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A characterization of the meager ideal

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Abstract. We give a classical proof of the theorem stating that the σ -ideal of meager sets is the unique σ -ideal on a Polish group, generated by closed sets which is invariant under translations and ergodic.

Keywords: Polish group; σ -ideal; meager sets

Classification: 03E15, 54H05

1. Introduction

The σ -ideal \mathcal{M} of meager subsets of \mathbb{R} has the following remarkable properties:

- \mathcal{M} is generated by closed sets,
- \mathcal{M} satisfies the countable chain condition (ccc),
- \mathcal{M} is invariant under translations,
- \mathcal{M} is \mathbb{Q} -ergodic, i.e., every \mathbb{Q} -invariant Borel subset of \mathbb{R} is either meager or comeager.

These properties are interrelated and conjunctions of some of them characterize \mathcal{M} .

Balcerzak and Rogowska [1] and, independently (using a different method), Reclaw and Zakrzewski [4] proved that if a σ -ideal \mathcal{I} on a Polish space X is generated by closed sets and ccc, then it is Borel isomorphic to \mathcal{M} . Both proofs are based on a deep theorem by Kechris and Solecki [3, Theorem 3] which provides a characterization of those σ -ideals on Polish spaces which are generated by closed sets and fulfil ccc. As a corollary, Kechris and Solecki [3] also showed that the σ -ideal of meager sets on a Polish group is the unique σ -ideal generated by closed sets which is invariant under translations and ccc.

Zapletal (see [7]) in turn proved that if a σ -ideal on \mathbb{R} (respectively, on a Polish space X) is generated by closed sets and \mathbb{Q} -ergodic (respectively, ergodic; see Section 2 for a general definition of ergodicity), then it is ccc.

Combining the last two statements we arrive at the following characterization of the σ -ideal of meager sets on Polish groups.

Theorem 1.1. *The σ -ideal of meager sets on a Polish group G is the unique σ -ideal on G which is generated by closed sets, invariant under translations by elements of G and ergodic.*

The original Zapletal's proof of the fact that ergodicity of a σ -ideal which is generated by closed sets implies countable chain condition used forcing (see [5, Lemma 1.3] or [6, Lemma 5.4.2]). The aim of this note is to give a "classical" proof of this result.

2. Preliminaries

Throughout the paper X (more precisely: (X, τ)) is an uncountable Polish (i.e., a separable, completely metrizable) topological space. The σ -algebra of Borel subsets of X is denoted by $\mathcal{B}(X)$.

By a σ -ideal \mathcal{I} on X we understand a collection of subsets of X , closed under countable unions and such that for any $A \in \mathcal{I}$, all subsets of A are in \mathcal{I} . Throughout the paper we assume that $X \notin \mathcal{I}$ and \mathcal{I} contains all singletons.

We say that a σ -ideal \mathcal{I} on X is *generated by closed sets* if there is a family $\mathcal{F} \subseteq \mathcal{I}$ consisting of sets closed in X such that each element of \mathcal{I} can be covered by countably many elements of \mathcal{F} .

Given a σ -ideal \mathcal{I} on X , we shall use the following notation and terminology:

- $\mathcal{I}^* = \{X \setminus A : A \in \mathcal{I}\}$,
- $A_1, A_2 \in \mathcal{B}(X) \setminus \mathcal{I}$ are *almost disjoint* if $A_1 \cap A_2 \in \mathcal{I}$,
- a family $\mathcal{A} \subseteq \mathcal{B}(X) \setminus \mathcal{I}$ is almost disjoint if it consists of pairwise almost disjoint sets.

A σ -ideal \mathcal{I} on X is:

- *ccc* if it satisfies the countable chain condition, i.e., if there is no uncountable almost disjoint family $\mathcal{A} \subseteq \mathcal{B}(X) \setminus \mathcal{I}$,
- *ergodic* if there is a countable Borel equivalence relation R on X such that every set $B \in \mathcal{B}(X)$ which is the union of a family of R -equivalence classes is either in \mathcal{I} or in \mathcal{I}^* ,
- *invariant* (under translations) if (X, \cdot) is a Polish group and $x \cdot A \in \mathcal{I}$ whenever $A \in \mathcal{I}$ and $x \in X$.

3. Classical proofs of Zapletal's results

We start with the following lemma which in the forcing terminology is closely related to the fact that "forcing with a σ -ideal generated by closed sets does not collapse \aleph_1 " (cf. [5] and [7]).

