. Since the doubling property allows the use of Wiener type covering lemmas, many of the basic results of harmonic analysis have been extended to the setting of spaces of homogeneous type.

On the other hand, the results in [17] show that typically the limit fractal provided by the Hutchinson iteration scheme (see [8]) and equipped with the invariant measure, is a (normal) space of homogeneous type. Some current attempts to extend notions of harmonic analysis and partial differential equations to fractals (see [18],

The research has been supported by CONICET, CAI+D (UNL) and ANPCyT.

[15], [14]) suggest that some results of real and functional analysis on these settings are of interest.

Since the classical self-similar fractals are actually obtained as fixed points of the iteration technique introduced by Hutchinson, one may ask for the preservation of the doubling property along the iteration. If this property is uniformly preserved, one can expect that the behavior of some operators on the limit fractal could be predicted from approximate versions defined on the simpler approximation spaces (see [2]). But the doubling property is generally not preserved by the iteration procedure. In [1] we prove that it may happen that no point, except for the first and last, of the orbit generated by a contraction is a space of homogeneous type, even when the starting point, and of course the limit point, both have the doubling property. On the other hand, the uniqueness of the (Banach) fixed point lead us to the same limit space no matter what is the initial space, and under the assumptions in [17] this limit space is doubling. Hence the question of how suddenly the doubling property of the limit appears seems natural.

In this note we prove that, in a precise sense, the elements of Hutchinson orbits become more and more doubling as the step of the iteration grows (Theorem 3.2), and the doubling property of the limit, in this sense, is gradual (Proposition 3.1). We use this result to prove the uniform upper doubling property of the orbits in Theorem 4.1.

For the sake of notational simplicty we have assumed the same contractivity coefficient for each of the contractive similitudes involved. As can easily be verified, the results in this paper still hold in the case of different coefficients.

In Section 2 we introduce notation and definitions, and some basic results. In Section 3 we find a gradual improvement for the doubling property of the orbit as the iteration step increases in Theorem 3.2, the proof of which is based on the construction of discrete approximations to the attractor. We consider the orbits starting from a mass point space, defined by a finite family of contractive similitudes, and prove the uniform normality, and hence the uniform doubling property, for the whole orbit (Lemmas 3.3 and 3.4). We also state some basic properties of iterated function systems (IFS) in Lemma 3.5. Finally, in Section 4, we use Theorem 3.2 in order to show that the approximating sequence of spaces is a uniform family of upper doubling spaces, in the sense of Hytönen.

2. Notation and basic results

Let us start by describing our general framework. Throughout this note, (X, d) is a given compact metric space. Without loss of generality we will assume that

$\operatorname{diam}(X) = 1$, where $\operatorname{diam}(X) = \sup\{d(x,y)\colon\ x,y \in X\}$. We define an open ball centered at x of radius r to be the set $B(x,r) = \{y \in X\colon\ d(x,y) < r\}$.

Let $\mathcal{K} = \{K \subseteq X\colon\ K \neq \emptyset,\ K \text{ compact}\}$. With $[A]_\varepsilon$ we shall denote the ε-enlargement of the set $A \subseteq X$; i.e. $[A]_\varepsilon = \bigcup_{x\in A} B(x,\varepsilon) = \{y \in X\colon\ d(y,A) < \varepsilon\}$. Here $d(x,A) = \inf\{d(x,y)\colon\ y \in A\}$. Given two sets A and B in $\mathcal{K}$ the Hausdorff distance from A to B is given by

$$d_H(A,B) = \inf\{\varepsilon > 0\colon\ A \subseteq [B]_\varepsilon \text{ and } B \subseteq [A]_\varepsilon\}.$$

Let us now introduce the Kantorovich-Hutchinson distance on the set of all Borel regular probability measures on the metric space (X,d). We denote

$$\mathcal{P} = \{\mu\colon\ \mu \text{ is a non-negative Borel measure on } X \text{ and } \mu(X) = 1\},$$

and Lip_1 as the space of all Lipschitz continuous functions defined on X with Lipschitz constant less than or equal to one, i.e., $f \in \mathrm{Lip}_1$ if and only if $|f(x) - f(y)| \leqslant d(x,y)$ for every $x,y \in X$.

Since (X,d) is compact, $d_K(\mu,\nu) = \sup\{|\int f\,\mathrm{d}\mu - \int f\,\mathrm{d}\nu|\colon\ f \in \mathrm{Lip}_1\}$ gives a distance on $\mathcal{P}$ such that the d_K-convergence of a sequence is equivalent to its weak star convergence to the same limit (see [7] for the Euclidean case and [3] for more general settings).

In [3] the following metric on $\mathcal{K} \times \mathcal{P}$ is considered

$$\delta((Y_1,\mu_1),(Y_2,\mu_2)) = d_H(Y_1,Y_2) + d_K(\mu_1,\mu_2),$$

with $(Y_i,\mu_i) \in \mathcal{K} \times \mathcal{P}$, $i = 1,2$. So $(\mathcal{K} \times \mathcal{P}, \delta)$ becomes a complete metric space. It is also proved that the set

$$\mathcal{E} = \{(Y,\mu) \in \mathcal{K} \times \mathcal{P}\colon\ \operatorname{supp}(\mu) \subseteq Y\}$$

is a closed subset of $\mathcal{K} \times \mathcal{P}$, so that $(\mathcal{E},\delta)$ is a complete metric space.

