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ON COINCIDENCE OF PETTIS AND MCSHANE INTEGRABILITY

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Dedicated to Jaroslav Kurzweil on the occasion of his 88th birthday and to the memory of Štefan Schwabik

Abstract. R. Deville and J. Rodríguez proved that, for every Hilbert generated space X, every Pettis integrable function $f\colon [0,1] \to X$ is McShane integrable. R. Avilés, G. Plebanek, and J. Rodríguez constructed a weakly compactly generated Banach space X and a scalarly null (hence Pettis integrable) function from $[0,1]$ into X, which was not McShane integrable. We study here the mechanism behind the McShane integrability of scalarly negligible functions from $[0,1]$ (mostly) into $C(K)$ spaces. We focus in more detail on the behavior of several concrete Eberlein (Corson) compact spaces K, that are not uniform Eberlein, with respect to the integrability of some natural scalarly negligible functions from $[0,1]$ into $C(K)$ in McShane sense.

Keywords: Pettis integral; McShane integral; MC-filling family; uniform Eberlein compact space; scalarly negligible function; Lebesgue injection; Hilbert generated space; strong Markuševič basis; adequate inflation

MSC 2010: 46G10, 46B26

1. Introduction

The so far largest known class of Banach spaces where Pettis integrability of vector valued functions coincides with McShane integrability is that of (subspaces) of Hilbert generated spaces. This is a result of Deville and Rodríguez [8]. On the other hand, Avilés, Plebanek and Rodríguez constructed a weakly compactly generated Banach space X and a scalarly negligible, hence Pettis integrable, vector function, with values in X, which is not McShane integrable [4]. A natural question is *if this*

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is a general phenomenon for all weakly compactly generated Banach spaces which are not subspaces of Hilbert generated spaces. A bit weaker question is *if, for every Eberlein compact space K, which is not uniform Eberlein, there exists a scalarly negligible vector function, with values in $C(K)$, which is not McShane integrable.* We do not know an answer. Just we should note that in the questions above, it is wise to require the validity of the continuum hypothesis (CH). Indeed, under Martin's axiom [20] and the negation of (CH), every subset of $\mathbb{R}$ of cardinality less than c is Lebesgue null. Thus, assuming this, every scalarly null vector function from $[0,1]$ into a Banach space X of density ω_1 is Lebesgue negligible, and hence even Bochner integrable, thus also McShane integrable [14], 1K theorem. If, in addition, such an X is weakly Lindelöf determined, then every Pettis integrable vector function from $[0,1]$ into X is McShane integrable, see Theorem 11. In particular, this applies to X of form $C(K)$ where K is a Corson compact spaces of density ω_1 and is such that every regular Borel measure on it has a separable support.

In this paper, we study the mechanism behind the McShane integrability of scalarly negligible vector functions from $[0,1]$ into $C(K)$ spaces and also to general Banach spaces. Then, we focus in more detail on the behavior of several concrete Eberlein (Corson) compact spaces, that are not uniform Eberlein, with respect to the integrability of some natural scalarly negligible vector functions in McShane sense. Thus, we believe that the questions raised above will be elucidated a bit.

2. Terminology and notation

Let λ and λ^* denote the Lebesgue measure and the outer Lebesgue measure. Let $(X, \|\cdot\|)$ be a Banach space and let $f\colon [0,1] \to X$ be a vector valued function. We say that f is *Pettis integrable* if the compound function $x^* \circ f$ is Lebesgue integrable for every $x^* \in X^*$, and for every Lebesgue measurable set $E \subset [0,1]$ there is a vector $x_E \in X$ such that $\int_E x^*(f(t))\,\mathrm{d}\lambda(t) = x^*(x_E)$ for all $x^* \in X^*$. We say that f is *McShane integrable* if there exists an $x \in X$ such that for every $\varepsilon > 0$ there is a *gauge function* $\delta(\cdot)$ assigning to every $t \in [0,1]$ an open subset $t \in \delta(t) \subset \mathbb{R}$ such that: for every sequence of points $t_1, t_2, \dots$ in $[0,1]$, and for every sequence of pairwise disjoint Lebesgue measurable subsets $E_1, E_2, \dots$ of $[0,1]$ such that $\sum\limits_{j=1}^{\infty} \lambda(E_j) = 1$, and $\delta(t_j) \supset E_j$, $j = 1, 2 \dots$, we have $\left\| \sum\limits_{j=1}^{r} \lambda(E_j) f(t_j) - x \right\| < \varepsilon$ for all large $r \in \mathbb{N}$; this x is then called the *McShane integral* of f. We recall that a predecessor of this concept—the *Henstock-Kurzweil integrability*—works with finitely many E_j's, each being an interval and satisfying $E_j \ni t_j$ for every j; see for instance [21]. According to Fremlin, a vector valued function from $[0,1]$ into X is McShane integrable if and only if it is simultaneously Pettis integrable and Henstock-Kurzweil integrable, see [21], Theorem 6.2.6. It is also well known that, for functions from $[0,1]$ into $\mathbb{R}$, the McShane integrability coincides with the Lebesgue integrability [14], 10 Theorem, and the Henstock-Kurzweil integrability coincides with the Perron integrability, see [18], Section 25. The vector function $f\colon [0,1] \to X$ is called *scalarly negligible* if the composition $x^* \circ f$ is a Lebesgue negligible function for every $x^* \in X^*$.

Abusing the language a bit, given a nonempty set A, by an *A-partition* we understand any formula like $M = \bigcup\limits_{\alpha\in A} M_\alpha$ where the M_α's are pairwise disjoint subsets of M. If A is countable, we say just *partition*. Given a (rather uncountable) set Γ and a nonempty set S, we put $\Sigma(S^\Gamma) = \{x \in S^\Gamma\colon \#\{\gamma \in \Gamma\colon x(\gamma) \neq 0\} \leqslant \omega\}$ and endow it with the topology inherited from the product topology of S^Γ. Instead of $\Sigma(\{0,1\}^\Gamma)$ we sometimes write $\Gamma^{\leqslant\omega}$ and consider then the elements of the latter as just at most countable subsets of Γ. Also $\Gamma^{<\omega}$ means the family of all finite subsets of Γ. Sometimes we have to add an empty set of summands. Then we put $\sum\emptyset = 0$. Given a set Γ and a subset $A \subset \Gamma$, the *characteristic function* 1_A on Γ is defined by $1_A(\gamma) = 1$ if $\gamma \in A$ and $1_A(\gamma) = 0$ if $\gamma \in \Gamma \setminus A$. For any set S, a function $\varphi\colon [0,1] \to S$ is called *Lebesgue injection* if $\lambda(\varphi^{-1}(s)) = 0$ for every $s \in S$. It should be noted that this concept says nothing about possible measurability of φ. Indeed, every one to one function $\varphi\colon [0,1] \to [0,1]$ is a Lebesgue injection.

3. Compact space setting

A compact space is called *Eberlein (uniform Eberlein)* if it is homeomorphic to a weakly compact subset of a Banach space (Hilbert space). We recall that *a compact space K is Eberlein (uniform Eberlein) if and only if the Banach space $C(K)$ is weakly compactly generated (Hilbert generated),* see [11], Theorems 14.9 and 14.15. Also, *a Banach space is a subspace of a Hilbert generated space if and only if its dual unit ball provided with the weak* topology is a uniform Eberlein compact space,* see [11], Theorem 14.15. We recall the following result due to Farmaki [12], Theorem 2.10, [10], Theorem 10 (a): *Given a (rather uncountable) set Γ, then a compact set K sitting in $\Sigma(\mathbb{R}^\Gamma)$ is a uniform Eberlein compact space if and only if for every $\varepsilon > 0$ there is a partition $\Gamma = \bigcup\limits_{n=1}^{\infty} \Gamma_n$ of Γ such that for every $k \in K$ and for every $n \in \mathbb{N}$ we have $\#\{\gamma \in \Gamma_n\colon |k(\gamma)| > \varepsilon\} < n$.* In the sequel, we shall frequently use a special case of this statement for compact sets sitting in $\Sigma(\{0,1\}^\Gamma)$ $(= \Gamma^{\leqslant\omega})$, due to Leiderman and Sokolov [16], Theorem 4.9:

Proposition 1. *Let Γ be an uncountable set and let $\mathcal{F} \subset \Gamma^{\leqslant\omega}$ be a family such that the corresponding space $K_\mathcal{F} := \{1_A\colon A \in \mathcal{F}\}$ is compact. Then $K_\mathcal{F}$ is uniform*

Eberlein compact if and only if there exists a partition $\Gamma = \bigcup\limits_{n=1}^{\infty} \Gamma_n$ *such that for every* $A \in \mathcal{F}$ *and for every* $n \in \mathbb{N}$ *we have* $\#(A \cap \Gamma_n) < n$.

Of some importance for us is the following proposition providing nontrivial scalarly negligible vector functions.

Proposition 2. *Assume that a compact subset* K *of* $\Sigma(\mathbb{R}^{[0,1]})$ *is Eberlein, or more generally, is such that every regular Borel measure on it has a separable support. Let* $\varphi\colon [0,1] \to [0,1]$ *be any Lebesgue injection and let* $a\colon [0,1] \to \mathbb{R}$ *be any function. Then the vector function* $g\colon [0,1] \to C(K)$ *defined by* $g(t)(k) = a(t)k(\varphi(t))$, $t \in [0,1]$, $k \in K$, *is scalarly negligible.*

P r o o f. Fix any $k \in K$. Let $\{t_1, t_2, \ldots\}$ denote the set $\{t \in [0,1]\colon\ k(t) \neq 0\}$. Then

$$\begin{aligned}\{t \in [0,1]\colon\ g(t)(k) \neq 0\} &= \{t \in [0,1]\colon\ a(t)k(\varphi(t) \neq 0\} \\ &\subset \{t \in [0,1]\colon\ k(\varphi(t) \neq 0\} \\ &= \bigcup_{n=1}^{\infty} \{t \in [0,1]\colon\ \varphi(t) = t_n\} = \bigcup_{n=1}^{\infty} \varphi^{-1}(t_n),\end{aligned}$$

and the latter set is Lebesgue negligible by the assumption. We proved that the assignment $[0,1] \ni t \mapsto \langle \delta_k, g(t)\rangle$ is a Lebesgue negligible function; here and below δ_k means the Dirac measure. It is well known that the absolute convex hull of the set $\{\delta_k\colon\ k \in K\}$ is weak* dense in $B_{C(K)^*}$. Now, the properties of K guarantee that the dual unit ball $(B_{C(K)^*}, w^*)$ is a Corson compact space by [11], Theorems 14.9 and 13.20, and [3], Theorem 3.5. Therefore, the set of all finite "rational" linear combinations of all δ_k, $k \in K$, is weak* sequentially dense in the whole dual $C(K)^*$. It then follows that g is scalarly negligible. $\square$

It seems that, behind the non-McShane integrability of (scalarly negligible) vector functions defined in the proposition above, there is a concept of the so called MC-filling family.

