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ON THE COMPLEXITY OF SOME CLASSES OF BANACH SPACES AND NON-UNIVERSALITY

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Abstract. These notes are dedicated to the study of the complexity of several classes of separable Banach spaces. We compute the complexity of the Banach-Saks property, the alternating Banach-Saks property, the complete continuous property, and the LUST property. We also show that the weak Banach-Saks property, the Schur property, the Dunford-Pettis property, the analytic Radon-Nikodym property, the set of Banach spaces whose set of unconditionally converging operators is complemented in its bounded operators, the set of Banach spaces whose set of weakly compact operators is complemented in its bounded operators, and the set of Banach spaces whose set of Banach-Saks operators is complemented in its bounded operators, are all non Borel in SB. At last, we give several applications of those results to non-universality results.

Keywords: Banach-Saks operator; Dunford-Pettis property; analytic Radon-Nikodym property; complete continuous property; Schur property; unconditionally converging operator; weakly compact operator; local structure; non-universality; ℓ_p-Baire sum; descriptive set theory; tree

MSC 2010: 46B20

1. Introduction

Our goal for these notes is to study the complexity of certain classes of Banach spaces, hence, these notes lie in the intersection of descriptive set theory and the theory of Banach spaces.

First, we study two problems related to special classes of operators on separable Banach spaces being complemented in the space of its bounded operators or not. Specifically, we will show that both the set of Banach spaces with its unconditionally converging operators complemented in its bounded operators, and the set of Banach spaces with its weakly compact operators complemented in its bounded operators, are non Borel. The first is actually complete coanalytic. In both of these problems, we will be using results of [4] concerning the complementability of those ideals in its space of bounded operators and the fact that the space itself contains c_0.

Next, we study the complexity of other classes of Banach spaces, namely, Banach spaces with the so called Banach-Saks property, alternating Banach-Saks property, and weak Banach-Saks property. We show that the first two of them are complete coanalytic sets in the class of separable Banach spaces, and that the third is at least non Borel (it is also shown that the weak Banach-Saks property is at most Π^1_2). In order to show some of these results we use the geometric sequential characterizations of Banach spaces with the Banach-Saks property and the alternating Banach-Saks property given by B. Beauzamy (see [5]). The stability under ℓ_2-sums of the Banach-Saks property shown by J. R. Partington [23] will also be of great importance in our proofs.

It is also shown that the set of Banach spaces whose set of Banach-Saks operators is complemented in its bounded operators is non Borel. For this, a result by J. Diestel and C. J. Seifert [11] that says that weakly compact operators $T\colon\ C(K) \to X$, where K is a compact Hausdorff space, are Banach-Saks operators, will be essential.

In order to show that the class of Banach spaces with the Schur property is non Borel we will rely on the stability of this property under ℓ_1-sums shown by B. Tanbay [30], and, when dealing with the Dunford-Pettis property, the same will be shown using one of its characterizations (see [28], and [13]) and Tanbay's result. It is also shown that the Schur property is at least Π^1_2.

Next, we show that the set of separable Banach spaces with the complete continuous property CCP is complete coanalytic. For this we use a characterization of this property in terms of the existence of a special kind of bush on the space (see [15]). Also, we show that the analytic Radon-Nikodym property is non Borel.

We also deal with the local structure of separable Banach spaces by showing that the set of Banach spaces with local unconditional structure is Borel.

At last, we give several applications of the theorems obtained in these notes to non-universality like results. In all the results proved in these notes we will be applying techniques related to descriptive set theory and its applications to the geometry of Banach spaces that can be found in [12], and [29]. Also, this work was highly motivated by D. Puglisi's paper on the position of $\mathscr{K}(X,Y)$ in $\mathscr{L}(X,Y)$, in which Puglisi shows that the set of pairs of separable Banach spaces (X,Y) such that the ideal of compact operators from X to Y is complemented in the bounded operators from X to Y is non Borel (see [26]).