Throughout this paper, a finite set $\Phi = \{\varphi_i\colon\ X \to X,\ i = 1,2,\ldots,M\}$ of contractive similitudes with the same contractivity coefficient is given. Precisely, each φ_i satisfies

$$d(\varphi_i(x),\varphi_i(y)) = \frac{1}{a} d(x,y)$$

for every $x,y \in X$ and some $a > 1$. The constant $1/a$ is called the *contractivity coefficient*.

We consider the transformation T induced by Φ, defined on $(Y,\mu) \in \mathcal{E}$ for $T(Y,\mu) = (T_1(Y), T_2(\mu)) = (Y', \mu')$ where

$$Y' = \bigcup_{i=1}^{M} \varphi_i(Y),$$

and

$$\mu'(B) = \frac{1}{M}\sum_{i=1}^{M} \mu(\varphi_i^{-1}(B)),$$

for every Borel subset B of X. It is easy to see that $T\colon \mathcal{E} \to \mathcal{E}$ is a δ-contraction.

From the (Banach) fixed point theorem we have that any δ-contractive mapping $T\colon \mathcal{E} \to \mathcal{E}$ has a unique fixed point. Moreover, the fixed point can be achieved as the limit for $n \to \infty$ of the n-th iteration T^n of T starting at any initial point $(Y_0, \mu_0) \in \mathcal{E}$.

Let us write (Y_∞, μ_∞) to denote the unique limit point of $T^n(Y_0, \mu_0)$, which depends only on T but not on the starting space (Y_0, μ_0). The fractal set Y_∞ is called the *attractor* of the system Φ, and is the only compact set in X satisfying

$$Y_\infty = \bigcup_{i=1}^{M} \varphi_i(Y_\infty).$$

On the other hand, μ_∞ is called the *invariant measure* and is the only probability Borel measure supported in Y_∞ such that

$$\mu_\infty(B) = \frac{1}{M}\sum_{i=1}^{M} \mu_\infty(\varphi_i^{-1}(B)),$$

for every Borel set B (see [7], [8]).

We shall say that the system $\Phi = \{\varphi_1, \ldots, \varphi_M\}$ satisfies the *open set condition* (OSC) if there exists a nonempty open set $U \subset X$ such that

$$\bigcup_{i=1}^{M} \varphi_i(U) \subseteq U,$$

and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ if $i \neq j$. We shall say that U is an *open set for the OSC* for Φ (see for example [8], [7] and [16]).

The basic examples of IFS are the systems generating the most classical and best-known fractal sets, such as the ternary Cantor set, the Sierpinski gasket and carpet, and the von Koch snowflake. For example, in the case of the ternary Cantor set C the system Φ is defined on $X = [0,1]$ equipped with the Euclidean distance $d(x,y) = |x-y|$, and consists of $M = 2$ contractive similitudes with contractivity coefficient equal to $1/3$ ($a = 3$). More precisely, $\varphi_1(x) = x/3$ and $\varphi_2(x) = x/3 + 2/3$. Also $U = (0,1)$ is an open set for the OSC for Φ. If we take $Y_0 = [0,1]$ and $\mu_0 =$ Lebesgue measure on Y_0, we have that $T_1(Y_0) = [0,1/3] \cup [2/3,1]$, $T_1^2(Y_0) = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$, and in general, $T_1^n(Y_0)$ is the union of 2^n disjoint intervals of the n-th step in the usual construction of the Cantor set. Denoting this union by C_n, we have that $T_2^n(\mu_0)$ coincides with the uniform measure on C_n normalized to a probability measure. But if we now take $Y_0 = \{0\}$ and μ_0 to be the Dirac delta concentrated at 0, then $T_1^n(Y_0)$ is the collection L_n of all the left points of each interval in C_n, and $T_2^n(\mu_0)$ is the counting measure on L_n divided by 2^n. As we have already mentioned, the fixed point depends only on T but not on the starting space (Y_0, μ_0). In this case the limit (Y_∞, μ_∞) is the Cantor set C with the s-dimensional Hausdorff measure on C, where $s = \log 2/\log 3$ (see [7], [8]).

Then we consider some subclasses of $\mathcal{E}$ introduced in [1].

Definition 2.1. Given $(Y,\mu) \in \mathcal{E}$, we say that (Y,μ) is a *space of homogeneous type*, or that μ is a *doubling measure* on Y, if there exists a constant $A \geqslant 1$ such that the inequalities

$$0 < \mu(B(y,2r)) \leqslant A\mu(B(y,r)) \tag{2.1}$$

hold for every $y \in Y$ and every $r > 0$. When $(Y,\mu) \in \mathcal{E}$ satisfies (2.1) we shall write $(Y,\mu) \in \mathcal{D}_A$ to keep record of the quantitative parameter of the doubling property. Set $\mathcal{D} = \bigcup\limits_{A \geqslant 1} \mathcal{D}_A$.

We make some remarks concerning the classes $\mathcal{D}_A$ defined above. First let us observe that $\mathcal{D}_1$ is not empty, since every set consisting of a single point equipped with any metric and with the counting measure, belongs to $\mathcal{D}_1$. Notice also that if $A_1 \leqslant A_2$ then $\mathcal{D}_{A_1} \subseteq \mathcal{D}_{A_2}$. Finally let us point out that if $(Y,\mu) \in \mathcal{D}_A$ then $\operatorname{supp}(\mu) = Y$. In fact, since $(Y,\mu) \in \mathcal{E}$ we have $\operatorname{supp}(\mu) \subseteq Y$. On the other hand, for $y \in Y \setminus \operatorname{supp}(\mu)$ there exists an open set G containing y with $\mu(G) = 0$. So for some ball B in Y we should have $\mu(B) = 0$, which is impossible.