Definition 3. A family $\mathcal{F}$ of subsets of $[0,1]$ is called MC-*filling* if there exists $\varepsilon \in (0,1)$ such that for every partition $[0,1] = \bigcup\limits_{m=1}^{\infty} \Omega_m$ there is $A \in \mathcal{F}$ so that $\lambda^*\big(\bigcup\{\Omega_m\colon\ m \in \mathbb{N} \text{ and } A \cap \Omega_m \neq \emptyset\}\big) > \varepsilon$. More generally, a subset K of $\mathbb{R}^{[0,1]}$ is called MC-*filling* if there exists an $\varepsilon \in (0,1)$ such that for every partition $[0,1] = \bigcup\limits_{m=1}^{\infty} \Omega_m$ there is $k \in K$ such that

$$\lambda^*\Big(\bigcup\{\Omega_m\colon\ m \in \mathbb{N} \text{ and } |k(t)| > \varepsilon \text{ for some } t \in \Omega_m\}\Big) > \varepsilon; \tag{1}$$

compare with [4], Definition 1.1.

Clearly, *a family* $\mathcal{F}$ *of subsets of* $[0,1]$ *is* MC-*filling if and only if the corresponding subset* $\{1_A\colon\ A\in\mathcal{F}\}\subset\{0,1\}^{[0,1]}$ *is* MC-*filling.*

If K is a uniform Eberlein compact space sitting in $\Sigma(\mathbb{R}^{[0,1]})$, then it is not difficult to show that K is not MC-filling. More generally, we have

Proposition 4. *Let* K *be any uniform Eberlein compact subset of* $\Sigma(\mathbb{R}^{[0,1]})$ *and let* $\varphi\colon\ [0,1]\to[0,1]$ *be any Lebesgue injection. Then the (compact) subset* $K\circ\varphi:=\{k\circ\varphi\colon\ k\in K\}$ *of* $\mathbb{R}^{[0,1]}$ *is (also uniform Eberlein and moreover it is) not* MC-*filling.*

Proof. The mapping $K\ni k\mapsto k\circ\varphi\in K\circ\varphi$ is obviously continuous. Hence, by the Benyamini-Rudin-Wage theorem (see also [10], page 422), $K\circ\varphi$ is a uniform Eberlein compact space. (If φ is surjective, this mapping is a homeomorphism and $K\circ\varphi$ is then automatically a uniform Eberlein compact space.)

It remains to prove the rest. Fix any $\varepsilon\in(0,1)$. From Farmaki's criterion, find a partition $[0,1]=\bigcup\limits_{n=1}^{\infty}\Gamma_n$ such that for every $n\in\mathbb{N}$ and every $k\in K$ we have $\#\{s\in\Gamma_n\colon\ |k(s)|>\varepsilon\}<n$. For every $n\in\mathbb{N}$ and every $s\in\Gamma_n$ find an open set $\varphi^{-1}(s)\subset G_s\subset\mathbb{R}$ such that $\lambda(G_s)<\varepsilon/(n2^n)$. Let $I_1,I_2,\ldots$ be an enumeration of all open intervals in $\mathbb{R}$ with "rational" endpoints. For every $t\in[0,1]$ find (a unique) $n\in\mathbb{N}$ such that $\varphi(t)\in\Gamma_n$, and then find $m\in\mathbb{N}$ such that $t\in I_m\subset G_{\varphi(t)}$; finally denote this I_m by $\delta(t)$. (Note that $\delta(\cdot)$ will be a gauge function.) Clearly,

$$[0,1]=\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\varphi^{-1}(\Gamma_n)\cap\delta^{-1}(\{I_m\})$$

and this is a partition. We shall show that this partition "works". Put, for simplicity

$$\Omega_{n,m}=\varphi^{-1}(\Gamma_n)\cap\delta^{-1}(\{I_m\}),\quad n,m\in\mathbb{N}.$$

Fix any $k\in K$. For $n\in\mathbb{N}$ put $F_n=\{s\in\Gamma_n\colon\ |k(s)|>\varepsilon\}$; thus $\#F_n<n$. We are ready to estimate

$$\begin{aligned}
&\lambda^*\Big(\bigcup\{\Omega_{n,m}\colon\ n,m\in\mathbb{N}\text{ and }|k\circ\varphi(t)|>\varepsilon\text{ for some }t\in\Omega_{n,m}\}\Big)\\
&\quad\leqslant\sum_{n=1}^{\infty}\lambda^*\Big(\bigcup\{\Omega_{n,m}\colon\ m\in\mathbb{N}\text{ and }|k\circ\varphi(t)|>\varepsilon\text{ for some }t\in\Omega_{n,m}\}\Big)\\
&\quad=\sum_{n=1}^{\infty}\lambda^*\Big(\bigcup\Big\{\bigcup\{\Omega_{n,m}\colon\ m\in\mathbb{N}\text{ and }\varphi^{-1}(s)\cap\delta^{-1}(\{I_m\})\neq\emptyset\}\colon\ s\in F_n\Big\}\Big)\\
&\quad\leqslant\sum_{n=1}^{\infty}\sum_{s\in F_n}\lambda^*\Big(\bigcup\{\Omega_{n,m}\colon\ m\in\mathbb{N}\text{ and }\varphi^{-1}(s)\cap\delta^{-1}(\{I_m\})\neq\emptyset\}\Big)=:(*).
\end{aligned}$$

Fix for a while any $n \in \mathbb{N}$ and then any $s \in F_n$. Consider any $m \in \mathbb{N}$ such that the set $\varphi^{-1}(s) \cap \delta^{-1}(\{I_m\})$ is nonempty. Pick some t in it. Then $\delta(t) = I_m \subset G_{\varphi(t)} = G_s$. Thus

$$\Omega_{n,m} \subset \delta^{-1}(\{I_m\}) \subset I_m \subset G_s, \quad \text{whenever } m \in \mathbb{N} \text{ and } \varphi^{-1}(s) \cap \delta^{-1}(\{I_m\}) \neq \emptyset.$$

Now, we are ready to finalize our estimate

$$(*) \leqslant \sum_{n=1}^{\infty} \sum_{s \in F_n} \lambda(G_s) \leqslant \sum_{n=1}^{\infty} \#F_n \frac{\varepsilon}{n2^n} < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

□

It should be noted that Proposition 4 can be proved indirectly as follows: If $K \circ \varphi$ were MC-filling, Proposition 6 below would guarantee that the mapping f therein is not McShane integrable. And this contradicts to [8], Lemma 3.3.

Question 1 (Main)**.** Is Proposition 4 an equivalence? More explicitly: *Assume that a compact set $K \subset \Sigma(\mathbb{R}^{[0,1]})$ is (Eberlein and) such that for every Lebesgue injection $\varphi\colon [0,1] \to [0,1]$ the set $K \circ \varphi := \{k \circ \varphi\colon k \in K\}$ is not* MC*-filling. Is then K necessarily a uniform Eberlein compact space?* Under the validity of Martin's axiom and simultaneously the non-validity of the continuum hypothesis (CH), there exists an Eberlein compact space H which is not uniform Eberlein and yet $H \circ \varphi := \{h \circ \varphi\colon h \in H\}$ is not MC-filling for every Lebesgue injection $\varphi\colon [0,1] \to [0,1]$, see the end of Example 16. Hence, we have to add in the question raised above some set-theoretical axioms, like (CH). If the answer were then affirmative, this would indicate that the coincidence of Pettis and McShane integrability for vector functions from $[0,1]$ into $C(K)$ means that the space K is uniform Eberlein, see Proposition 6. If it were negative for some Eberlein compact subset $K \subset \Sigma(\mathbb{R}^{[0,1]})$, we would be able to construct a non-separable subspace X of the WCG (but not Hilbert-generated) space $C(K)$ such that all Pettis integrable vector functions from $[0,1]$ into X are McShane integrable, see Remark 12.

Lemma 5. *Given any $\varepsilon > 0$ and any sequence of sets $\Omega_1, \Omega_2, \ldots$ in $[0,1]$, then for every $m \in \mathbb{N}$ there exist countably many pairwise disjoint open intervals $I_1^m, I_2^m, \ldots$ in $\mathbb{R}$ such that $\bigcup\limits_{i=1}^{\infty} I_i^m \supset \Omega_m$ and that*

$$(2) \qquad \lambda\Big(\bigcup\{I_i^m\colon\ m \in J,\ i \in \mathbb{N}\}\Big) < \lambda^*\Big(\bigcup_{m \in J} \Omega_m\Big) + \varepsilon \text{ for every finite } J \subset \mathbb{N}.$$

P r o o f. For every $J \in \mathbb{N}^{<\omega}$ find $B_J \in \mathcal{L}$ so that $B_J \supset \bigcup_{m\in J} \Omega_m$ and $\lambda(B_J) < \lambda^*\Bigl(\bigcup_{m\in J} \Omega_m\Bigr) + \varepsilon/2$. For $m \in \mathbb{N}$ put then $A_m = \bigcap\{B_J\colon\ J \in \mathbb{N}^{<\omega},\ J \ni m\}$. Clearly, $\Omega_m \subset A_m \in \mathcal{L}$ for every $m \in \mathbb{N}$. Now, for every $J \in \mathbb{N}^{<\omega}$ we have $\bigcup_{m\in J} A_m \subset B_J$, and hence $\lambda\Bigl(\bigcup_{m\in J} A_m\Bigr) \leqslant \lambda(B_J) < \lambda^*\bigl(\bigcup_{m\in J} \Omega_m\bigr) + \varepsilon/2$. For every $m \in \mathbb{N}$ find an open set $A_m \subset G_m \subset [0,1]$ such that $\lambda(G_m \setminus A_m) < 2^{-m}(\varepsilon/2)$. We observe that for every $J \in \mathbb{N}^{<\omega}$ we have $\bigcup_{m\in J} G_m \setminus \bigcup_{m\in J} A_m \subset \bigcup_{m\in J} (G_m \setminus A_m)$, and hence

$$\begin{aligned}(3)\qquad \lambda\Bigl(\bigcup_{m\in J} G_m\Bigr) &\leqslant \lambda\Bigl(\bigcup_{m\in J} A_m\Bigr) + \sum_{m\in J} \lambda(G_m \setminus A_m)\\ &< \lambda\Bigl(\bigcup_{m\in J} A_m\Bigr) + \frac{\varepsilon}{2} < \lambda^*\Bigl(\bigcup_{m\in J} \Omega_m\Bigr) + \varepsilon.\end{aligned}$$

We realize that for every $m \in \mathbb{N}$ there are pairwise disjoint open intervals I_i^m, $i \in \mathbb{N}$, (we assume that $\emptyset$ is also an open interval) such that $G_m = \bigcup_{i=1}^{\infty} I_i^m$. Hence the conclusion follows. $\square$

The equivalence that follows is in the spirit of [4], Proposition 3.3 (ii).

Proposition 6. *Let K be a compact subset of $[-1,1]^{[0,1]}$. Then the following statements are mutually equivalent:*

(i) *K is not* MC*-filling;*

(ii) *given any function $a\colon\ [0,1] \to [-1,1]$, the McShane integral of the vector function $f_a\colon\ [0,1] \to C(K)$ defined by $f_a(t)(k) = a(t)k(t)$, $k \in K$, $t \in [0,1]$, is equal to* 0.