Lemma 3.1 (Main Lemma). *Assume that \mathcal{I} is a σ -ideal on X generated by closed sets. Let $\langle \mathcal{A}_n : n \in \mathbb{N} \rangle$ be a sequence of maximal almost disjoint subfamilies of $\mathcal{B}(X) \setminus \mathcal{I}$. Then there exists a set $E \in \mathcal{B}(X) \setminus \mathcal{I}$ such that for every $n \in \mathbb{N}$ we have*

$$|\{A \in \mathcal{A}_n : E \cap A \notin \mathcal{I}\}| \leq \aleph_0.$$

PROOF: For each $n \in \mathbb{N}$, using the maximality of \mathcal{A}_n , fix a function $\psi_n : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$ such that

$$B \in \mathcal{B}(X) \setminus \mathcal{I} \Rightarrow (\psi_n(B) \in \mathcal{A}_n \wedge B \cap \psi_n(B) \notin \mathcal{I})$$

and let $\varphi_n : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$ be the function defined by

$$\varphi_n(B) = B \cap \psi_n(B) \quad \text{for } B \in \mathcal{B}(X).$$

Note that if $B \in \mathcal{B}(X) \setminus \mathcal{I}$, then $\varphi_n(B) \notin \mathcal{I}$ and $\psi_n(B)$ is the only $A \in \mathcal{A}_n$ such that $\varphi_n(B) \cap A \notin \mathcal{I}$.

Recall that τ is the topology of X and let (U_n) be a countable basis of (X, τ) .

Sublemma 3.2. *There exist a field \mathcal{C} of subsets of X , a Polish topology $\bar{\tau}$ extending τ and a countable base \mathcal{V} of $\bar{\tau}$ satisfying the following conditions:*

- (1) $\mathcal{C} \subseteq \mathcal{B}(X)$,
- (2) \mathcal{C} is countable,
- (3) $\varphi_n(B) \in \mathcal{C}$ for every $B \in \mathcal{C}$ and $n \in \mathbb{N}$,
- (4) $\mathcal{C} \subseteq \bar{\tau}$,
- (5) $\mathcal{V} \subseteq \mathcal{C}$.

PROOF OF SUBLEMMA 3.2: We construct inductively fields \mathcal{C}_n of subsets of X and Polish topologies τ_n on X with associated countable bases \mathcal{V}_n , $n \in \mathbb{N}$, so that:

- $n > 0$ implies τ_n is zero-dimensional,
- $\mathcal{V}_n \subseteq \mathcal{C}_n$,
- $\mathcal{C}_n \subseteq \text{clop}(X, \tau_n)$, the field of clopen subsets of (X, τ_n) ,
- $n < m$ implies $\mathcal{C}_n \subseteq \mathcal{C}_m$,
- $n < m$ implies $\tau_n \subseteq \tau_m$,
- \mathcal{C}_n is countable,
- $B \in \mathcal{C}_n$ and $m \in \mathbb{N}$ implies $\varphi_m(B) \in \mathcal{C}_{n+1}$.

Let \mathcal{C}_0 be the field of subsets of X generated by $\{U_k : k \in \mathbb{N}\}$ and $\mathcal{V}_0 = \{U_k : k \in \mathbb{N}\}$.

If \mathcal{C}_n , τ_n and \mathcal{V}_n have been defined, let

$$\mathcal{R}_{n+1} = \mathcal{C}_n \cup \bigcup_{m \in \mathbb{N}} \varphi_m[\mathcal{C}_n]$$

and extend τ_n to a Polish zero-dimensional topology τ_{n+1} on X such that $\mathcal{R}_{n+1} \subseteq \text{clop}(X, \tau_{n+1})$. Then let \mathcal{V}_{n+1} be a countable base of τ_{n+1} consisting of sets clopen in τ_{n+1} . Finally, let \mathcal{C}_{n+1} be the field of subsets of X generated by $\mathcal{R}_{n+1} \cup \mathcal{V}_{n+1}$. This completes the construction.

Now let $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ and let $\bar{\tau}$ be the topology generated by $\bigcup_{n \in \mathbb{N}} \tau_n$.

The topology $\bar{\tau}$ is Polish and finite intersections of elements of $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ form a countable base \mathcal{V} of $\bar{\tau}$ (cf. [2, Lemma 13.3]). Since \mathcal{C} is closed under finite intersections, we have $\mathcal{V} \subseteq \mathcal{C}$.

It is easy to see that \mathcal{C} , $\bar{\tau}$ and \mathcal{V} satisfy conditions (1)–(5) above which completes the proof of Sublemma 3.2. \square

Continuing the proof of Main Lemma enumerate \mathcal{V} as $\{V_k : k \in \mathbb{N}\}$ and let

$$D = X \setminus \bigcup \{V_k : k \in \mathbb{N} \text{ and } V_k \in \mathcal{I}\}.$$

Note that

- $D \in \mathcal{I}^*$,
- D is closed in $\bar{\tau}$, so uncountable Polish in the relative topology; *in the rest of the proof all topological notions concerning subsets of D will refer, unless stated otherwise, to this topology*,
- no nonempty open subset of D is in \mathcal{I} ,
- if P is closed in D and $P \in \mathcal{I}$, then P is nowhere dense in D .

For every $n \in \mathbb{N}$ let

$$\mathcal{O}_n = D \cap \bigcup_{k \in \mathbb{N}} \varphi_n(V_k).$$

We claim that each \mathcal{O}_n is open and dense in D .

To see that \mathcal{O}_n is open, use (5), (3) and (4).