It is proved in [17] that, under the open set condition for the system Φ, the limit set (attractor) equipped with the invariant measure and the usual Euclidean distance, is a (normal or Ahlfors regular) space of homogeneous type. With our notation $(Y_\infty, \mu_\infty) \in \mathcal{D}$. However, the examples in [1] show that it may happen that the only points in the orbit satisfying the doubling property are (Y_0, μ_0) and (Y_∞, μ_∞), but no other $(Y_n, \mu_n) := T^n(Y_0, \mu_0)$ is a space of homogeneous type. These examples seem to suggest that, if we take a sequence $\{\varepsilon_n\}$ whose elements tend to zero when $n \to \infty$, then there exists a constant $A \geqslant 1$ such that (2.1) holds for every n, every $y \in Y_n$ and every $r \geqslant \varepsilon_n$. This leads us to define the ε-*doubling* condition for a measure and consequently another subclass of $\mathcal{E}$.

Definition 2.2. For $\varepsilon > 0$, we shall denote by $\mathcal{D}_A^\varepsilon$ the class of all those couples $(Y,\mu) \in \mathcal{E}$ for which (2.1) holds for every $y \in Y$ and every $r \geqslant \varepsilon$. Set $\mathcal{D}^\varepsilon = \bigcup\limits_{A \geqslant 1} \mathcal{D}_A^\varepsilon$.

We shall now give a representative example of the above definition.

Example 2.3. Take $Y = [0,1]$ and $\mu(E) = 1/10 \ \mathrm{card}\{E \cap Z\}$, where the set Z is defined by $Z = \{j/10,\ j = 0,1,\dots,9\}$. It is easy to see that $(Y,\mu) \notin \mathcal{D}$ since $\mu(B(1/20,1/20)) = 0$ but $\mu(B(1/20,1/10)) = 1/5$. Nevertheless, (2.1) becomes true for every $y \in Y$, taking $r \geqslant 1/10$.

3. Uniform gradual doubling orbits starting at any point of $\mathcal{E}$

When the elements of the approximating sequence become more and more doubling in a sense that will be made precise, below, then the doubling property of the limit (Y_∞,μ_∞) does not appear suddenly but naturally. This result is stated in the next proposition.

Proposition 3.1. *Let* $((Y_n,\mu_n))_{n\geqslant 1}$ *be a sequence in* $\mathcal{E}$ *such that* $(Y_n,\mu_n) \in \mathcal{D}_A^{\varepsilon_n}$ *for some sequence* $\varepsilon_n \to 0$ *when* $n \to \infty$. *If* $(Y_n,\mu_n) \xrightarrow{\delta} (Y,\mu)$ *then* $(Y,\mu) \in \mathcal{D}_{A^4}$.

Proof. Take $y \in Y$ and $r > 0$. Let ψ be the continuous function defined on $\mathbb{R}_0^+$ by $\psi \equiv 1$ on $[0,1]$, $\psi \equiv 0$ on $[2,\infty)$, and by assuming it to be linear on $[1,2]$. For $t > 0$ we denote $\psi_{y,t}(x) = \psi(d(y,x)/t)$ for $x \in X$. Since $Y_n \xrightarrow{d_H} Y$, we can choose $y_n \in Y_n$ such that $d(y_n,y) \to 0$ when $n \to \infty$. Then, since there exists n_0 such that $y_n \in B(y,r/16)$ and $\varepsilon_n < 5r/16$ for every $n \geqslant n_0$, we have

$$\begin{aligned}
\mu(B(y,2r)) &\leqslant \int \psi_{y,2r}(x)\,\mathrm{d}\mu(x) = \lim_{n\to\infty} \int \psi_{y,2r}(x)\,\mathrm{d}\mu_n(x) \\
&\leqslant \liminf_{n\to\infty} \mu_n(B(y,4r)) \leqslant \liminf_{n\to\infty} \mu_n(B(y_n,5r)) \\
&\leqslant \liminf_{n\to\infty} A^4 \mu_n\Big(B\Big(y_n,\frac{5r}{16}\Big)\Big) \leqslant A^4 \liminf_{n\to\infty} \mu_n\Big(B\Big(y,\frac{r}{2}\Big)\Big) \\
&\leqslant A^4 \lim_{n\to\infty} \int \psi_{y,r/2}(x)\,\mathrm{d}\mu_n(x) = A^4 \int \psi_{y,r/2}(x)\,\mathrm{d}\mu(x) \\
&\leqslant A^4 \mu(B(y,r)).
\end{aligned}$$

□

Let us state the additional hypothesis that we are assuming for the results in the remainder of this section. We will assume that (X, d) has furthermore *finite metric* (or *Assouad*) *dimension*. This means that there exists a constant $N \in \mathbb{N}$, called a constant for the Assouad dimension of X, such that for every $x \in X$, every $r > 0$ and every r-disperse subset E of X, we have that $\operatorname{card}(E \cap B(x, 2r)) \leqslant N$. A set E is said to be *r-disperse* if $d(x, y) \geqslant r$ for every $x, y \in E$, $x \neq y$. If (X, d) has finite metric dimension, then every r-disperse subset of X has at most N^m points in each ball of radius $2^m r$, with m a positive integer (see [6] and [4]).

The following theorem is the main result of our paper. This theorem proves that the elements of Hutchinson orbits generated by a transformation induced by IFS become uniformly more and more doubling in the following sense: the doubling property is satisfied with the same doubling constant except perhaps for radii which decrease to zero as the step of the iteration grows.