P r o o f. (i) $\Rightarrow$ (ii). Fix any $\varepsilon > 0$. From Definition 3, find a partition $[0,1] = \bigcup_{m=1}^{\infty} \Omega_m$ such that for every $k \in K$

$$(4)\qquad \lambda^*\Bigl(\bigcup\{\Omega_m\colon\ m \in \mathbb{N} \text{ and } |k(t)| > \varepsilon \text{ for some } t \in \Omega_m\}\Bigr) \leqslant \varepsilon.$$

To these Ω_m's find, by Lemma 5, the corresponding open intervals I_i^m's. Further, for every $m, i \in \mathbb{N}$ we put $\tilde I_i^m = \emptyset$ if $I_i^m = \emptyset$ and $\tilde I_i^m = (a - 2^{-m-i}\varepsilon, b + 2^{-m-i}\varepsilon)$ if $I_i^m = (a,b)$ and $a < b$. Then $\lambda(\tilde I_i^m) \leqslant \lambda(I_i^m) + 2\varepsilon 2^{-m-i}$. Now, for every finite $J \subset \mathbb{N}$ we have $\lambda\Bigl(\bigcup_{m\in J}\bigcup_{i=1}^{\infty}(\tilde I_i^m \setminus I_i^m)\Bigr) \leqslant \sum_{m\in J}\sum_{i=1}^{\infty} 2\varepsilon 2^{-m-i} < 2\varepsilon$ and $\bigcup_{m\in J}\bigcup_{i=1}^{\infty} \tilde I_i^m \setminus \bigcup_{m\in J}\bigcup_{i=1}^{\infty} I_i^m \subset$

$\bigcup\limits_{m\in J}\bigcup\limits_{i=1}^{\infty}(\tilde{I}_i^m \setminus I_i^m)$; hence

$$\lambda\Big(\bigcup_{m\in J}\bigcup_{i=1}^{\infty}\tilde{I}_i^m\Big) < \lambda\Big(\bigcup_{m\in J}\bigcup_{i=1}^{\infty}I_i^m\Big) + 2\varepsilon,$$

and using (2),

$$\lambda\Big(\bigcup_{m\in J}\bigcup_{i=1}^{\infty}\tilde{I}_i^m\Big) < \lambda^*\Big(\bigcup_{m\in J}\Omega_m\Big) + 3\varepsilon. \tag{5}$$

Now we are ready to integrate f_a in the McShane sense. We define a gauge function $\delta(\cdot)$ as follows. Fix any $t \in [0,1]$. Find a (unique) $m \in \mathbb{N}$ such that $\Omega_m \ni t$. Find a (unique) $i \in \mathbb{N}$ such that $I_i^m \ni t$. Put then $\delta(t) = (t - \varepsilon 2^{-m-i}, t + \varepsilon 2^{-m-i})$. Now, let $t_1, t_2, \ldots \in [0,1]$, and $E_1, E_2, \ldots \subset [0,1]$ be any sequences considered in the definition of McShane integrability. Fix any $r \in \mathbb{N}$. Fix any $k \in K$. We shall show that $\Big|\Big(\sum\limits_{j=1}^{r}\lambda(E_j)f_a(t_j)\Big)(k) - 0\Big| < 5\varepsilon$. Put $J = \{j \in \{1,\ldots,r\}\colon\ |k(t_j)| > \varepsilon\}$. Then

$$\Big|\Big(\sum_{j=1}^{r}\lambda(E_j)f_a(t_j)\Big)(k) - 0\Big| \leqslant \sum_{j=1}^{r}\lambda(E_j)|k(t_j)| \leqslant \varepsilon + \sum_{j\in J}\lambda(E_j). \tag{6}$$

For every $j \in J$ find an $m_j \in \mathbb{N}$ such that $\Omega_{m_j} \ni t_j$ (note that the m_j's may not be necessarily pairwise distinct) and then find $i_j \in \mathbb{N}$ so that $I_{i_j}^{m_j} \ni t_j$; thus

$$\begin{aligned} E_j \subset \delta(t_j) &= (t_j - \varepsilon 2^{-m_j-i_j}, t_j + \varepsilon 2^{-m_j-i_j}) = \{t_j\} + (-\varepsilon 2^{-m_j-i_j}, \varepsilon 2^{-m_j-i_j}) \\ &\subset I_{i_j}^{m_j} + (-\varepsilon 2^{-m_j-i_j}, \varepsilon 2^{-m_j-i_j}) = \tilde{I}_{i_j}^{m_j}. \end{aligned}$$

Then $\bigcup\limits_{j\in J}E_j \subset \bigcup\limits_{j\in J}\bigcup\limits_{i=1}^{\infty}\tilde{I}_i^{m_j}$, and (6), (5) yield

$$\begin{aligned} \Big|\Big(\sum_{j=1}^{r}\lambda(E_j)f_a(t_j)\Big)(k)\Big| &\leqslant \varepsilon + \lambda\Big(\bigcup_{j\in J}E_j\Big) \leqslant \varepsilon + \lambda\Big(\bigcup_{j\in J}\bigcup_{i=1}^{\infty}\tilde{I}_i^{m_j}\Big) \\ &< \varepsilon + \lambda^*\Big(\bigcup_{j\in J}\Omega_{m_j}\Big) + 3\varepsilon. \end{aligned}$$

But (4) says that $\lambda^*\Big(\bigcup\limits_{j\in J}\Omega_{m_j}\Big) \leqslant \varepsilon$. Therefore $\Big|\Big(\sum\limits_{j=1}^{r}\lambda(E_j)f_a(t_j)\Big)(k)\Big| < 5\varepsilon$, and so $\Big\|\sum\limits_{j=1}^{r}\lambda(E_j)f_a(t_j) - 0\Big\| \leqslant 5\varepsilon$. This holds for every $r \in \mathbb{N}$. We proved that f_h is McShane integrable, with McShane integral equal to 0.

$\neg$(i) $\Rightarrow$ $\neg$(ii). Let $\varepsilon > 0$ witness for the MC-filling of K. Define $f\colon [0,1] \to C(K)$ by $f(t)(k) = |k(t)|$, $t \in [0,1]$, $k \in K$. Fix any gauge function $\delta\colon [0,1] \to \{$open subsets of $\mathbb{R}\}$, with $\delta(t) \ni t$ for every $t \in [0,1]$. It is enough to find points $t_1, t_2 \ldots \in [0,1]$, and pairwise disjoint Lebesgue measurable subsets $E_1, E_2, \ldots$ of $[0,1]$ such that $\sum\limits_{j=1}^{\infty} \lambda(E_j) = 1$, $\delta(t_j) \supset E_j$ for every $j \in \mathbb{N}$, and $\Big\|\sum\limits_{j=1}^{r} \lambda(E_j) f(t_j)\Big\| > \varepsilon^2$ for infinitely many $r \in \mathbb{N}$. Let $U_1, U_2, \ldots$ be a countable base for the standard topology of $\mathbb{R}$. For every $t \in [0,1]$ find $m \in \mathbb{N}$ such that $t \in U_m \subset \delta(t)$ and put then $\tilde{\delta}(t) = U_m$. For every $m \in \mathbb{N}$ put $\Omega_m = \{t \in [0,1]\colon\ \tilde{\delta}(t) = U_m\}$. (Note that these sets may not be Lebesgue measurable.) Clearly, $\Omega_m \subset U_m$ for every $m \in \mathbb{N}$ and $[0,1] = \bigcup\limits_{m=1}^{\infty} \Omega_m$ is a partition of $[0,1]$. Find $k \in K$ so that (1) holds. Denote $J = \{m \in \mathbb{N}\colon\ |k(t)| > \varepsilon$ for some $t \in \Omega_m\}$. Then there exist $l \in \mathbb{N}$ and pairwise distinct numbers $m(1), \ldots, m(l) \in J$ so that $\lambda^*(\Omega_{m(1)} \cup \ldots \cup \Omega_{m(l)}) > \varepsilon$. Indeed, if J is finite, it is enough to enumerate the elements of J; otherwise, we profit from the inequality $\lambda^*\Big(\bigcup\limits_{m\in J} \Omega_m\Big) > \varepsilon$. Enumerate the set $\mathbb{N} \setminus \{m(1), \ldots, m(l)\}$ as $\{m(l+1), m(l+2), \ldots\}$. Define then

$$E_1 = U_{m(1)},\ E_2 = U_{m(2)} \setminus U_{m(1)},\ E_3 = U_{m(3)} \setminus (U_{m(1)} \cup U_{m(2)}),\ \ldots;$$

these sets are Lebesgue measurable and $\sum\limits_{j=1}^{\infty} \lambda(E_j) = 1$. For $j \in \{1, \ldots, l\}$ find $t_j \in \Omega_{m(j)}$ so that $|k(t_j)| > \varepsilon$, and for $j \in \{l+1, l+2, \ldots\}$ pick some $t_j \in \Omega_{m(j)}$. Then

$$\delta(t_j) \supset \tilde{\delta}(t_j) = U_{m(j)} \supset E_j \quad \text{for every } j \in \mathbb{N}.$$

Now, for $r \in \mathbb{N}$, with $r \geqslant l$, we are ready to estimate

$$\begin{aligned}\Big\|\sum_{j=1}^{r} \lambda(E_j) f(t_j) - 0\Big\| &\geqslant \Big(\sum_{j=1}^{r} \lambda(E_j) f(t_j)\Big)(k) = \sum_{j=1}^{r} \lambda(E_j)|k(t_j)| \\ &\geqslant \varepsilon \sum_{j=1}^{l} \lambda(E_j) = \varepsilon\lambda(E_1 \cup \ldots \cup E_l) \\ &= \varepsilon\lambda(U_{m(1)} \cup \ldots \cup U_{m(l)}) \geqslant \varepsilon\lambda^*(\Omega_{m(1)} \cup \ldots \cup \Omega_{m(l)}) > \varepsilon^2.\end{aligned}$$

We proved that 0 cannot be the McShane integral of the vector function f. $\square$

Remark 7. Assume that the compact space K in Proposition 6 sits even in $\Sigma([-1,1]^{[0,1]})$ and is Eberlein, or more generally, is such that every regular Borel measure on it has a countable support. Then the vector function f_a considered in (ii) is scalarly negligible by Proposition 2, and hence, the only candidate for the McShane integral of f_a is 0.

Proposition 8. *Let $K \subset \Sigma([-1,1]^{[0,1]})$ be any uniform Eberlein compact subset, let $\varphi\colon [0,1] \to [0,1]$ be any Lebesgue injection, and let $a\colon [0,1] \to \mathbb{R}$ be any function. Then the vector function $g\colon [0,1] \to C(K)$ defined by $g(t)(k) = a(t)k(\varphi(t))$, $k \in K$, $t \in [0,1]$, is scalarly negligible and McShane integrable.*

Proof. Proposition 4 says that the compact subset $K \circ \varphi := \{k \circ \varphi\colon k \in K\}$ in $[-1,1]^{[0,1]}$ is not MC-filling. Proposition 6 then says that the vector function $\overline{g}\colon [0,1] \to C(K \circ \varphi)$ defined by

$$\overline{g}(t)(k \circ \varphi) = a(t)(k \circ \varphi)(t) \ (= g(t)(k)), \quad k \in K,\ t \in [0,1],$$

is McShane integrable. But $C(K \circ \varphi)$ is isometrical with a subspace of $C(K)$. Therefore, $g\colon [0,1] \to C(K)$ is also McShane integrable. That g is scalarly negligible follows from Remark 7. □

4. Banach space setting

Let X be a Banach space. We say that a set $\Delta \subset X$ *countably supports* $x^* \in X^*$ if the set $\{x \in \Delta\colon x^*(x) \neq 0\}$ is at most countable. The utility of this concept (for us) can be demonstrated by a simple observation that if $g\colon [0,1] \to X$ is any Lebesgue injection such that the image $g([0,1])$ countably supports every $x^* \in X^*$, then g is scalarly negligible (and hence Pettis integrable). Indeed, given a fixed $x^* \in X^*$, put $\Delta_0 = \{x \in g([0,1])\colon x^*(x) \neq 0\}$; this is an at most countable set. Then $\lambda(\{t \in [0,1]\colon x^* \circ g(t) \neq 0\}) = \sum\limits_{x \in \Delta_0} \lambda(g^{-1}(x)) = 0$. Of particular interest (for us) are then big subsets of X that countably support every $x^* \in X^*$. If X is a subspace of a weakly compactly generated space, then there exists a linearly dense set $\Delta \subset X$ that countably supports every $x^* \in X^*$. We actually have the equivalence: *A Banach space X admits a linearly dense subset that countably supports every $x^* \in X^*$ if and only if X is weakly Lindelöf determined,* see [10], Theorem 5, for the details.