To prove that \mathcal{O}_n is dense in D , take a basic open subset of D of the form $V_k \cap D \neq \emptyset$.

Then $V_k \in \mathcal{C} \setminus \mathcal{I}$ hence $\varphi_n(V_k) \in \mathcal{C} \setminus \mathcal{I}$.

Consequently, $\varphi_n(V_k)$ being a member of \mathcal{C} is $\bar{\tau}$ -open and $\varphi_n(V_k) \cap D \neq \emptyset$ since $D \in \mathcal{I}^*$.

But $\varphi_n(V_k) \subseteq V_k$ and $D \cap \varphi_n(V_k) \subseteq \mathcal{O}_n$ which implies that

$$(V_k \cap D) \cap \mathcal{O}_n \supseteq V_k \cap (D \cap \varphi_n(V_k)) = \varphi_n(V_k) \cap D \neq \emptyset,$$

completing the proof that \mathcal{O}_n is dense in D .

Finally, let

$$E = \bigcap_{n \in \mathbb{N}} \mathcal{O}_n.$$

To complete the proof of Main Lemma it suffices to prove the following

Claim.

- (6) $E \notin \mathcal{I}$,
- (7) $\forall n \quad \{A \in \mathcal{A}_n : E \cap A \notin \mathcal{I}\} \subseteq \{\psi_n(V_k) : k \in \mathbb{N}\}$.

To prove (6), we shall use the fact that \mathcal{I} is generated by closed sets. So let (D_n) be a sequence of τ -closed sets from \mathcal{I} . Our aim is to show that

$$E \not\subseteq \bigcup_{n \in \mathbb{N}} D_n.$$

Note that:

- $E = \bigcap_{n \in \mathbb{N}} \mathcal{O}_n$ is a dense G_δ subset of D .
- Each D_n being τ -closed is also closed in $\bar{\tau}$, so $D_n \cap D$ is closed nowhere dense in D .

By the Baire category theorem, we are done.

To prove (7), recall that for each n :

- if $B \in \mathcal{B}(X) \setminus \mathcal{I}$, then $\psi_n(B)$ is the only $A \in \mathcal{A}_n$ such that $\varphi_n(B) \cap A \notin \mathcal{I}$,

$$\bullet E = \bigcap_{m \in \mathbb{N}} \mathcal{O}_m \subseteq \mathcal{O}_n = D \cap \bigcup_{k \in \mathbb{N}} \varphi_n(V_k) \subseteq \bigcup_k \varphi_n(V_k).$$

Fix n and let $A \in \mathcal{A}_n$ be such that $E \cap A \notin \mathcal{I}$. Then there is k with $\varphi_n(V_k) \cap A \notin \mathcal{I}$. But the only $A \in \mathcal{A}_n$ with this property is $A = \psi_n(V_k)$ which shows (7) and completes the proof of Main Lemma. \square

With the help of Main Lemma we are now ready to finish our proof of Zapletal’s theorem (cf. [5, Lemma 1.3] and [6, Lemma 5.4.2]).

Theorem 3.3 (Zapletal). *If a σ -ideal \mathcal{I} on X is generated by closed sets and ergodic, then \mathcal{I} is ccc.*

PROOF: Recall that ergodicity of \mathcal{I} means that there is a countable Borel equivalence relation R on X such that every set $B \in \mathcal{B}(X)$ which is the union of a family of R -equivalence classes is either in \mathcal{I} or in \mathcal{I}^* .

By the Feldman–Moore theorem, R is the orbit equivalence relation for a certain countable group $G = \{g_n : n \in \mathbb{N}\}$ of Borel automorphisms of X .

So, ergodicity of \mathcal{I} means that

$$B \in \mathcal{B}(X) \setminus \mathcal{I} \Rightarrow \bigcup_n g_n B \in \mathcal{I}^*.$$

Suppose that \mathcal{I} is *not* ccc and let $\{A_\alpha : \alpha < \omega_1\}$ be a disjoint family of sets in $\mathcal{B}(X) \setminus \mathcal{I}$.

For each n let

$$\mathcal{A}_n = \{g_n A_\alpha : \alpha < \omega_1\} \setminus \mathcal{I}.$$

\mathcal{A}_n is a disjoint (perhaps empty) collection of sets in $\mathcal{B}(X) \setminus \mathcal{I}$ hence by Main Lemma, there is $E \in \mathcal{B}(X) \setminus \mathcal{I}$ such that

$$(*) \quad \forall n \quad |\{\alpha < \omega_1 : E \cap g_n A_\alpha \notin \mathcal{I}\}| \leq \aleph_0.$$

On the other hand, by ergodicity, for every $\alpha < \omega_1$ there is $n \in \mathbb{N}$ such that

$$E \cap g_n A_\alpha \notin \mathcal{I},$$

so there is a single $n \in \mathbb{N}$ with

$$|\{\alpha < \omega_1 : E \cap g_n A_\alpha \notin \mathcal{I}\}| = \aleph_1,$$

contradicting (*). \square

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