Theorem 3.2. *Let $\Phi = \{\varphi_1, \dots, \varphi_M\}$ be a family of contractive similitudes on X with the same contractivity coefficient $1/a$ and satisfying the OSC. Let U be an open set for the OSC of Φ. Let $(Y_0, \mu_0) \in \mathcal{E}$ be such that $Y_0 \subseteq U$, and for each non-negative integer n, set $(Y_n, \mu_n) = T^n(Y_0, \mu_0)$. Then there exists a constant $A \geqslant 1$ such that $(Y_n, \mu_n) \in \mathcal{D}_A^{5a^{-n}}$ for each n.*

Notice that the above theorem and Proposition 3.1 with $\varepsilon_n = 5a^{-n}$ show that, even when no point of the orbit is a space of homogeneous type, the doubling property of the limit space (Y_∞, μ_∞) does not appear suddenly.

In order to prove Theorem 3.2, we shall use the following three lemmas. In Lemmas 3.3 and 3.4 we assume the hypothesis $(Y_0, \mu_0) \in \mathcal{E}$ with $Y_0 \subseteq U$ and use the definitions given below.

Let us fix $u \in Y_0$, and for each non-negative integer n set

$$T^n(\{u\}, \lambda_u) = (\Delta_n, \nu_n),$$

where λ_u is the unit mass at u. So Δ_n has M^n elements and for every $x \in \Delta_n$ we have $\nu_n(\{x\}) = M^{-n}$. In other words, ν_n is the measure on X counting the points of Δ_n, normalized to a probability measure.

Notice that due to the OSC each Y_n can be written as a disjoint union of M^n Borel pieces $Y_n^{\boldsymbol{i}}$, where $\boldsymbol{i} \in \{1, 2, \dots, M\}^n$. Also $\Delta_n \subseteq Y_n$ and $\operatorname{card}(\Delta_n \cap Y_n^{\boldsymbol{i}}) = 1$ for every $\boldsymbol{i} \in \{1, 2, \dots, M\}^n$.

Lemma 3.3. *There exists a constant $C \geqslant 1$, depending on $\varrho = \operatorname{dist}(Y_0, \partial U) > 0$, such that the inequalities*

$$C^{-1} r^s \leqslant \nu_n(B(x,r)) \leqslant C r^s$$

hold for every $r \geqslant \varrho a^{-n}$, every $x \in \Delta_n$ and every $n \in \mathbb{N}$, where $s = \log_a M$.

The above lemma states that each (Δ_n, ν_n) is an Ahlfors s-regular space for every $r \geqslant \varrho a^{-n}$ with a constant which does not depend on n.

Lemma 3.4. *The elements of the sequence $((\Delta_n, \nu_n))_{n \geqslant 1}$ are uniform spaces of homogeneous type. In other words, there exists a constant $\widetilde{A} \geqslant 1$ such that $((\Delta_n, \nu_n))_{n \geqslant 1} \subseteq \mathcal{D}_{\widetilde{A}}$.*

Finally we shall state some basic results about IFS. Given $\boldsymbol{i} = (i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, M\}^k$ we denote by $\boldsymbol{\varphi_i}$ the composition $\varphi_{i_k} \circ \varphi_{i_{k-1}} \circ \ldots \circ \varphi_{i_2} \circ \varphi_{i_1}$. Also, if $i_0 \in \{1, 2, \ldots, M\}$ we write $\boldsymbol{i}' = (i_0, \boldsymbol{i})$ to denote the $(k+1)$-tuple $(i_0, i_1, i_2, \ldots, i_k)$.

Lemma 3.5. *With U an open set for the OSC for Φ, we have*

(a) *if $\boldsymbol{i}, \boldsymbol{j} \in \{1, 2, \ldots, M\}^k$ and $\boldsymbol{i} \neq \boldsymbol{j}$, then $\boldsymbol{\varphi_i}(U) \cap \boldsymbol{\varphi_j}(U) = \emptyset$;*

(b) *if $\boldsymbol{i} = (i, \boldsymbol{i}')$ with $\boldsymbol{i}' \in \{1, 2, \ldots, M\}^k$ and $i \in \{1, 2, \ldots, M\}$, then $\boldsymbol{\varphi_i}(U) \subseteq \boldsymbol{\varphi_{i'}}(U)$;*

(c) *if $\boldsymbol{i}'$ and $\boldsymbol{j}'$ are two different elements in $\{1, 2, \ldots, M\}^k$ and $\boldsymbol{i} = (i, \boldsymbol{i}')$ where $i \in \{1, 2, \ldots, M\}$, then $\boldsymbol{\varphi_i}(U) \cap \boldsymbol{\varphi_{j'}}(U) = \emptyset$;*

(d) *for any fixed $u \in U$ and each positive integer n, if we define*

$$\Delta_n = \{\boldsymbol{\varphi_j}(u) \colon \boldsymbol{j} \in \{1, 2, \ldots, M\}^n\},$$

then we have

$$\operatorname{card}(\boldsymbol{\varphi_l}(U) \cap \Delta_n) = M^{n-k}$$

for every $k \leqslant n$ and every $\boldsymbol{l} \in \{1, 2, \ldots, M\}^k$.

The proof of Lemma 3.5 is contained in [1]. We shall postpone the proofs of Lemmas 3.3 and 3.4 and give the proof of the theorem.