2. Background

A separable metric space is said to be a *Polish space* if there exists an equivalent metric in which it is complete. A continuous image of a Polish space into another Polish space is called an *analytic set* and a set whose complement is analytic is called *coanalytic*. A measure space $(X, \mathscr{A})$, where X is a set and $\mathscr{A}$ is a σ-algebra of subsets of X, is called a *standard Borel space* if there exists a Polish topology on this set whose Borel σ-algebra coincides with $\mathscr{A}$. We define Borel, analytic and coanalytic sets in standard Borel spaces by saying that these are the sets that, by considering a compatible Polish topology, are Borel, analytic, and coanalytic, respectively. Observe that this is well defined, i.e., this definition does not depend on the Polish topology itself but on its Borel structure. A function between two standard Borel spaces is called *Borel measurable* if the inverse image of each Borel subset of its codomain is Borel in its domain. We usually refer to Borel measurable functions just as Borel functions. Notice that, if you consider a Borel function between two standard Borel spaces, its inverse image of analytic or coanalytic sets is analytic or coanalytic, respectively, (see [29], Proposition 1.3, page 50).

Given a Polish space X the set of analytic or coanalytic subsets of X is denoted by $\Sigma^1_1(X)$ or $\Pi^1_1(X)$, respectively. Hence, the terminology Σ^1_1-set or Π^1_1-set is used to refer to analytic sets or coanalytic sets, respectively.

Let X be a standard Borel space. An analytic or coanalytic subset $A \subset X$ is said to be *complete analytic* or *complete coanalytic* if for each standard Borel space Y and each $B \subset Y$ analytic or coanalytic, respectively, there exists a Borel function $f \colon Y \to X$ such that $f^{-1}(A) = B$. This function is called a *Borel reduction* from B to A, and B is said to be *Borel reducible* to A.

Let X be a standard Borel space. We call a subset $A \subset X$ Σ^1_1*-hard* (Π^1_1*-hard*) if for each standard Borel space Y and each $B \subset Y$ analytic (coanalytic) there exists a Borel reduction from B to A. Therefore, to say that a set $A \subset X$ is Σ^1_1-hard (Π^1_1-hard) means that A is at least as complex as Σ^1_1-sets (Π^1_1-sets) in the projective hierarchy. With this terminology we have that $A \subset X$ is complete analytic (complete coanalytic) if and only if A is Σ^1_1-hard (Π^1_1-hard) and analytic (coanalytic).

As there exist analytic non Borel and coanalytic non Borel sets we have that Σ^1_1-hard and Π^1_1-hard sets are non Borel. Also, if X is a standard Borel space, $A \subset X$, and there exists a Borel reduction from a Σ^1_1-hard or Π^1_1-hard subset B of a standard Borel space Y to A, then A is Σ^1_1-hard or Π^1_1-hard, respectively. We refer to [29], page 56, and [21], Section 26, for more on complete analytic and coanalytic sets. Complete analytic sets or complete coanalytic sets are also called Σ^1_1-complete sets or Π^1_1-complete, respectively.

Consider a Polish space X and let $\mathscr{F}(X)$ be the set of all its non empty closed sets. We endow $\mathscr{F}(X)$ with the *Effros-Borel structure*, i.e., the σ-algebra generated by

$$\{F \subset X;\ F \cap U \neq \emptyset\},$$

where U varies among the open sets of X. It can be shown that $\mathscr{F}(X)$ with the Effros-Borel structure is a standard Borel space ([21], Theorem 12.6). The following well-known lemma (see [21], Theorem 12.13) will be crucial in some of our proofs.