We have a statement in the spirit of the implication (i) $\Rightarrow$ (ii) in Proposition 6.

Proposition 9. *Let X be a Banach space and let $g\colon [0,1] \to B_X$ be a vector function such that for every $\varepsilon > 0$ there is a partition $[0,1] = \bigcup\limits_{m=1}^{\infty} \Omega_m$ such that for every $x^* \in B_{X^*}$*

$$\lambda^*\Big(\bigcup\{\Omega_m\colon\ m \in \mathbb{N} \text{ and } |x^*(g(t))| > \varepsilon \text{ for some } t \in \Omega_m\}\Big) < \varepsilon. \tag{7}$$

Then g is McShane integrable with McShane integral equal to 0.

P r o o f. Let Z denote the closed linear span of $f([0,1])$. Define a (continuous injection) $\psi\colon (B_{Z^*}, w^*) \hookrightarrow [-1,1]^{[0,1]}$ by $\psi(z^*)(t) = z^*(g(t))$, $t \in [0,1]$, $z^* \in B_{Z^*}$, and put $K = \psi(B_{Z^*})$; this is a compact space. We shall show that K is not MC-filling. So fix any $\varepsilon \in (0,1)$. Let $[0,1] = \bigcup\limits_{m=1}^{\infty} \Omega_m$ be a partition such that (7) holds. Take any $k \in K$. Find $z^* \in B_{Z^*}$ so that $\psi(z^*) = k$. Then $k(t) = \psi(z^*)(t) = z^*(g(t))$ for every $t \in [0,1]$, and hence we have

$$\begin{aligned}
&\lambda^*\Bigl(\bigcup\{\Omega_m\colon\ m \in \mathbb{N} \text{ and } |k(t)| > \varepsilon \text{ for some } t \in \Omega_m\}\Bigr)\\
&\quad = \lambda^*\Bigl(\bigcup\{\Omega_m\colon\ m \in \mathbb{N} \text{ and } |z^*(g(t))| > \varepsilon \text{ for some } t \in \Omega_m\}\Bigr) < \varepsilon
\end{aligned}$$

by (7). Once we know that K is not MC-filling, Proposition 6 says that the vector function $f\colon [0,1] \to C(K)$ defined by $f(t)(k) = k(t)$, $t \in [0,1]$, $k \in K$, is McShane integrable, with McShane integral equal to 0. Hence, realizing that

$$\begin{aligned}
\Bigl\|\sum_{j=1}^{r} \lambda(E_j) g(t_j)\Bigr\|_X &= \sup_{z^* \in B_{Z^*}} \sum_{j=1}^{r} \lambda(E_j) z^*(g(t_j)) = \sup_{z^* \in B_{Z^*}} \sum_{j=1}^{r} \lambda(E_j)\psi(z^*)(t_j)\\
&= \sup_{k \in K} \sum_{j=1}^{r} \lambda(E_j) k(t_j) = \sup_{k \in K} \sum_{j=1}^{r} \lambda(E_j) f(t_j)(k)\\
&= \Bigl\|\sum_{j=1}^{r} \lambda(E_j) f(t_j)\Bigr\|_{C(K)}
\end{aligned}$$

for any $r \in \mathbb{N}$, any points $t_1, t_2, \ldots \in [0,1]$, and any pairwise disjoint Lebesgue measurable sets $E_1, E_2, \ldots$ in $[0,1]$, we can conclude that g is also McShane integrable, with McShane integral equal to 0. □

Proposition 10. *Let X be a (subspace of a) Hilbert generated space, let $a\colon [0,1] \to \mathbb{R}$ be any function and let $g\colon [0,1] \to B_X$ be any vector function such that the image $g([0,1])$ countably supports every $x^* \in X^*$ and that $\lambda(g^{-1}(x) \setminus a^{-1}(0)) = 0$ for every $x \in B_X$. Then the vector function $f\colon [0,1] \to X$ defined by $f(\cdot) = a(\cdot)g(\cdot)$ is scalarly negligible, and is McShane integrable.*

P r o o f. Fix any $0 \neq x^* \in X^*$. The set $S := \{x \in g([0,1])\colon\ x^*(x) \neq 0\}$ is at most countable. Thus the set $\{t \in [0,1]\colon\ x^* \circ f(t) \neq 0\} = \bigcup\limits_{x \in S} (g^{-1}(x) \setminus a^{-1}(0))$ is Lebesgue null, and hence $x^* \circ f$ is a negligible function. We have proved that f is scalarly negligible.

Assume first that a is bounded. Put $c = \sup\{|a(t)|\colon\ t \in [0,1]\}$. Fix any $\varepsilon \in (0,1)$. We shall verify (7) by imitating the proof of Proposition 4. From [10], Theorem 6, find a partition $g([0,1]) = \bigcup_{n=1}^{\infty} \Gamma_n$ such that for every $n \in \mathbb{N}$ and every $x^* \in B_{X^*}$ we have that $\#\{x \in \Gamma_n \colon |x^*(x)| > \varepsilon\} < n$. For every $n \in \mathbb{N}$ and every $x \in \Gamma_n$ find an open set $g^{-1}(x) \setminus a^{-1}(0) \subset G_x \subset \mathbb{R}$ such that $\lambda(G_x) < \varepsilon/(n2^n)$. Let $I_1, I_2, \ldots$ be an enumeration of all open intervals in $\mathbb{R}$ with "rational" endpoints. For every $t \in [0,1]$ find (a unique) $n \in \mathbb{N}$ such that $g(t) \in \Gamma_n$, and then find $m \in \mathbb{N}$ such that $t \in I_m \subset G_{g(t)}$; finally denote this I_m by $\delta(t)$. Clearly,

$$[0,1] = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (g^{-1}(\Gamma_n) \setminus a^{-1}(0)) \cap \delta^{-1}(\{I_m\}) \cup h^{-1}(0)$$

and this is a partition. We shall show that this partition "works". For $n, m \in \mathbb{N}$ put $\Omega_{n,m} = (g^{-1}(\Gamma_n) \setminus a^{-1}(0)) \cap \delta^{-1}(\{I_m\})$. Fix any $x^* \in B_{X^*}$. For $n \in \mathbb{N}$ put $F_n = \{x \in \Gamma_n \colon |x^*(x)| > \varepsilon\}$; thus $\#F_n < n$. We are ready to estimate

$$\begin{aligned}
&\lambda^*\Big(\bigcup\Big\{\Omega_{n,m} \colon n, m \in \mathbb{N} \text{ and } \Big|x^*\Big(\frac{1}{c}f(t)\Big)\Big| > \varepsilon \text{ for some } t \in \Omega_{n,m}\Big\}\Big)\\
&\leqslant \lambda^*\Big(\bigcup\{\Omega_{n,m} \colon n, m \in \mathbb{N} \text{ and } |x^*(g(t))| > \varepsilon \text{ for some } t \in \Omega_{n,m}\}\Big)\\
&\leqslant \sum_{n=1}^{\infty} \lambda^*\Big(\bigcup\{\Omega_{n,m} \colon m \in \mathbb{N} \text{ and } |x^*(g(t))| > \varepsilon \text{ for some } t \in \Omega_{n,m}\}\Big)\\
&= \sum_{n=1}^{\infty} \lambda^*\Big(\bigcup\Big\{\bigcup\{\Omega_{n,m} \colon m \in \mathbb{N} \text{ and } (g^{-1}(x) \setminus a^{-1}(0)) \cap \delta^{-1}(\{I_m\}) \neq \emptyset\} \colon x \in F_n\Big\}\Big)\\
&\leqslant \sum_{n=1}^{\infty} \sum_{x \in F_n} \lambda^*\Big(\bigcup\{\Omega_{n,m} \colon m \in \mathbb{N} \text{ and } (g^{-1}(x) \setminus a^{-1}(0)) \cap \delta^{-1}(\{I_m\}) \neq \emptyset\}\Big)\\
&\leqslant \sum_{n=1}^{\infty} \sum_{x \in F_n} \lambda(G_x) \leqslant \sum_{n=1}^{\infty} \#F_n \frac{\varepsilon}{n2^n} < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.
\end{aligned}$$

Now it remains to apply Proposition 9. If h is unbounded, we use the "bounded" case proved above together with [14], 4A Theorem. □

We conclude this section with the following statement taken more or less from [8].

Theorem 11. *Given a weakly Lindelöf determined Banach space X, then the following assertions are mutually equivalent:*

(i) *Every Pettis integrable vector function $f \colon [0,1] \to X$ is McShane integrable.*
(ii) *Every scalarly negligible vector function $f \colon [0,1] \to X$ is McShane integrable.*
(iii) *For some strong Markuševič basis $\{(\gamma, \xi_\gamma) \colon \gamma \in \Gamma\}$ in X every scalarly negligible vector function $f \colon [0,1] \to \mathbb{R}\Gamma$ is McShane integrable.*

P r o o f. The chain (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial, see [8]. For (ii) $\Rightarrow$ (i) see [7], page 1184, or the proof of [8], Theorem 3.7. (iii) $\Rightarrow$ (ii) is a combination of [8], Lemmas 3.4, 3.5, and 3.6. $\square$

Putting together Proposition 10 and Theorem 11 we get [8], Theorem 3.7.

The adjective "strong" at Markuševič basis means that every $x \in X$ belongs to the closed linear span of the set $\{\gamma \in \Gamma \colon\ \xi_\gamma(x) \neq 0\}$. For instance, Schauder bases in separable Banach spaces and unconditional bases in any Banach spaces (if they exist) are strong. The existence of a strong Markuševič basis for separable Banach spaces is a rather deep statement due to Terenzi [15], Theorem 1.36; an extension of this fact to (nonseparable) weakly Lindelöf determined spaces, can then be done via a standard transfinite induction argument using projectional resolutions of the identity; see [15], Corollary 5.2. It should be noted that Proposition 10 easily implies that (iii) above is satisfied if X is a subspace of a Hilbert generated Banach space, see also [8].