Proof of Theorem 3.2. Let $(Y_0, \mu_0) \in \mathcal{E}$ be such that $Y_0 \subseteq U$. Notice that

$$\mu_n(Y_n^{\boldsymbol{j}}) = \mu_n(\boldsymbol{\varphi_j}(Y_0)) = M^{-n}$$

for every $n \in \mathbb{N}$ and every $\boldsymbol{j} \in \{1, 2 \ldots, M\}^n$. In fact, for a fixed $\boldsymbol{j} \in \{1, 2 \ldots, M\}^n$ we have

$$\begin{aligned}\mu_n(\boldsymbol{\varphi_j}(Y_0)) &= M^{-n} \sum_{\boldsymbol{i} \in \{1,2\ldots,M\}^n} \mu_0(\boldsymbol{\varphi_i}^{-1}(\boldsymbol{\varphi_j}(Y_0))) \\ &= M^{-n}\mu_0(Y_0) + M^{-n} \sum_{\substack{\boldsymbol{i} \in \{1,2\ldots,M\}^n \\ \boldsymbol{i} \neq \boldsymbol{j}}} \mu_0(\boldsymbol{\varphi_i}^{-1}(\boldsymbol{\varphi_j}(Y_0)).\end{aligned}$$

Since $\mu_0(Y_0) = 1$ and $\boldsymbol{\varphi_i}^{-1}(\boldsymbol{\varphi_j}(Y_0)) = \emptyset$ for every choice of $\boldsymbol{i} \neq \boldsymbol{j}$ (see Lemma 3.5 (a)), we have the claim.

Fix $n \in \mathbb{N}$, $y \in Y_n$ and $r \geqslant 5a^{-n}$. There exists one and only one $\boldsymbol{i} \in \{1, \ldots, M\}^n$ such that $y \in Y_n^{\boldsymbol{i}}$. Let us write $x_n^{\boldsymbol{i}}$ to denote the unique point in $\Delta_n \cap Y_n^{\boldsymbol{i}}$. Then $d(y, x_n^{\boldsymbol{i}}) \leqslant a^{-n}$. For $t > 2a^{-n}$ denote

$$B_k = B(x_n^{\boldsymbol{i}}, t + (k-2)a^{-n}),$$

$k = 0, 1, 3, 4$. Notice that

$$B_1 \subseteq B(y, t) \subseteq B_3,$$

and then

$$\mu_n(B_1) \leqslant \mu_n(B(y, t)) \leqslant \mu_n(B_3).$$

We claim that the comparison of the measure μ_n with the counting measure ν_n on Δ_n is

$$\mu_n(B_1) \geqslant \nu_n(B_0) \quad \text{and} \quad \mu_n(B_3) \leqslant \nu_n(B_4). \tag{3.1}$$

If the claim holds, then

$$\nu_n(B_0) \leqslant \mu_n(B(y, t)) \leqslant \nu_n(B_4)$$

for every $y \in Y_n^{\boldsymbol{i}}$ and $t > 2a^{-n}$. Let $\widetilde{A} \geqslant 1$ be a constant such that $\{(\Delta_n, \nu_n) \colon n \in \mathbb{N}\} \subseteq \mathcal{D}_{\widetilde{A}}$ (see Lemma 3.4). Then

$$\begin{aligned}\mu_n(B(y, 2r)) &\leqslant \nu_n(B(x_n^{\boldsymbol{i}}, 2r + 2a^{-n})) \\ &\leqslant \widetilde{A}^2 \nu_n(B(x_n^{\boldsymbol{i}}, (r + a^{-n})/2)) \\ &\leqslant \widetilde{A}^2 \mu_n(B(y, (r + 5a^{-n})/2)) \\ &\leqslant \widetilde{A}^2 \mu_n(B(y, r)),\end{aligned}$$

and the result holds with $A = \widetilde{A}^2$. Then it only remains to prove the inequalities contained in (3.1). To show the first we define the set

$$\mathcal{J} = \{\boldsymbol{j} \in \{1, \dots, M\}^n \colon\ Y_n^{\boldsymbol{j}} \subseteq B_1\}.$$

Let us prove that if $x_n^{\boldsymbol{j}} \in B_0 \cap \Delta_n$ then $\boldsymbol{j} \in \mathcal{J}$. Since $x_n^{\boldsymbol{j}} \in B_0$ we have that $d(x_n^{\boldsymbol{j}}, x_n^{\boldsymbol{i}}) < t - 2a^{-n}$. To see that $Y_n^{\boldsymbol{j}} \subseteq B_1$ fix $z \in Y_n^{\boldsymbol{j}}$. Since $\operatorname{diam}(Y_n^{\boldsymbol{j}}) = a^{-n}$ we have that $d(z, x_n^{\boldsymbol{j}}) \leqslant a^{-n}$. Then

$$d(z, x_n^{\boldsymbol{i}}) \leqslant d(z, x_n^{\boldsymbol{j}}) + d(x_n^{\boldsymbol{j}}, x_n^{\boldsymbol{i}}) < a^{-n} + t - 2a^{-n} = t - a^{-n},$$

and hence $Y_n^{\boldsymbol{j}} \subseteq B_1$. So

$$\mu_n(B_1) \geqslant \sum_{\boldsymbol{j} \in \mathcal{J}} \mu_n(Y_n^{\boldsymbol{j}}) = \sum_{\boldsymbol{j} \in \mathcal{J}} M^{-n} = \sum_{\boldsymbol{j} \in \mathcal{J}} \nu_n(\{x_n^{\boldsymbol{j}}\}) \geqslant \nu_n(B_0).$$

To prove the second inequality let us now define the set

$$\mathcal{Q} = \{\boldsymbol{q} \in \{1, \dots, M\}^n \colon\ Y_n^{\boldsymbol{q}} \cap B_3 \neq \emptyset\}.$$