Lemma 1 (Kuratowski-Ryll-Nardzewski selection principle)**.** *Let X be a Polish space. There exists a sequence of Borel functions $(S_n)_{n\in\mathbb{N}}\colon \mathscr{F}(X) \to X$ such that $\{S_n(F)\}_{n\in\mathbb{N}}$ is dense in F for all closed $F \subset X$.*

In these notes we will only work with separable Banach spaces. We denote the closed unit ball of a Banach space X and its unit sphere by B_X and S_X, respectively. It is well known that every separable Banach space is isometrically isomorphic to a closed linear subspace of $C(\Delta)$ (see [21], page 79), where Δ denotes the Cantor set. Therefore, $C(\Delta)$ is called *universal* for the class of separable Banach spaces and we can code the class of separable Banach spaces by $\mathrm{SB} = \{X \subset C(\Delta);\ X \text{ is a closed linear subspace of } C(\Delta)\}$. As $C(\Delta)$ is clearly a Polish space we can endow $\mathscr{F}(C(\Delta))$ with the Effros-Borel structure. It can be shown that SB is a Borel set in $\mathscr{F}(C(\Delta))$ and hence it is also a standard Borel space (see [12], Theorem 2.2). It now makes sense to wonder if specific sets of separable Banach spaces are Borel or not.

Throughout these notes we will denote by $\{S_n\}_{n\in\mathbb{N}}$ the sequence of Borel functions $S_n\colon \mathrm{SB} \to C(\Delta)$ given by Lemma 1 (more precisely, the restriction of those functions to SB). Hence, for all $X \in \mathrm{SB}$, $\{S_n(X)\}_{n\in\mathbb{N}}$ is dense in X. By taking rational linear combinations of the functions $\{S_n\}$, we can (and will) assume that, for all $X \in \mathrm{SB}$, all $n, k \in \mathbb{N}$, and all $p, q \in \mathbb{Q}$, there exists $m \in \mathbb{N}$ such that $qS_n(X) + pS_k(X) = S_m(X)$.

Denote by $\mathbb{N}^{<\mathbb{N}}$ the set of all finite tuples of natural numbers plus the empty set. Given $s = (s_0, \dots, s_{n-1}), t = (t_0, \dots, t_{m-1}) \in \mathbb{N}^{<\mathbb{N}}$ we say that the length of s is $|s| = n$, $s_{|i} = (s_0, \dots, s_{i-1})$ for all $i \in \{1, \dots, n\}$, $s_0 = \emptyset$, $s \leqslant t$ *iff* $n \leqslant m$ and $s_i = t_i$ for all $i \in \{0, \dots, n-1\}$, i.e., if t is an extension of s. We define $s < t$ analogously. Define the concatenation of s and t as $s^\frown t = (s_0, \dots, s_{n-1}, t_0, \dots, t_{m-1})$. A subset T of $\mathbb{N}^{<\mathbb{N}}$ is called a *tree* if $t \in T$ implies $t_{|i} \in T$ for all $i \in \{0, \dots, |t|\}$. We denote the set of trees on $\mathbb{N}$ by Tr. A subset I of a tree T is called a segment if I is completely ordered and if $s, t \in I$ with $s \leqslant t$, then $l \in I$ for all $l \in T$ such that $s \leqslant l \leqslant t$. Two segments I_1, I_2 are called completely incomparable if neither $s \leqslant t$ nor $t \leqslant s$ hold if $s \in I_1$ and $t \in I_2$.

As $\mathbb{N}^{<\mathbb{N}}$ is countable, $2^{\mathbb{N}^{<\mathbb{N}}}$ (the power set of $\mathbb{N}^{<\mathbb{N}}$) is Polish with its standard product topology. If we think about Tr as a subset of $2^{\mathbb{N}^{<\mathbb{N}}}$ it is easy to see that Tr is a closed set in $2^{\mathbb{N}^{<\mathbb{N}}}$, so it is a standard Borel space. A $\beta \in \mathbb{N}^{\mathbb{N}}$ is called a *branch* of a tree T if $\beta_{|i} \in T$ for all $i \in \mathbb{N}$, where $\beta_{|i}$ is defined analogously as above. We call a tree T *well-founded* if T has no branches and *ill-founded* otherwise, we denote the set of well-founded and ill-founded trees by WF and IF, respectively. It is well known that WF is a complete coanalytic set of Tr, hence IF is complete analytic (see [21], Theorem 27.1).