Remark 12. We mention one situation when we do not need the result of Terenzi; then we shall be able to apply Theorem 11 more directly. *Let $K \subset \Sigma([0,1]^{[0,1]})$ be any Eberlein compact set such that for every $t \in [0,1]$ there is $k \in K$ such that $k(t) > 0$, and that for every Lebesgue injection $\varphi\colon\ [0,1] \to [0,1]$ the (compact) space $K \circ \varphi := \{k \circ \varphi\colon\ k \in K\}$ is not* MC*-filling.* Define

$$H_K := \{h \in [-1,1]^{[0,1]} \colon\ |a(\cdot)| \leqslant k(\cdot) \text{ for some } k \in K\};$$

this is a compact space. For $t \in [0,1]$ and $h \in H_K$ define $\pi_t(h) = h(t)$; clearly $0 \neq \pi_t \in C(H_K)$. Let $X_K \subset C(H_K)$ be the closed linear span of all such π_t's. For every finite set $F \subset [0,1]$ and every $a_t \in \mathbb{R}$, $t \in F$, we have $\Big\|\sum\limits_{t\in F} a_t\pi_t\Big\| = \max\Big\{\sum\limits_{t\in F} |a_t| k(t)\colon\ k \in K\Big\}$. From this we can conclude that for every $x \in X_K$ there exist unique $a_t \in \mathbb{R}$, $t \in [0,1]$, such that $x = \sum\limits_{t\in[0,1]} a_t\pi_t$; this means that for every $\varepsilon > 0$ there is a finite set $F \subset [0,1]$ such that $\Big\|x - \sum\limits_{t\in J} a_t\pi_t\Big\| < \varepsilon$ whenever $F \subset J \subset [0,1]$ is a finite set. For suggestions how to get this, see the proof of [17], Proposition 1.a.3. Now, for every $s \in [0,1]$ and every $x = \sum\limits_{t\in[0,1]} a_t\pi_t \in X_K$ we put $\xi_s(x) = a_s$. It is easy to show that this ξ_s is an element of the dual ${X_K}^*$. Then, clearly $\{(\pi_t, \xi_t)\colon\ t \in [0,1]\}$ is a strong Markuševič basis in ${X_K}^*$.

It remains to verify that (iii) in Theorem 11 holds for this basis. Consider any scalarly negligible vector function $f\colon\ [0,1] \to \mathbb{R}\{\pi_s\colon\ s \in [0,1]\}$. For $t \in f^{-1}(0)$ put $a(t) := 0 \in \mathbb{R}$ and $\varphi(t) := t$. For $t \in [0,1] \setminus f^{-1}(0)$ find $a(t) \in \mathbb{R}$ and $\varphi(t) \in [0,1]$ such that $f(t) = a(t)\pi_{\varphi(t)}$. Then $f(t) = a(t)\pi_{\varphi(t)}$ for every $t \in [0,1]$. Fix any $s \in [0,1]$.

Find $k \in K$ so that $k(s) > 0$. Define $h(s) = k(s)$ and $h(t) = 0$ for $t \in [0,1] \setminus \{s\}$. Note that $h \in H_K$. Since f is scalarly negligible, the function $[0,1] \ni t \mapsto f(t)(h)$ is Lebesgue negligible. Hence

$$\begin{aligned} 0 &= \lambda(\{t \in [0,1]\colon\ f(t)(h) \neq 0\}) \\ &= \lambda(\{t \in [0,1]\colon\ a(t) \neq 0 \text{ and } \pi_{\varphi(t)}(h) \neq 0\}) \\ &= \lambda(\{t \in [0,1] \setminus a^{-1}(0)\colon\ s = \varphi(t)\}) = \lambda(\varphi^{-1}(s) \setminus a^{-1}(0)). \end{aligned}$$

Also $\varphi^{-1}(s) \cap a^{-1}(0) \subset \{s\}$. Therefore $\lambda(\varphi^{-1}(s)) = 0$. We proved that our $\varphi\colon [0,1] \to [0,1]$ is a Lebesgue injection.

Since K is not MC-filling, the compact space H_K is not MC-filing as well. Proposition 6 says that the vector function $\overline{f}\colon\ [0,1] \to C(H_K \circ \varphi)$ defined by

$$\overline{f}(t)(h \circ \varphi) = a(t)(h \circ \varphi)(t)\ (= f(t)(h)), \quad h \in H_K,\ t \in [0,1],$$

is McShane integrable. But $C(H_K \circ \varphi)$ is isometrical with a subspace of $C(H_K)$. Therefore, $f\colon\ [0,1] \to C(H_K)$ is also McShane integrable. We thus verified (iii). Now, Theorem 11 guarantees that every Pettis integrable vector function $f\colon\ [0,1] \to X_K$ is McShane integrable.

We can perform the reasoning above if K is a uniform Eberlein compact space, see Proposition 4. However, if there exists a non-uniform Eberlein compact space $K \subset \Sigma([0,1]^{[0,1]})$ which is Eberlein (more generally, which is such that every regular Borel measure on the corresponding H_K has a separable support) and is such that $K \circ \varphi$ is not MC-filling for every Lebesgue injection $\varphi\colon\ [0,1] \to [0,1]$ (so far we do not know if such a K exists), then we would have that the corresponding Banach space X_K constructed above is a space where Pettis and McShane integrability coincide and yet X_K is not a subspace of a Hilbert generated space. Let us check the latter statement. Assume that X_K is a subspace of a Hilbert generated space. We observe that the set $\{\pi_t\colon\ t \in [0,1]\}$ countably supports all elements of the dual ${X_K}^*$. This is so since the dual unit ball of this space endowed with the weak* topology is a Corson compact space. Now, for every $\varepsilon > 0$ [10], Theorem 6, yields a partition $[0,1] = \bigcup\limits_{n=1}^{\infty} T_n^{\varepsilon}$ such that for every $n \in \mathbb{N}$ and every $h \in H_K$ the set $\{t \in T_n^{\varepsilon}\colon\ |\pi_t(h)| > \varepsilon\}$ has cardinality less than n. But then [10], Theorem 10 (a), would yield that the compact space H_K, hence also K, were uniformly Eberlein; a contradiction.

5. Examples

In this section, we inspect how several known compact spaces behave with respect to the concept of MC-filling.

Example 13. This example is a formal variant of the compact space of Benyamini-Starbird [5], simplified a bit by Argyros-Farmaki [2], Example 1.10: Here, instead of the "triangle" $\{1\} \times \{1,2\} \times \{1,2,3\} \times \ldots$ used in [5] and [2], we shall prefer working with the interval $[0,1]$. For $n \in \mathbb{N}$ put $\mathcal{S}_n = \{s \in \{1,2,\ldots,n\}^n\colon\ s(1) = 1, s(2) \leqslant 2, \ldots, s(n) \leqslant n\}$; thus $\#\mathcal{S}_n = n!$. Put then $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \ldots$. For $n \in \mathbb{N}$ and $s \in \mathcal{S}_n$ we put $|s| = n$ and further $s\widehat{\ }j = (s(1), s(2), \ldots, s(n), j)$ for $j \in \{1,2,\ldots,n,n+1\}$. For $s \in \mathcal{S}$ we shall construct intervals $T_s \subset [0,1)$, of form $[a,b)$, as follows. Put $T_{(1)} = [0,1)$. Now, let $n \in \mathbb{N}$ be fixed for a while and assume that we have already constructed T_s, $s \in \mathcal{S}_n$. Fix for a while any $s \in \mathcal{S}_n$. Thus $T_s = [a,b)$ where $0 \leqslant a < b \leqslant 1$. Insert "equidistantly" numbers $a = c_0 < c_1 < c_2 < \ldots < c_n < c_{n+1} = b$ and define then $T_{s\widehat{\ }j} = [c_{j-1}, c_j)$, $j = 1, \ldots, n+1$. Do so for every $s \in \mathcal{S}_n$. This way, we defined T_s for every $s \in \mathcal{S}_{n+1}$. Do so for every $n \in \mathbb{N}$. This way we defined T_s for every $s \in \mathcal{S}$. We observe that $\lambda(T_s) = 1/|s|!$ for every $s \in \mathcal{S}$. Also, we can easily observe that if $t, s \in \mathcal{S}$ and $|t| \geqslant |s|$, then either $T_t \subset T_s$ or $T_t \cap T_s = \emptyset$. Moreover, $[0,1] = \bigcup\{T_s\colon\ s \in \mathcal{S}_n\} \cup \{1\}$ is a (finite) partition for every $n \in \mathbb{N}$.

Now, for any *"handle"* $h \in \mathcal{S}$ let $\mathcal{B}_h$ be the family consisting of all *"brooms"* $B \subset [0,1)$ such that $B \subset T_h$ and that $\#(B \cap T_{h\widehat{\ }j}) \leqslant 1$ for every $j = 1, \ldots, |h|+1$; thus $\#B \leqslant |h|+1$. Put then $\mathcal{B} = \bigcup\{\mathcal{B}_h\colon\ h \in \mathcal{S}\}$; this is an adequate family on $[0,1)$, i.e., $\mathcal{B}$ contains all singletons, if $A \subset B \in \mathcal{B}$, then $A \in \mathcal{B}$, and a set $B \subset [0,1)$ belongs to $\mathcal{B}$ whenever $A \in \mathcal{B}$ for every finite $A \subset B$. Define $K_{\mathcal{B}} = \{1_B\colon\ B \in \mathcal{B}\}$. Since $\mathcal{B} \subset [0,1]^{<\omega}$, it follows that $K_{\mathcal{B}}$ is a weakly compact subset of $c_0([0,1])$ and so *is an Eberlein compact space.* Assume for a while that $K_{\mathcal{B}}$ is a uniform Eberlein compact space. By Proposition 1, there is a partition $[0,1] = \bigcup\limits_{n=1}^{\infty} \Gamma_n$ such that for every $B \in \mathcal{B}$ and for every $n \in \mathbb{N}$ we have $\#(B \cap \Gamma_n) \leqslant n$. Baire's theorem yields $n \in \mathbb{N}$ such that $\operatorname{int} \overline{\Gamma}_n \neq \emptyset$. Find $s \in \mathcal{S}$ so that $T_s \subset \overline{\Gamma}_n$. We may and do assume that $|s| \geqslant n$. For every $j = 1, \ldots, n+1$ we have $T_{s\widehat{\ }j} \subset T_s \subset \overline{\Gamma}_n$, hence there is $t_j \in T_{s\widehat{\ }j} \cap \Gamma_n$. Putting then $B = \{t_1, \ldots, t_{n+1}\}$, we have $B \in \mathcal{B}_s$ and $\#B \cap \Gamma_n = n+1$, a contradiction. Therefore, *$K_{\mathcal{B}}$ is not a uniform Eberlein compact space.*

Yet the *compact space $K_{\mathcal{B}}$*, equivalently, *the family $\mathcal{B}$, is not* MC-*filling!* In order to prove this, let $\varepsilon \in (0,1)$ be any fixed number. Find $n \in \mathbb{N}$ so big that $(n-1)! > 1/\varepsilon$. We shall show that the (even finite) partition $[0,1] = \bigcup\{T_s\colon\ s \in \mathcal{S}_n\} \cup \{1\}$ "works". Indeed, take any broom $B \in \mathcal{B}$. Let h be the handle of B, i.e., $B \in \mathcal{B}_h$. First, assume that $|h| < n$. Then

$$\lambda\Big(\bigcup\{T_s\colon\ s \in \mathcal{S}_n,\ B \cap T_s \neq \emptyset\}\Big) \leqslant \sum\{\lambda(T_s)\colon\ s \in \mathcal{S}_n,\ B \cap T_s \neq \emptyset\}$$
$$\leqslant \#B \cdot \frac{1}{n!} \leqslant \frac{|h|+1}{n!} \leqslant \frac{n}{n!} = \frac{1}{(n-1)!} < \varepsilon.$$