Observe that if $\boldsymbol{q} \in \mathcal{Q}$ then $Y_n^{\boldsymbol{q}} \subseteq B_4$. In fact, if $\boldsymbol{q} \in \mathcal{Q}$ there exists $z_n^{\boldsymbol{q}} \in Y_n^{\boldsymbol{q}} \cap B(x_n^{\boldsymbol{i}}, t + a^{-n})$. Then for every $z \in Y_n^{\boldsymbol{q}}$ we have

$$d(z, x_n^{\boldsymbol{i}}) \leqslant d(z, z_n^{\boldsymbol{q}}) + d(z_n^{\boldsymbol{q}}, x_n^{\boldsymbol{i}}) < a^{-n} + t + a^{-n} = t + 2a^{-n},$$

and then $z \in B_4$. Hence

$$\mu_n(B_3) \leqslant \sum_{\boldsymbol{q} \in \mathcal{Q}} \mu_n(Y_n^{\boldsymbol{q}}) = \sum_{\boldsymbol{q} \in \mathcal{Q}} \nu_n(\{x_n^{\boldsymbol{q}}\}) \leqslant \nu_n(B_4),$$

as desired. □

Proof of Lemma 3.3. Fix $n \in \mathbb{N}$, $x \in \Delta_n = \{\boldsymbol{\varphi_i}(u) \colon\ \boldsymbol{i} \in \{1, 2, \dots, M\}^n\}$ and $r \geqslant \varrho a^{-n}$, where $\varrho = \operatorname{dist}(Y_0, \partial U) > 0$. Let us start with the following two remarks. The first is that if $\boldsymbol{l} \in \{1, 2, \dots, M\}^k$ and $k \leqslant n$, then from Lemma 3.5 (d) we have

$$\nu_n(\boldsymbol{\varphi_l}(U)) = M^{-n} \operatorname{card}(\boldsymbol{\varphi_l}(U) \cap \Delta_n) = M^{-k} = a^{-ks},$$

with $s = \log_a M$.

The second remark is that the OSC implies that Δ_n is a ϱa^{-n}-disperse set. In fact, take $\boldsymbol{j}, \boldsymbol{i} \in \{1, \dots, M\}^n$ with $\boldsymbol{j} \neq \boldsymbol{i}$, and set $x_{n,\boldsymbol{j}} = \boldsymbol{\varphi_j}(u)$ and $x_{n,\boldsymbol{i}} = \boldsymbol{\varphi_i}(u)$. Since U is an open set, we have that $B(u, \varrho) \subseteq U$. Then

$$B(x_{n,\boldsymbol{j}}, \varrho a^{-n}) = \boldsymbol{\varphi_j}(B(u, \varrho)) \subseteq \boldsymbol{\varphi_j}(U),$$
$$B(x_{n,\boldsymbol{i}}, \varrho a^{-n}) = \boldsymbol{\varphi_i}(B(u, \varrho)) \subseteq \boldsymbol{\varphi_i}(U),$$

and since $\boldsymbol{\varphi_j}(U)$ and $\boldsymbol{\varphi_i}(U)$ are disjoint, we have $B(x_{n,\boldsymbol{j}}, \varrho a^{-n}) \cap B(x_{n,\boldsymbol{i}}, \varrho a^{-n}) = \emptyset$. This implies that $d(x_{n,\boldsymbol{j}}, x_{n,\boldsymbol{i}}) \geqslant \varrho a^{-n}$.

Assume first that $r > a^{-n}$. Set k to denote the only integer less than or equal to n for which $a^{-k} < r \leqslant a^{-k+1}$. For the lower bound, with $x = \boldsymbol{\varphi_i}(u)$, $\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$ and $\boldsymbol{i}' = (i_{n-k+1}, i_{n-k+2}, \ldots, i_n)$ we have

$$\boldsymbol{\varphi}_{\boldsymbol{i}'}(U) \cap \Delta_n \subseteq B(x,r) \cap \Delta_n.$$

In fact, if $y \in \boldsymbol{\varphi}_{\boldsymbol{i}'}(U) \cap \Delta_n$ then $y = \boldsymbol{\varphi_l}(u)$, where

$$\boldsymbol{l} = (l_1, l_2, \ldots, l_{n-k}, i_{n-k+1}, i_{n-k+2}, \ldots, i_n),$$

for some $l_1, l_2, \ldots, l_{n-k} \in \{1, 2, \ldots, M\}$. Then

$$d(x,y) \leqslant a^{-k} < r.$$

Hence

$$\nu_n(B(x,r)) \geqslant \nu_n(\boldsymbol{\varphi}_{\boldsymbol{i}'}(U)) = a^{-ks} \geqslant a^{-s} r^s.$$

For the upper bound, we define

$$\mathcal{J} = \{\boldsymbol{j} \in \{1, 2, \ldots, M\}^k \colon\ B(x,r) \cap \boldsymbol{\varphi_j}(U) \neq \emptyset\}.$$

Since $\{\boldsymbol{\varphi_j}(U),\ \boldsymbol{j} \in \{1, 2, \ldots, M\}^k\}$ is a covering of Δ_n we have

$$B(x,r) \cap \Delta_n \subseteq \bigcup_{\boldsymbol{j} \in \mathcal{J}} \boldsymbol{\varphi_j}(U).$$

From the first remark at the beginning of the proof,

$$\nu_n(B(x,r)) \leqslant \sum_{\boldsymbol{j} \in \mathcal{J}} \nu_n(\boldsymbol{\varphi_j}(U)) = \operatorname{card}(\mathcal{J}) a^{-ks} \leqslant \operatorname{card}(\mathcal{J}) r^s.$$