There is a really important index that can be defined on the set of trees called the *order index* of a tree. For a given tree $T \in \mathrm{Tr}$ we define the *derived tree* of T as

$$T' = \{s \in T;\ \exists t \in T,\ s < t\}.$$

By transfinite induction we define $(T_\xi)_{\xi \in \mathrm{ON}}$, where ON denotes the ordinal numbers, as follows:

$$\begin{aligned} &T^0 = T, \\ &T^\alpha = (T^\beta)', \quad \text{if } \alpha = \beta + 1 \text{ for some } \beta \in \mathrm{ON}, \\ &T^\alpha = \bigcap_{\beta < \alpha} T^\beta, \quad \text{if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

We now define the order index on Tr. If there exists an ordinal number $\alpha < \omega_1$, where ω_1 denotes the smallest uncountable ordinal such that $T^\alpha = \emptyset$ we say the order index of T is $o(T) = \min\{\alpha < \omega_1;\ T^\alpha = \emptyset\}$. If there is no such countable ordinal we set $o(T) = \omega_1$. The reason why we introduce this index is because of the way it interacts with the notion of well-founded and ill-founded trees. We have the following easy proposition (see [29], Chapter 3, Section 2).

Proposition 2. *A tree $T \in \mathrm{Tr}$ on the natural numbers is well-founded if and only if its order index is countable, i.e., if and only if $o(T) < \omega_1$.*

For a tree $T \in \mathrm{Tr}$ and $k \in \mathbb{N}$, let $T(k) = \{s \in \mathbb{N}^{<\mathbb{N}};\ (k)^\frown s \in T\}$ and $T_k = \{s \in T;\ (k) \leqslant s\}$. We have another useful application of the order index to well-founded trees (see [29], Chapter 3, Section 2).

Proposition 3. *Let $T \in \mathrm{WF}$ with $o(T) > 1$, then $o(T(k)) < o(T)$ for all $k \in \mathbb{N}$.*

Now that we have seen all the descriptive set theoretical background we need in order to understand our results and their proofs let us start with the real math.

3. ℓ_p-Baire sums

We now treat ℓ_p-Baire sums of basic sequences; this tool will be crucial in many of our results in these notes. For each $p \in [1, \infty)$ and each basic sequence $\mathscr{E} = (e_n)_{n \in \mathbb{N}}$, we define a Borel function $\varphi_{\mathscr{E},p} \colon \mathrm{Tr} \to \mathrm{SB}$ in the following manner. For each $\theta \in \mathrm{Tr}$, and $x = (x(s))_{s \in \theta} \in c_{00}(\theta)$ we define

$$\|x\|_{\mathscr{E},p,\theta} = \sup\Bigg\{ \Bigg(\sum_{i=1}^{n} \Bigg\| \sum_{s \in I_i} x(s) e_{|s|} \Bigg\|_{\mathscr{E}}^{p} \Bigg)^{1/p}; \quad n \in \mathbb{N},\ I_1, \dots,\ I_n \text{ incomparable segments of } \theta \Bigg\},$$

where $\|\cdot\|_{\mathscr{E}}$ is the norm of $\overline{\operatorname{span}}\{\mathscr{E}\}$. We define $\varphi_{\mathscr{E},p}(\theta)$ as the completion of $c_{00}(\theta)$ under the norm $\|\cdot\|_{\mathscr{E},p,\theta}$. The space $\varphi_{\mathscr{E},p}(\theta)$ is known as the ℓ_p-*Baire sum* of $\overline{\operatorname{span}}\{\mathscr{E}\}$ (index by θ). Pick $Y \subset C(\Delta)$ such that $\varphi_{\mathscr{E},p}(\mathbb{N}^{<\mathbb{N}})$ is isometric to Y. If we consider the natural isometries of $\varphi_{\mathscr{E},p}(\theta)$ into $\varphi_{\mathscr{E},p}(\mathbb{N}^{<\mathbb{N}})$, we can see $\varphi_{\mathscr{E},p}$ as a Borel function from Tr into SB. With this in mind, we have (see [29], Proposition 3.1, page 79):