Second, assume $|h| \geqslant n$. Then a moment's reflection yields that $B \subset T_u$ for some $u \in \mathcal{S}_n$. Hence

$$\lambda\Big(\bigcup\{T_s\colon\ s \in \mathcal{S}_n,\ B \cap T_s \neq \emptyset\}\Big) = \lambda(T_u) = \frac{1}{|u|!} = \frac{1}{n!} < \varepsilon.$$

□

Example 14 (Marciszewski [19])**.** Instead of $\{0,1\}^\omega$ used in [19], we shall work with $[0,1)$. Consider "dyadic" intervals T_s, $s \in \{0,1\}^{<\omega}$, in $[0,1)$, that is, put

$$T_{(0)} = [0,1/2),\ T_{(1)} = [1/2,1),\ T_{(0,0)} = [0,1/4),\ T_{(0,1)} = [1/4,1/2),$$
$$T_{(1,0)} = [1/2,3/4),\ T_{(1,1)} = [3/4,1),\ T_{(0,0,0)} = [0,1/8),\ \ldots.$$

For any "handle" $h \in \{0,1\}^{<\omega}$ let $\mathcal{B}_h$ consist of all "brooms" $B \subset T_h$ with $\#B \leqslant |h|$. Put then $\mathcal{B} = \bigcup\{\mathcal{B}_h\colon\ h \in \{0,1\}^{<\omega}\}$. It is easy to check that this family is adequate on $[0,1)$. Indeed, let $A \subset [0,1)$ be a set such that $B \in \mathcal{B}$ for every finite subset $B \subset A$. If A is finite, then clearly $A \in \mathcal{B}$. So, assume A is infinite. Find a pairwise distinct sequence $t_1, t_2, \ldots$ in A. For every $m \in \mathbb{N}$, $m > 1$, we (already) know that $\{t_1, \ldots, t_m\} \in \mathcal{B}$; find then $h_m \in \{0,1\}^{<\omega}$, of maximal possible length $|h_m|$, such that $\{t_1, \ldots, t_m\} \subset T_{h_m}$. Then, necessarily, $|h_2| \geqslant |h_3| \geqslant \ldots$. Hence, there must exist $m \in \mathbb{N}$ such that $m > |h_m|$, which is impossible because each element of $\mathcal{B}_{h_m}$ has cardinality at most $|h_m|$. It follows that A must be finite. Thus $K_{\mathcal{B}} := \{1_B\colon\ B \in \mathcal{B}\}$ *is an (Eberlein) compact space.* That $K_{\mathcal{B}}$ *is not uniform Eberlein* can be seen as in Example 13. That $K_{\mathcal{B}}$ *is not* MC*-filling* either, can also be seen as in Example 13. Indeed, given any $\varepsilon \in (0,1)$, find $n \in \mathbb{N}$ so large that $n2^{-n} < \varepsilon$; then the finite partition $[0,1] = \bigcup\{T_s\colon\ s \in \{0,1\}^{<\omega}$ and $|s| = n\} \cup \{1\}$ "works". □

Example 15 (Talagrand [22], Théorème 4.3)**.** This construction is based on an (adequate) family sitting on the whole Baire space $\mathbb{N}^{\mathbb{N}}$ (instead of just the "triangle" $\{1\} \times \{1,2\} \times \{1,2,3\} \times \ldots$ considered in Example 13). Again, instead of $\mathbb{N}^{\mathbb{N}}$, we shall be working in $[0,1]$. For $s \in \mathbb{N}^{<\omega}$ we shall construct intervals $T_s \subset (0,1]$ as follows. Put $T_\emptyset = (0,1]$. Further we proceed by induction. Assume that $s \in \mathbb{N}^{<\omega}$ is fixed and that we already defined T_s of the form $(a,b]$. Then we define $T_{s^\frown j} = (a + 2^{-j}(b-a), a + 2^{-j+1}(b-a)]$, $j \in \mathbb{N}$. This way we define T_s for every $s \in \mathcal{S}$.

We observe that $\lambda(T_s) = 2^{-s(1)-\dots-s(n)}$ for every $n \in \mathbb{N}$ and every $s \in \mathbb{N}^n$. Also, we can easily check that, if $t, s \in \mathcal{S}$ and $|t| \geqslant |s|$, then either $T_t \subset T_s$ or $T_t \cap T_s = \emptyset$. Moreover, $[0,1] = \bigcup\{T_s\colon\ s \in \mathbb{N}^n\} \cup \{0\}$ is a (countable) partition.

Now, for any *handle* $h \in \mathbb{N}^{<\omega}$ let $\mathcal{B}_h$ be the family consisting of all *brooms* $B \subset (0,1]$ such that $B \subset T_h$ and that $\#(B \cap T_{t\frown j}) \leqslant 1$ for every $j \in \mathbb{N}$; thus B is at most countable. Put then $\mathcal{B} = \bigcup\{\mathcal{B}_h\colon\ h \in \mathbb{N}^{<\omega}\}$; this is again an *adequate family* on $(0,1]$. Define $K_{\mathcal{B}} = \{1_B\colon\ B \in \mathcal{B}\}$; then $K_{\mathcal{B}} \hookrightarrow \Sigma(\{0,1\}^{[0,1]})$ $(= [0,1]^{\leqslant\omega})$ continuously and so $K_{\mathcal{B}}$ *is a Corson compact space*. More specifically, we can say that this $K_{\mathcal{B}}$ *is a Talagrand (hence Gul'ko) compact space* [22], Théorème 4.1. Using a criterion of Talagrand's, see [16], Theorem 4.8, together with a Baire category argument, we can check that $K_{\mathcal{B}}$ *is not an Eberlein compact space*; hence, a fortiori, $K_{\mathcal{B}}$ *is not a uniform Eberlein compact space.*

Yet *the space $K_{\mathcal{B}}$, equivalently, the family $\mathcal{B}$, is not* MC-*filling!* In order to prove this, let $\varepsilon > 0$ be any fixed number. Find $n \in \mathbb{N}$ so large that $2^{-n+1} < \varepsilon$. We shall show that the countable partition $[0,1] = \bigcup\{T_s\colon\ s \in \mathbb{N}^n\} \cup \{0\}$ "works". Indeed, take any $B \in \mathcal{B}$. Let h be the handle of B. First, assume that $|h| < n$. Find mutually disjoint $s_1, s_2, \ldots \in \mathbb{N}^n$ such that $B \subset \bigcup\limits_{j=1}^{\infty} T_{s_j}$ and that $\#(B \cap T_{s_j}) \leqslant 1$ for every $j \in \mathbb{N}$. Then

$$\begin{aligned}\lambda\Big(\bigcup\{T_s\colon\ s \in \mathbb{N}^n,\ B \cap T_s \neq \emptyset\}\Big) &\leqslant \sum_{j=1}^{\infty} \lambda(T_{s_j})\\ &= \sum_{j=1}^{\infty} 2^{-h(1)-\ldots-h(|h|)-s_j(|h|+1)-\ldots-s_j(n)}\\ &< 2^{-(n-1)} \sum_{j=1}^{\infty} 2^{-j} = 2^{-(n-1)} < \varepsilon.\end{aligned}$$

Second, assume $|h| \geqslant n$. Then a moment's reflexion yields that $B \subset T_u$ for some $u \in \mathcal{S}_n$, and hence

$$\lambda\Big(\bigcup\{T_s\colon\ s \in \mathbb{N}^n,\ B \cap T_s \neq \emptyset\}\Big) = \lambda(T_u) \leqslant 2^{-n} < \varepsilon.$$

$\square$

Question 2. Does there exist a Lebesgue injection $\varphi\colon\ [0,1] \to [0,1]$ such that the family $\{\varphi^{-1}(B)\colon\ B \in \mathcal{B}\}$, where $\mathcal{B}$ is from Example 13, 14, or 15, would be MC-filling? If the answer were affirmative, then we would have, by Proposition 6, a scalarly negligible vector function from $[0,1]$ into $C(K_{\mathcal{B}})$ which is not McShane integrable, and thus Question 1 would remain open. If the answer were negative for some $\mathcal{B}$, we would be able to construct, using Remark 12, a Banach space $X_{\mathcal{B}}$ which is not a subspace of a Hilbert generated space and yet every Pettis integrable vector function $f\colon [0,1] \to X_{\mathcal{B}}$ would be McShane integrable; thus we would get beyond [8], Theorem 3.7. The negative answer to Question 2 would imply the negative answer to Question 1.

Example 16 (Siberian [16], Example 5.2)**.** Let ω_1 denote the first uncountable ordinal. Consider the family $\mathcal{F}$ of all $A \subset {\omega_1}^2$ such that whenever $(\alpha,\beta),(\alpha',\beta')$ are distinct elements of A, then either $\alpha < \alpha'$ and $\beta > \beta'$, or $\alpha > \alpha'$ and $\beta < \beta'$. This is an adequate family consisting of finite (!) sets. Hence the corresponding $K_{\mathcal{F}} := \{1_A\colon\ A \in \mathcal{F}\}$ *is an Eberlein compact space.* Using Proposition 1, we can check that $K_{\mathcal{F}}$ *is not uniform Eberlein.* We shall show that, *assuming the continuum hypothesis* (CH)*, this $K_{\mathcal{F}}$, after a suitable continuous injection into* $\Sigma(\{0,1\}^{[0,1]})$ $(= [0,1]^{\leqslant\omega})$*, is* MC*-filling.*

Below, we were partially inspired by the proof of [4], Theorem 3.5. Fix (even any) $\varepsilon \in (0,1)$ and any partition $[0,1] = \bigcup\limits_{m=1}^{\infty} \Omega_m$. According to [13], 419I, there exists an ω_1-partition $[0,1] = \bigcup\limits_{\alpha<\omega_1} Z_\alpha$ such that $\lambda^*(Z_\alpha) = 1$ for every $\alpha \in \Omega_1$. Clearly, each Z_α must be uncountable. Using (CH), for every $\alpha < \omega_1$, we enumerate the set Z_α as $\{t^\alpha_\beta\colon\ \beta < \omega_1\}$. Now, define $\kappa\colon\ {\omega_1}^2 \to [0,1]$ by ${\omega_1}^2 \ni (\alpha,\beta) \mapsto t^\alpha_\beta$; this is an injective surjection. Define then $\tilde\kappa\colon\ K_{\mathcal{F}} \to \{0,1\}^{<{\omega_1}^2}$ by $K \ni 1_A \mapsto 1_{\kappa(A)}$; this is a continuous injection. Now, for every $\alpha < \omega_1$ we have $\lambda^*(Z_\alpha) = 1$; hence there is $s(\alpha) \in \mathbb{N}$ such that $\lambda^*\Bigl(Z_\alpha \cap \bigcup\limits_{m=1}^{s(\alpha)} \Omega_m\Bigr) > \varepsilon$. For $n \in \mathbb{N}$ put $\Gamma_n = \{\alpha < \omega_1\colon\ s(\alpha) = n\}$. Then $\omega_1 = \bigcup\limits_{n=1}^{\infty} \Gamma_n$ is a partition of ω_1. Find an $n \in \mathbb{N}$ so that Γ_n is infinite. Thus we have $\lambda^*\Bigl(Z_\alpha \cap \bigcup\limits_{m=1}^{n} \Omega_m\Bigr) > \varepsilon$ for every $\alpha \in \Gamma_n$. Pick some mutually distinct $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Gamma_n$. We may and do assume that $\alpha_1 > \alpha_2 > \ldots > \alpha_n$. Thus we have

$$\lambda^*\Bigl(Z_{\alpha_j} \cap \bigcup_{m=1}^{n} \Omega_m\Bigr) > \varepsilon \quad \text{for every } j = 1, \ldots, n.$$