We only have to show that $\operatorname{card}(\mathcal{J})$ is bounded by a constant which does not depend on x and r. In order to prove it, let us identify each $\boldsymbol{j} \in \mathcal{J}$ with the point $\boldsymbol{\varphi_j}(u) \in \boldsymbol{\varphi_j}(U)$, and let us define the set $\mathcal{A} = \{\boldsymbol{\varphi_j}(u) \colon\ \boldsymbol{j} \in \mathcal{J}\}$. Since $\boldsymbol{\varphi_j}(U)$ are pairwise disjoint for $\boldsymbol{j}$ ranging on the set of indices with fixed length, we have that $\operatorname{card}(\mathcal{J}) = \operatorname{card}(\mathcal{A})$. Notice that $\mathcal{A} \subseteq B(x, 2r)$. In fact, if $\boldsymbol{j} \in \mathcal{J}$ then there exists $y \in B(x,r) \cap \boldsymbol{\varphi_j}(U)$, and

$$d(\boldsymbol{\varphi_j}(u), x) \leqslant d(\boldsymbol{\varphi_j}(u), y) + d(y, x) < a^{-k} + r \leqslant 2r.$$

Since, being a subset of Δ_k, the set $\mathcal{A}$ is ϱa^{-k}-disperse, we have that

$$\operatorname{card}(\mathcal{A}) = \operatorname{card}(B(x, 2r) \cap \mathcal{A}) \leqslant \operatorname{card}(B(x, 2a^{-k+1}) \cap \mathcal{A}) \leqslant N^l,$$

where l is a positive integer such that $2^l \geqslant 2a/\varrho$ and N is a constant for the finite Assouad dimension of X.

Let us finally check the case $\varrho a^{-n} \leqslant r \leqslant a^{-n}$. Notice first that

$$\nu_n(B(x,r)) \geqslant \nu_n(B(x,\varrho a^{-n})) \geqslant M^{-n} = a^{-ns} \geqslant r^s,$$

and on the other hand,

$$\begin{aligned}\nu_n(B(x,r)) &\leqslant \nu_n(B(x,a^{-n})) = M^{-n}\mathrm{card}(\Delta_n \cap B(x,a^{-n})) \\ &\leqslant N^{l-1}a^{-ns} \leqslant N^{l-1}\varrho^{-s}r^s,\end{aligned}$$

with l and N as before.

Hence the result holds with $C = N^l \varrho^{-s}$. $\square$

P r o o f of Lemma 3.4. Fix $n \in \mathbb{N}$, $x \in \Delta_n$ and $r > 0$. If $2r < \varrho a^{-n}$, since Δ_n is ϱa^{-n}-disperse (see proof of Lemma 3.3) we have that $B(x,2r)\cap\Delta_n = B(x,r)\cap\Delta_n = \{x\}$ and the result holds taking $\widetilde{A} = 1$. Otherwise, if $2r \geqslant \varrho a^{-n}$, we shall consider the cases $r \geqslant \varrho a^{-n}$ and $r < \varrho a^{-n}$ and in both cases we shall use Lemma 3.3. In the first case we obtain

$$\nu_n(B(x,2r)) \leqslant C2^s r^s \leqslant C^2 2^s \nu_n(B(x,r)).$$

On the other hand, when $r < \varrho a^{-n} \leqslant 2r$, since $a^s = M$ we have

$$\nu_n(B(x,2r)) \leqslant C2^s r^s < C2^s\varrho^s a^{-ns} = C2^s\varrho^s M^{-n} = C2^s\varrho^s\nu_n(B(x,r)).$$

Hence the lemma holds with $\widetilde{A} = C2^s\varrho^s$. $\square$

As we already mentioned, for the sake of notational simplicity we have assumed the same contractivity coefficient $1/a$ for all contractive similitude φ_i, for $i = 1,\ldots,M$. Nevertheless, the results in this paper still hold in the case of different coefficients $1/a_i$, with $a_i > 1$ for every i. In this case, Lemma 3.3 holds with $a_{\max}$ instead of a, where $a_{\max} := \max\limits_{1\leqslant i\leqslant M} a_i$, proof of Lemma 3.4 follows the same lines, and Theorem 3.2 holds with $a_{\min}$ instead of a, where $a_{\min} := \min\limits_{1\leqslant i\leqslant M} a_i$.

4. Uniform upper doubling property of the orbits

A new class of metric measure spaces is introduced in [9], which generalizes the spaces of homogeneous type as well as power-bounded measures on $\mathbb{R}^n$, have been in the centre of the recent developments of non-doubling harmonic analysis theory.

Given a metric space (X,d) and $Y \subseteq X$, a Borel measure μ on Y is called *upper doubling* if there exist a function $\lambda\colon Y \times \mathbb{R}^+ \to \mathbb{R}^+$ and a constant C_λ such that

$$\begin{aligned} \lambda(y,r) &\leqslant \lambda(y,s), \\ \lambda(y,2r) &\leqslant C_\lambda \lambda(y,r), \\ \mu(B(y,r)) &\leqslant \lambda(y,r), \end{aligned}$$

for every $y \in Y$, $r > 0$ and $s \geqslant r$. The function λ is called a *dominating function*, and we say that (Y,μ) is an *upper doubling space*.

Then, a doubling measure is a special case of upper doubling measure, with $\lambda(x,r) = \mu(B(x,r))$. On the other hand, non-doubling harmonic analysis has recently been developed in the spaces $(\mathbb{R}^n,\mu)$ with $\mu(B(x,r)) \leqslant Cr^t$ for some $t \in (0,n]$, which are upper doubling spaces with the dominating function $\lambda(x,r) = Cr^t$.

The interest in upper doubling spaces has been growing during the last few years because these spaces provide an adequate framework for an abstract extension of results of harmonic and real analysis (see for example [9], [10], [11], [12] and [13]).