Proposition 4. *Let $\varphi_{\mathscr{E},p} \colon \mathrm{Tr} \to \mathrm{SB}$ be the function defined above. Then $\varphi_{\mathscr{E},p}$ is a Borel function. The same is true if we define $\|\cdot\|_{\mathscr{E},0,\theta}$ as*

$$\|x\|_{\mathscr{E},0,\theta} = \sup\Bigg\{ \Bigg\| \sum_{s \in I} x(s) e_{|s|} \Bigg\|_{\mathscr{E}}; \ I \text{ segment of } \theta \Bigg\},$$

and let $\varphi_{\mathscr{E},0}(\theta)$ to be the completion of $(c_{00}(\theta), \|\cdot\|_{\mathscr{E},0,\theta})$.

Let $\theta \in \mathrm{Tr}$, $p \in [1, \infty)$, and let $\mathscr{E} = (e_n)_{n \in \mathbb{N}}$ be a basic sequence. We denote by $\mathscr{E}^*$ the same sequence as $\mathscr{E}$ but with the first term deleted. We clearly have that $\varphi_{\mathscr{E},p}(\theta)$ is isomorphic to

$$\mathbb{R} \oplus \Bigg(\bigoplus_{\lambda \in \Lambda} \varphi_{\mathscr{E}^*,p}(\theta(\lambda)) \Bigg)_{\ell_p},$$

where $\Lambda = \{\lambda \in \mathbb{N};\ (\lambda) \in \theta\}$, and the term $\mathbb{R}$ appears because of the empty coordinate of θ. The following lemma is of great importance for understanding the geometry of $\varphi_{\mathscr{E},p}(\theta)$.

Lemma 5. *The Borel function $\varphi_{\mathscr{E},p} \colon \mathrm{Tr} \to \mathrm{SB}$ defined above has the following properties:*

(i) *If $\theta \in \mathrm{IF}$, then $\varphi_{\mathscr{E},p}(\theta)$ contains $\overline{\operatorname{span}}\{\mathscr{E}\}$.*
(ii) *If $\theta \in \mathrm{WF}$, then $\varphi_{\mathscr{E},p}(\theta)$ is ℓ_p-saturated, i.e., every infinite dimensional subspace of $\varphi_{\mathscr{E},p}(\theta)$ contains a copy of ℓ_p.*

The analogous is true for $\varphi_{\mathscr{E},0}\colon \mathrm{Tr} \to \mathrm{SB}$, i.e.:

(i) *If $\theta \in \mathrm{IF}$, then $\varphi_{\mathscr{E},0}(\theta)$ contains $\overline{\mathrm{span}}\{\mathscr{E}\}$.*

(ii) *If $\theta \in \mathrm{WF}$, then $\varphi_{\mathscr{E},0}(\theta)$ is c_0-saturated, i.e., every infinite dimensional subspace of $\varphi_{\mathscr{E},0}(\theta)$ contains a copy of c_0.*

Before we prove this lemma, let us show a simple lemma that will be important in our proof.

Lemma 6. *A finite sum of ℓ_p-saturated or c_0-saturated spaces is ℓ_p-saturated or c_0-saturated, respectively.*

Proof. Say $(X_1, \|\cdot\|_1), \ldots, (X_n, \|\cdot\|_n)$ are ℓ_p-saturated. Let $(X, \|\cdot\|_X)$ be the sum of those spaces. As this is a finite sum, we can assume $X = \Big(\bigoplus_{j=1}^{n} X_j\Big)_{\ell_1}$, i.e., if $(x_1, \ldots, x_n) \in X$, then $\|x\|_X = \sum_j \|x_j\|_j$. Denote by $P_j\colon X \to X_j$ the standard projection on the j-th coordinate. Let $E \subset X$ be an infinite dimensional subspace.