Hence, there exists $m(1) \in \{1,\ldots,n\}$ so that $\lambda^*(Z_{\alpha_1} \cap \Omega_{m(1)}) > 0$. If $\lambda^*(\Omega_{m(1)}) > \varepsilon$, put $r = 1$ and stop the process. Further assume the opposite. Then

$$\begin{aligned}\varepsilon < \lambda^*\Bigl(Z_{\alpha_2} \cap \bigcup_{m=1}^{n} \Omega_m\Bigr) &\leqslant \lambda^*(Z_{\alpha_2} \cap \Omega_{m(1)}) + \lambda^*\Bigl(Z_{\alpha_2} \cap \bigcup_{m\neq m(1)}^{n} \Omega_m\Bigr)\\ &\leqslant \varepsilon + \lambda^*\Bigl(Z_{\alpha_2} \cap \bigcup_{m\neq m(1)}^{n} \Omega_m\Bigr).\end{aligned}$$

Hence, there exists $m(2) \in \{1, \ldots, n\} \setminus \{m(1)\}$ such that $\lambda^*(Z_{\alpha_2} \cap \Omega_{m(2)}) > 0$. If $\lambda^*(\Omega_{m(1)} \cup \Omega_{m(2)}) > \varepsilon$, put $r = 2$ and stop the process. If not, then a similar reasoning yields $m(3) \in \{1, \ldots, n\} \setminus \{m(1), m(2)\}$ such that $\lambda^*(Z_{\alpha_3} \cap \Omega_{m(3)}) > 0$. Proceeding on in an obvious way, our process must once stop, at the latest when $r = n$, since we know that $\lambda^*\Big(\bigcup_{m=1}^{n} \Omega_m\Big) > \varepsilon$. Now, pick some $\beta_1 < \omega_1$ so that $t^{\alpha_1}_{\beta_1} \in \Omega_{m(1)}$. Pick then some $\beta_1 < \beta_2 < \omega_1$ such that $t^{\alpha_2}_{\beta_2} \in \Omega_{m(2)}$; here we used (CH). Similarly, the (CH) enables us to choose subsequently $\beta_3 < \omega_1$ so that $t^{\alpha_3}_{\beta_3} \in \Omega_{m(3)}$, with $\beta_3 > \beta_2$, etc., until $\beta_r < \omega_1$ with $t^{\alpha_r}_{\beta_r} \in \Omega_{m(r)}$ and $\beta_r > \beta_{r-1} > \ldots > \beta_2 > \beta_1$. Finally, putting $A = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r)\}$, we get that $A \in \mathcal{F}$ and that

$$\lambda^*\Big(\bigcup\{\Omega_m\colon\ m \in \mathbb{N} \text{ and } |1_{\kappa(A)}(t)| > \varepsilon \text{ for some } t \in \Omega_m\}\Big)$$
$$= \lambda^*\Big(\bigcup\{\Omega_m\colon\ m \in \mathbb{N} \text{ and } \kappa(A) \cap \Omega_m \neq \emptyset\}\Big) \geqslant \lambda^*\Big(\bigcup_{i=1}^{r} \Omega_{m(i)}\Big) > \varepsilon.$$

We proved that *the set* $\tilde{\kappa}(K_{\mathcal{F}})$ *is* MC-*filling*, i.e., that *the family* $\{\kappa(A)\colon\ A \in \mathcal{F}\}$ *is* MC-*filling*. Thus, by Proposition 6 and its proof, the vector function $f\colon\ [0,1] \to C(\tilde{\kappa}(K_{\mathcal{F}}))$ defined by $f(t)(l) = l(t)$, $t \in [0,1]$, $l \in \tilde{\kappa}(K_{\mathcal{F}})$, does not have 0 as its McShane integral. On the other hand, by Proposition 2, f is scalarly negligible. Hence, f cannot be McShane integrable.

Denote $\mathcal{H} = \{\kappa(A)\colon\ A \in \mathcal{F}\}$. We shall find an injective surjection $\pi\colon\ [0,1] \to [0,1]$ such that *the space* $\{1_{\pi(B)}\colon\ B \in \mathcal{H}\}$ *is not* MC-*filling*, that is, *the family* $\{\pi(B)\colon\ B \in \mathcal{H}\}$ *is not* MC-*filling*. Let C denote the Cantor set; note that $\#C = \mathfrak{c}$ and that $\lambda(C) = 0$. Find any injecton $\pi\colon\ [0,1] \to [0,1]$ that maps Z_1 onto $[0,1] \setminus C$ and $\bigcup_{1<\alpha<\omega_1} Z_\alpha$ onto C. Now, fix any $\varepsilon > 0$. Pick $s \in \mathbb{N}$ so that $s > 1/\varepsilon + 1$ and put $\Omega_m = [0,1] \cap [m\varepsilon - \varepsilon, m\varepsilon) \setminus C$, $m = 1, \ldots, s-1$ and $\Omega_s = C$. Then $\lambda(\Omega_s) = 0$ and $[0,1] = \bigcup_{m=1}^{s} \Omega_m$ is a partition of $[0,1]$. Take any $B \in \mathcal{H}$. Assume there is $m \in \{1, \ldots, s-1\}$ so that $\pi(B) \cap \Omega_m \neq \emptyset$. Then B must be of form $\{t^{\alpha_1}_{\beta_1}, t^{\alpha_2}_{\beta_2}, \ldots, t^{\alpha_{n-1}}_{\beta_{n-1}}, t^{1}_{\beta_n}\}$ where $n \in \mathbb{N}$, $\alpha_1 > \alpha_2 > \ldots > \alpha_{n-1} > 1$ and $\beta_1 < \ldots < \beta_n$, and we have $\pi(B) \cap \Omega_m = \{\pi(t^1_{\beta_n})\}$. Clearly, $\pi(B) \cap \Omega_{m'} = \emptyset$ for every $m' \in \{1, \ldots, s-1\} \setminus \{m\}$. Therefore, $\lambda\Big(\bigcup\{\Omega_m\colon\ m \in \{1, \ldots, s\} \text{ and } \pi(B) \cap \Omega_m \neq \emptyset\}\Big) \leqslant \varepsilon$. We have proved that $\{\pi(B)\colon\ B \in \mathcal{H}\}$ is not MC-filling. Put $\varphi = \pi^{-1}$, this is clearly a (Lebesgue) injection from $[0,1]$ into $[0,1]$. Then the compact space

$$\tilde{\kappa}(K_{\mathcal{F}}) \circ \varphi := \{1_{\kappa(A)} \circ \varphi\colon\ A \in \mathcal{F}\} = \{1_{\varphi^{-1}(\kappa(A))}\colon\ A \in \mathcal{F}\} = \{1_{\pi(B)}\colon\ B \in \mathcal{H}\}$$

is not MC-filling.

If Martin's axiom [20] holds and simultaneously (CH) does not hold, then for any mapping $\kappa\colon\ {\omega_1}^2 \to [0,1]$ the set $\tilde{\kappa}(K_{\mathcal{F}})$ is not MC-filling. Indeed, in this case $\kappa({\omega_1}^2)$

is a Lebesgue null set, and hence the 2-partition $[0,1] = \kappa(\omega_1{}^2) \cup ([0,1] \setminus \kappa(\omega_1{}^2))$ demonstrates this. More generally, if $\varphi\colon\ [0,1] \to [0,1]$ is any Lebesgue injection, then the 2-partition $[0,1] = \varphi^{-1}(\kappa(\omega_1{}^2)) \cup ([0,1] \setminus \varphi^{-1}(\kappa(\omega_1{}^2)))$ shows that the space $\tilde{\kappa}(K_{\mathcal{F}}) \circ \varphi$ is not MC-filling.

Variants of this example work also under other set-theoretical axioms. In particular, assume that Martin's axiom [20] holds; then every subset of $[0,1]$ of cardinality less than c is Lebesgue null. Hence, if our family $\mathcal{F}$ is built of subsets of $\omega_1 \times \mathfrak{c}$ (instead of $\omega_1{}^2$), we get that the space $\tilde{\kappa}(K_{\mathcal{F}})$ is MC-filling for our injection $\kappa\colon\ \omega_1 \times \mathfrak{c} \to [0,1]$. We thank O. Kalenda for telling us these two remarks related to set-theoretical axioms.

6. Adequate inflations

The technology developed in the proof of [4], Theorem 3.5, can be used for constructing, from a given compact set $K \subset \Sigma(\mathbb{R}^\Gamma)$ which is not uniform Eberlein, a compact overspace $K \subset H \subset \Sigma(\mathbb{R}^{[0,1]})$ which will be MC-filling (provided that $\#\Gamma \leqslant \mathfrak{c}$).

Let Γ be a fixed uncountable set with $\#\Gamma \leqslant \mathfrak{c}$. Let $[0,1] = \bigcup\limits_{\gamma\in\Gamma} Z_\gamma$ be a fixed Γ-partition of the interval $[0,1]$ such that $\lambda^*(Z_\gamma) = 1$ for every $\gamma \in \Gamma$, see, for instance, in [13], 419I.

Consider any $k \in \Sigma(\mathbb{R}^\Gamma)$. Take any, possibly empty, set $S \subset \operatorname{supp} k$ and for every $\gamma \in S$ pick some $t_\gamma \in Z_\gamma$. Define then $h(t_\gamma) = k(\gamma)$ if $\gamma \in S$ and $h(t) = 0$ if $t \in [0,1] \setminus \{t_\gamma\colon\ \gamma \in S\}$; clearly, h is an element of $\Sigma(\mathbb{R}^{[0,1]})$ and $\|h\|_\infty \leqslant \|k\|_\infty$. Any h constructed in this way will be called an *adequate extension of* k (subordinated to our Γ-partition). Clearly, every $0 \not\equiv k \in \Sigma(\mathbb{R}^\Gamma)$ has plenty of adequate extensions. Define $\varphi\colon\ [0,1] \to \Gamma$ by $\varphi_{|Z_\gamma} \equiv \gamma$, $\gamma \in \Gamma$.

Now, consider a (rather compact) set $K \subset \Sigma(\mathbb{R}^\Gamma)$ and let H denote the set of all $h \in \Sigma(\mathbb{R}^{[0,1]})$ which are adequate extensions of elements of K. This H will be called the *adequate inflation of* K (subordinated to our Γ-partition). Clearly, H is a norm-bounded set, if so is K. Also, H is adequate, that is, $h \cdot 1_A \in H$ whenever $h \in H$ and $A \subset \Gamma$. Moreover, H can be understood as an over-space of K. Indeed, if for every $\gamma \in \Gamma$ we pick some $t_\gamma \in Z_\gamma$, and then for every $k \in K$ we put

$$j(k)(t_\gamma) = k(\gamma) \quad \text{if } \gamma \in \Gamma, \quad \text{and} \quad j(k)(t) = 0 \quad \text{if } t \in [0,1] \setminus \{t_\gamma\colon\ \gamma \in \Gamma\},$$

then, clearly, $j(k) \in H$ and $j\colon\ K \hookrightarrow H$ will be a homeomorphism into.