In this section we shall apply Theorem 3.2 in order to show that the orbits are uniformly upper doubling spaces. We shall keep the assumption of finite metric dimension of (X,d) and the hypothesis of Theorem 3.2, and as before we shall use (Y_n,μ_n) to denote the space $T^n(Y_0,\mu_0)$.

For a fixed n, we define a dominating function $\lambda_n\colon Y_n \times \mathbb{R}^+ \to \mathbb{R}^+$ in the following way:

$$\lambda_n(y,r) = \begin{cases} \mu_n(B(y,r)), & \text{if } r \geqslant 5a^{-n}; \\ \mu_n(B(y,5a^{-n})), & \text{if } r < 5a^{-n}. \end{cases}$$

The following result states that the orbits (Y_n,μ_n) are uniformly upper doubling spaces. Here the "uniformity" refers to the doubling constant C_{λ_n} for the dominating function λ_n.

Theorem 4.1. *The sequence $((Y_n,\mu_n))_{n\geqslant 1}$ is a uniform family of upper doubling spaces, in the sense that the doubling constant of all dominating functions is the constant A of Theorem 3.2.*

Proof. Let us fix a natural number n, $y \in Y_n$ and $r > 0$. It is clear that $\mu_n(B(y,r)) \leqslant \lambda_n(y,r)$ and that $\lambda_n(y,r) \leqslant \lambda_n(y,s)$ if $s \geqslant r$. Also, if $r \geqslant 5a^{-n}$ from

Theorem 3.2 we have that

$$\lambda_n(y,2r) = \mu_n(B(y,2r)) \leqslant A\mu_n(B(y,r)) = A\lambda_n(y,r).$$

On the other hand, if $r < 5a^{-n}$, we shall consider the cases $2r < 5a^{-n}$ and $2r \geqslant 5a^{-n}$. In the former case we have $\lambda_n(y,2r) = \lambda_n(y,r)$, and in the latter we have

$$\lambda_n(y,2r) \leqslant \lambda_n(y,10a^{-n}) \leqslant A\lambda_n(y,5a^{-n}) = A\lambda_n(y,r).$$

So we can take $C_{\lambda_n} = A$ for every n. □

Remark 4.2. Since from Proposition 3.1 the limit measure have the doubling property with constant A^4, then μ is trivially upper doubling with the dominating function $\lambda(y,r) = \mu(B(y,r))$ for $y \in Y$ and for all $r > 0$. On the other hand, since A is greater than 1 we can conclude, using the above theorem, that all spaces of the orbit and its limit space are upper doubling spaces, where each corresponding dominating function has the doubling constant A^4.

References

[1] *H. Aimar, M. Carena, B. Iaffei*: Boundedness of the Hardy-Littlewood maximal operator along the orbits of contractive similitudes. J. Geom. Anal. *23* (2013), 1832–1850. zbl MR
[2] *H. Aimar, M. Carena, B. Iaffei*: On approximation of maximal operators. Publ. Math. *77* (2010), 87–99. zbl MR
[3] *H. Aimar, M. Carena, B. Iaffei*: Discrete approximation of spaces of homogeneous type. J. Geom. Anal. *19* (2009), 1–18. zbl MR
[4] *P. Assouad*: Étude d'une dimension métrique liée à la possibilité de plongements dans $\mathbb{R}^n$. C. R. Acad. Sci., Paris, Sér. A *288* (1979), 731–734. (In French.) zbl MR
[5] *R. R. Coifman, M. de Guzman*: Singular integrals and multipliers on homogeneous spaces. Rev. Un. Mat. Argentina *25* (1970), 137–143. zbl MR
[6] *R. R. Coifman, G. Weiss*: Non-Commutative Harmonic Analysis on Certain Homogeneous Spaces. Study of Certain Singular Integrals. Lecture Notes in Mathematics 242, Springer, Berlin, 1971. zbl MR
[7] *K. Falconer*: Techniques in Fractal Geometry. John Wiley, Chichester, 1997. zbl MR
[8] *J. E. Hutchinson*: Fractals and self similarity. Indiana Univ. Math. J. *30* (1981), 713–747. zbl MR
[9] *T. Hytönen*: A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa. Publ. Mat., Barc. *54* (2010), 485–504. zbl MR
[10] *T. Hytönen, S. Liu, D. Yang, D. Yang*: Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces. Can. J. Math. *64* (2012), 892–923. zbl MR
[11] *T. Hytönen, H. Martikainen*: Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces. J. Geom. Anal. *22* (2012), 1071–1107. zbl MR
[12] *T. Hytönen, D. Yang, D. Yang*: The Hardy space H^1 on non-homogeneous metric spaces. Math. Proc. Camb. Philos. Soc. *153* (2012), 9–31. zbl MR
[13] *B. Iaffei, L. Nitti*: Riesz type potentials in the framework of quasi-metric spaces equipped with upper doubling measures. ArXiv:1309.3755 (2013).
[14] *J. Kigami*: Analysis on Fractals. Cambridge University Press, Cambridge, 2001. zbl MR

[15] *J. Kigami*: A harmonic calculus on the Sierpiński spaces. Japan J. Appl. Math. *6* (1989), 259–290. zbl MR
[16] *P. A. P. Moran*: Additive functions of intervals and Hausdorff measure. Proc. Camb. Philos. Soc. *42* (1946), 15–23. zbl MR
[17] *U. Mosco*: Variational fractals. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. *25* (1997), 683–712. zbl MR
[18] *R. S. Strichartz*: Differential Equations on Fractals. A Tutorial. Princeton University Press, Princeton, 2006. zbl MR

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