Claim: $P_{j_0}\colon E \to X_{j_0}$ is not strictly singular, for some $j_0 \in \{1, \ldots, n\}$.

Once the claim is proved, the result trivially follows. Assume P_j is strictly singular for all $j \in \{1, \ldots, n\}$. By a classical property of strictly singular operators (see [12], Proposition B.5), we know that for every $\varepsilon > 0$ there exists an infinite dimensional subspace $A \subset E$ such that $\|P_{j|A}\| < \varepsilon$, for all $j \in \{1, \ldots, n\}$. Pick $x \in A$ of norm one. Then, as $x = (P_1(x), \ldots, P_n(x))$, we have $\|x\|_X \leqslant n\varepsilon$. By choosing $\varepsilon < 1/n$ we get a contradiction. $\square$

Proof of Lemma 5. If $\theta \in \mathrm{IF}$, clearly $\varphi_{\mathscr{E},p}(\theta)$ contains $\overline{\mathrm{span}}\{\mathscr{E}\}$. Indeed, let β be a branch of θ, then $\overline{\mathrm{span}}\{\mathscr{E}\} \cong \varphi_{\mathscr{E},p}(\beta) \hookrightarrow \varphi_{\mathscr{E},p}(\theta)$, where by $\varphi_{\mathscr{E},p}(\beta)$ we mean $\varphi_{\mathscr{E},p}$ applied to the tree $\{s \in \mathbb{N}^{<\mathbb{N}};\ s < \beta\}$.

Say $\theta \in \mathrm{WF}$. Let us proceed by transfinite induction on the order of θ. If $o(\theta) = 1$ the result is clear. Indeed, if $o(\theta) = 1$, $\varphi_{\mathscr{E},p}(\theta)$ is finite dimensional, so it has no infinite dimensional subspaces. Assume $\varphi_{\mathscr{E},p}(\theta)$ is ℓ_p-saturated for all basic sequences $\mathscr{E}$, and all $\theta \in \mathrm{WF}$ with $o(\theta) < \alpha$ for some $\alpha < \omega_1$. Fix $\theta \in \mathrm{WF}$ with $o(\theta) = \alpha$.

Let $\Lambda = \{\lambda \in \mathbb{N};\ (\lambda) \in \theta\}$, and enumerate Λ, say $\Lambda = \{\lambda_i;\ i \in \mathbb{N}\}$. For each $\lambda \in \Lambda$, let $\theta_\lambda = \{s \in \theta;\ (\lambda) \leqslant s\}$. As $\theta \in \mathrm{WF}$, Proposition 3 gives us

$$o(\theta(\lambda_j)) < o(\theta) = \alpha, \quad \forall j \in \mathbb{N}.$$

Consider now the projections

$$P_{\lambda_n}\colon\ \varphi_{\mathscr{E},p}(\theta) \to \varphi_{\mathscr{E},p}\Big(\bigcup_{j=1}^{n} \theta_{\lambda_j}\Big),$$
$$(a_s)_{s\in\theta} \to (a_s)_{s\in\bigcup_{j=1}^{n}\theta_{\lambda_j}}.$$

As $\varphi_{\mathscr{E},p}\Big(\bigcup_{j=1}^{n}\theta_{\lambda_j}\Big)$ is the direct sum of $\bigoplus_{j=1}^{n}\varphi_{\mathscr{E}^*,p}(\theta(\lambda_j))$ with a finite dimensional space, our inductive hypothesis holds for it as well. Indeed, it is clear that

$$\varphi_{\mathscr{E},p}\Big(\bigcup_{j=1}^{n}\theta_{\lambda_j}\Big) \cong \mathbb{R} \oplus \Big(\bigoplus_{j=1}^{n}\varphi_{\mathscr{E}^*,p}(\theta(\lambda_j))\Big).$$

Hence, the inductive hypothesis and Lemma 6, give us that $\varphi_{\mathscr{E},p}\Big(\bigcup_{j=1}^{n}\theta_{\lambda_j}\Big)$ is ℓ_p-saturated as well.