Fact 17. The set $H \subset \Sigma(\mathbb{R}^{[0,1]})$ is compact provided that $K \subset \Sigma(\mathbb{R}^\Gamma)$ is compact.

P r o o f. Consider any net $(h_\tau)_{\tau\in T}$ in H. Find a net $(k_\tau)_{\tau\in T}$ in K such that h_τ is an adequate extension of k_τ for every $\tau \in T$. When going to subnets, we may and do assume that $k_\tau \to k \in K$ and $h_\tau \to x \in \mathbb{R}^{[0,1]}$ in the pointwise topologies. If $x \equiv 0$, we are done. Assume further that $x \not\equiv 0$. We shall show that x is an adequate extension of k, which will finish the proof. Fix any $t \in \operatorname{supp} x$. Find the (unique) $\gamma \in \Gamma$ so that $Z_\gamma \ni t$. We observe that for all τ's large enough we have $h_\tau(t) \neq 0$, and hence $h_\tau(t) = k_\tau(\gamma)$. Thus $x(t) = \lim_\tau h_\tau(t) = \lim_\tau k_\tau(\gamma)$ $(= k(\gamma))$. We proved that $\varphi(\operatorname{supp} x) \subset \operatorname{supp} k$. Further, consider any distinct $t, t' \in \operatorname{supp} x$ and find $\gamma, \gamma' \in \Gamma$ such that $t \in Z_\gamma$ and $t' \in Z_{\gamma'}$. Take a $\tau \in T$ so big that $h_\tau(t) \neq 0$ and $h_\tau(t') \neq 0$. Then, necessarily, $\gamma \neq \gamma'$. This all together implies that x is an adequate extension of k and that $x \in \Sigma(\mathbb{R}^{[0,1]})$. □

Proposition 18. *Let Γ be an uncountable set with $\#\Gamma \leqslant \mathfrak{c}$ and let $[0,1] = \bigcup_{\gamma\in\Gamma} Z_\gamma$ be a Γ-partition, with $\lambda^*(Z_\gamma) = 1$, for every $\gamma \in \Gamma$. Let $K \subset \Sigma(\mathbb{R}^\Gamma)$ be a compact set which is not uniform Eberlein. Then any adequate inflation $H \subset \Sigma(\mathbb{R}^{[0,1]}))$ of K subordinated to our Γ-partition is* MC*-filling; moreover, there exists an injective surjection $\pi\colon [0,1] \to [0,1]$, independent of H, such that the (compact) space $\tilde{\pi}(H) := \{h \circ \pi^{-1}\colon h \in H\}$ is not* MC*-filling.*

P r o o f. By Farmaki's criterion mentioned before Proposition 1, there exists $\varepsilon > 0$ such that for every partition $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ there are $k \in K$ and $n \in \mathbb{N}$ such that $\#\{\gamma \in \Gamma_n\colon |k(\omega)| > \varepsilon\} \geqslant n$. Further, we shall imitate the proof of [4], Theorem 3.5. Let $[0,1] = \bigcup_{m=1}^{\infty} \Omega_m$ be any partition. For every $\gamma \in \Gamma$ we have $\lambda^*(Z_\gamma) = 1$; hence, there is $s(\gamma) \in \mathbb{N}$ such that $\lambda^*\Big(Z_\gamma \cap \bigcup_{m=1}^{s(\gamma)} \Omega_m\Big) > \varepsilon$. For $n \in \mathbb{N}$ put $\Gamma_n = \{\gamma \in \Gamma\colon s(\gamma) = n\}$. Then $\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ is a partition of Γ. Find pairwise distinct $\gamma_1, \ldots, \gamma_n \in \Gamma_n$ such that $|k(\gamma_j)| > \varepsilon$ for every $j = 1, \ldots, n$. Thus we have that

$$\lambda^*\Big(Z_{\gamma_j} \cap \bigcup_{m=1}^{n} \Omega_m\Big) > \varepsilon \quad \text{for every } j = 1, \ldots, m.$$

Choose $m(1) \in \{1, \ldots, n\}$ so that $Z_{\gamma_1} \cap \Omega_{m(1)}$ is a nonempty set. If $\lambda^*(\Omega_{m(1)}) > \varepsilon$, put $r = 1$ and stop the process. Further assume the opposite. Then

$$\begin{aligned} \varepsilon < \lambda^*\Big(Z_{\gamma_2} \cap \bigcup_{m=1}^{n} \Omega_m\Big) &\leqslant \lambda^*(Z_{\gamma_2} \cap \Omega_{m(1)}) + \lambda^*\Big(Z_{\gamma_2} \cap \bigcup_{m\neq m(1)}^{n} \Omega_m\Big) \\ &\leqslant \varepsilon + \lambda^*\Big(Z_{\gamma_2} \cap \bigcup_{m\neq m(1)}^{n} \Omega_m\Big). \end{aligned}$$

Hence, there exists $m(2) \in \{1,\ldots,n\} \setminus \{m(1)\}$ such that $Z_{\gamma_2} \cap \Omega_{m(2)} \neq \emptyset$. If $\lambda^*(\Omega_{m(1)} \cup \Omega_{m(2)}) > \varepsilon$, put $r = 2$ and stop the process. If not, then a similar reasoning yields $m(3) \in \{1,\ldots,n\} \setminus \{m(1), m(2)\}$ such that $Z_{\gamma_3} \cap \Omega_{m(3)} \neq \emptyset$. Proceeding on in a similar way, our process must once stop, at the latest when $r = n$, since we know that $\lambda^*\Big(\bigcup\limits_{m=1}^{n} \Omega_m\Big) > \varepsilon$.

We recall that for every $j = 1,\ldots,r$ the set $Z_{\gamma_j} \cap \Omega_{m(j)}$ was nonepmty; pick some t_j in it. Put finally $h(t_j) = k(\gamma_j)$, $j = 1,\ldots,r$, and $h(t) = 0$ if $t \in [0,1] \setminus \{t_1,\ldots,t_r\}$. This h is an adequate extension of k, and hence $h \in H$. Now, we are ready to estimate

$$\lambda^*\Big(\bigcup\{\Omega_m \colon\ m \in \mathbb{N} \text{ and } |h(t)| > \varepsilon \text{ for some } t \in \Omega_m\}\Big) = \lambda^*\Big(\bigcup_{j=1}^{r} \Omega_{m(j)}\Big) > \varepsilon.$$

We proved that *the overspace space H is* MC*-filling.*

Finally, consider the injective surjection $\pi\colon\ [0,1] \to [0,1]$ used in Example 16. Let $\varepsilon \in (0,1)$ be arbitrary. Let $s \in \mathbb{N}$ and the partition $[0,1] = \bigcup\limits_{m=1}^{s} \Omega_m$ be those found for our ε in the same place. Assume there are distinct $m, m' \in \{1,\ldots,s-1\}$ such that $|h \circ \pi^{-1}(t)| > \varepsilon$ for some $t \in \Omega_m$ and $|h \circ \pi^{-1}(t')| > \varepsilon$ for some $t' \in \Omega_{m'}$. Then necessarily $\pi^{-1}(t) \in Z_1$, $\pi^{-1}(t') \in Z_1$, and hence $\pi^{-1}(t) = \pi^{-1}(t')$, that is, $t = t'$, by the very definition of H. Therefore, $\lambda\big(\bigcup\{\Omega_m \colon\ m \in \{1,\ldots,s\}$ and $|h \circ \pi^{-1}(t)| > \varepsilon$ for some $t \in \Omega_m\}\big) \leqslant \varepsilon$, which means that *the space $\tilde{\pi}(H)$ is not* MC*-filling.* $\square$

Corollary 19. *There exists an adequate family $\mathcal{H} \subset [0,1]^{<\omega}$ such that $K_{\mathcal{H}} := \{1_A\colon\ A \in \mathcal{H}\}$ is (automatically) an Eberlein compact space, $K_{\mathcal{H}}$ is not a uniform Eberlein compact space, $K_{\mathcal{H}}$ is* MC*-filling, and $\{1_{\pi(A)}\colon\ A \in \mathcal{H}\}$ is not* MC*-filling for a suitable injective surjection $\pi\colon\ [0,1] \to [0,1]$.*

P r o o f. Let $\mathcal{F} \subset [0,1]^{<\omega}$ be any adequate family such that $\{1_A\colon\ A \in \mathcal{F}\}$ is not a uniform Eberlein compact space, e.g., let $\mathcal{F}$ be the family $\mathcal{B}$ from Example 13 or 14. Inflate $\mathcal{F}$ to an $\mathcal{H}$ by Proposition 18. (Under (CH), it is enough to take for $\mathcal{H}$ the family from Example 16, without any further inflation.) $\square$

7. Concluding remarks

1. The (Eberlein) compact space made from the hereditary family constructed by Fremlin in [4], Lemma 3.4, cannot be uniform Eberlein. This follows easily from Farmaki's criterion, or, indirectly, from [8], Theorem 3.7. Propositions 18 and 6 thus cover, as a special case, the result of Avilés, Plebanek, and Rodríguez [4], Theorem 3.6.

2. Let Γ be an uncountable set, with $\#\Gamma \leqslant \mathfrak{c}$, and let K be a compact subset of $\Sigma(\mathbb{R}^\Gamma)$. Let $H \subset \Sigma(\mathbb{R}^{[0,1]})$ be an adequate inflation of K made in Proposition 18. Using [10], Theorem 10, we can easily verify that K *is uniform Eberlein, Eberlein, Talagrand, or Gul'ko if and only if so is, respectively,* H. Moreover, *if* L *is a Gul'ko compact space, then the dual* $C(L)^*$ *injects linearly and weak** *to pointwise continuously into* $\Sigma(\mathbb{R}^\Delta)$ *where* $\#\Delta$ *is equal to the density of* L [9], Theorem 7.2.5. And it should be noted that there do exist compact spaces which are, say: Gul'ko and not Talagrand, Talagrand and not Eberlein, and so on; see, for instance, [1], [6], and the references therein. Thus, using Proposition 6, we get several examples of compact spaces H such that the canonical vector function $f\colon [0,1] \to C(H)$ is scalarly negligible and not McShane integrable.

3. Let Γ be an uncountable set, with $\#\Gamma \leqslant \mathfrak{c}$, and let K be any compact subset of $\Sigma(\mathbb{R}^\Gamma)$; this means that K is Corson. Let H be an adequate inflation of K; then $H \subset \Sigma(\mathbb{R}^{[0,1]})$. In order to guarantee that the vector function $f\colon [0,1] \to C(H)$ defined by $f(t)(h) = h(t)$, $h \in H$, $t \in [0,1]$ is scalarly negligible, we need to know that $(B_{C(H)^*}, w^*)$ is angelic. However this may not be true in general, see [3]. This is the case if and only if $C(H)$ is weakly Lindelöf determined (WLD), if and only if every regular Borel measure on H has a separable support, see [3], Theorem 3.5. We do not know if $C(H)$ is WLD provided that $C(K)$ is such.

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