Say $E \subset \varphi_{\mathscr{E},p}(\theta)$ is an infinite dimensional subspace.

Case 1: There exists $j \in \mathbb{N}$ such that $P_{\lambda_j}\colon E \to \varphi_{\mathscr{E},p}\Big(\bigcup_{i=1}^{j}\theta_{\lambda_i}\Big)$ is not strictly singular.

Then there exists an infinite dimensional subspace $\widetilde{E} \subset E$ such that $P_{\lambda_j}{}_{|\widetilde{E}}$ is an isomorphism onto its image. By our inductive hypothesis, $\varphi_{\mathscr{E},p}\Big(\bigcup_{i=1}^{j}\theta_{\lambda_i}\Big)$ is ℓ_p-saturated, so we are done.

Case 2: $P_{\lambda_j}\colon E \to \varphi_{\mathscr{E},p}\Big(\bigcup_{i=1}^{j}\theta_{\lambda_i}\Big)$ is strictly singular for all $j \in \mathbb{N}$.

Claim: There exists $(x_n)_{n\in\mathbb{N}}$ a normalized sequence in E such that $P_{\lambda_j}(x_n) \to 0$ as $n \to \infty$, for all $j \in \mathbb{N}$.

Indeed, by a well-known consequence of the definition of strictly singular operators for all $j \in \mathbb{N}$ there exists a normalized sequence $(x_n^j)_{n\in\mathbb{N}}$ such that $\|P_{\lambda_j}(x_n^j)\| < 1/n$ for all $n \in \mathbb{N}$. Let $(x_n)_{n\in\mathbb{N}}$ be the diagonal sequence of the sequences $(x_n^j)_{n\in\mathbb{N}}$, i.e., $x_n = x_n^n$ for all $n \in \mathbb{N}$. As $i \leqslant j$ implies $\|P_{\lambda_i}(x)\| \leqslant \|P_{\lambda_j}(x)\|$ for all $x \in E$, $(x_n)_{n\in\mathbb{N}}$ has the required property.

Say $(\varepsilon_i)_{i\in\mathbb{N}}$ is a sequence of positive numbers converging to zero. Using the claim above and the fact that $P_{\lambda_j}(x) \to x$ as $n \to \mathbb{N}$, for all $x \in \varphi_{\mathscr{E},p}(\theta)$, we can pick increasing sequences of natural numbers $(n_k)_{k\in\mathbb{N}}$ and $(l_k)_{k\in\mathbb{N}}$ such that

i) $\|P_{\lambda_{l_k}}(x_{n_k}) - x_{n_k}\|_\theta < \varepsilon_k$ for all $k \in \mathbb{N}$, and

ii) $\|P_{\lambda_{l_k}}(x_{n_{k+1}})\|_\theta < \varepsilon_k$ for all $k \in \mathbb{N}$.

For all $k \in \mathbb{N}$, let $y_k = P_{\lambda_{l_k}}(x_{n_k}) - P_{\lambda_{l_{k-1}}}(x_{n_k})$. Choosing ε_k small enough we can assume $\|y_k\|_\theta^p \in (1/2, 2)$. It is easy to see that $(y_k)_{k\in\mathbb{N}}$ is equivalent to $(\tilde{e}_k)_{k\in\mathbb{N}}$, where $(\tilde{e}_k)_{k\in\mathbb{N}}$ is the standard ℓ_p-basis. Indeed, picking $a_1, \ldots, a_N \in \mathbb{R}$, then

$$\frac{1}{2}\sum_{i=1}^{N}|a_i|^p \leqslant \sum_{i=1}^{N}\|a_i y_i\|_\theta^p = \Big\|\sum_{i=1}^{N}a_i y_i\Big\|_\theta^p = \sum_{i=1}^{N}\|a_i y_i\|_\theta^p \leqslant 2\sum_{i=1}^{N}|a_i|^p,$$

where the equalities above only hold because the supports of $(y_k)_{k\in\mathbb{N}}$ are completely incomparable. Therefore, by choosing $(\varepsilon_k)_{k\in\mathbb{N}}$ converging to zero fast enough, the principle of small perturbations (see [1], Theorem 1.3.9) gives us that $(x_{n_k})_{k\in\mathbb{N}}$ is equivalent to $(y_k)_{k\in\mathbb{N}} \sim (\tilde{e}_k)_{k\in\mathbb{N}}$. So E contains a copy of ℓ_p.

The proof that $\varphi_{\mathscr{E},0}(\theta)$ is c_0-saturated of $\theta \in \mathrm{WF}$ is analogous. □

By letting $\mathscr{E}$ be a basis for the universal space $C(\Delta)$ we get the following corollary.

Corollary 7. *The set of universal spaces cannot be separated by a Borel set from the set of ℓ_p-saturated spaces, for all $p \in [1,\infty)$, i.e., there is no Borel subset $U \subset \mathrm{SB}$ such that all the universal spaces (of SB) are in U and all the ℓ_p-saturated spaces (of SB) are not in U.*

4. Complementability of ideals of $\mathscr{L}(X)$, Part I

4.1. Unconditionally converging operators. We say that an operator $T\colon X \to Y$ is *unconditionally converging* (see [25]) if it maps weakly unconditionally Cauchy series into unconditionally converging series. Let X and Y be Banach spaces. We let $\mathscr{U}(X)$ be the set of unconditionally converging operators from X to itself.

We write $Y \overset{\perp}{\hookrightarrow} X$ if Y is isomorphic to a complemented subspace of X.

Theorem 8. *Let $\mathscr{U} = \{X \in \mathrm{SB};\ \mathscr{U}(X) \overset{\perp}{\hookrightarrow} \mathscr{L}(X)\}$. Then $\mathscr{U}$ is complete coanalytic.*

P r o o f. In order to show this we only need to use that $\mathscr{U}(X)$ is complemented in $\mathscr{L}(X)$ if and only if c_0 does not embed in X (see [4], page 452). Therefore, $\mathscr{U} = \mathrm{NC}_{c_0}$ (where $\mathrm{NC}_X = \{Y \in \mathrm{SB};\ X \not\hookrightarrow Y\}$ for $X \in \mathrm{SB}$). Applying Lemma 5 to $p = 2$, and letting $\mathscr{E}$ be the standard basis of c_0, we obtain that $\varphi^{-1}_{\mathscr{E},p}(\mathscr{U}) = \mathrm{WF}$. As NC_X is well known to be coanalytic for all $X \in \mathrm{SB}$, we are done. We would like to point out that NC_X was shown to be complete coanalytic, for all infinite dimensional $X \in \mathrm{SB}$, in [6], so this result is actually just a corollary of [6] and [4]. □

4.2. Weakly compact operators. We say that an operator $T\colon X \to Y$ is *weakly compact* if it maps bounded sets into relatively weakly compact sets. For $X \in \mathrm{SB}$ we let $\mathscr{W}(X)$ be the set of weakly compact operators on X to itself.

Theorem 9. *Let $\mathscr{W} = \{X \in \mathrm{SB};\ \mathscr{W}(X) \overset{\perp}{\hookrightarrow} \mathscr{L}(X)\}$. Then $\mathscr{W}$ is Π^1_1-hard. In particular, $\mathscr{W}$ is non Borel.*

This result is a simple consequence of the following lemma (whose statement and part of its proof can be found in [29], Proposition 2.2, page 78).

Lemma 10. *Let* $\mathscr{E} = (e_n)_{n\in\mathbb{N}}$ *be a basic sequence, and* $p \in (1, \infty)$*. Then* $\varphi_{\mathscr{E},p}(\theta)$ *is reflexive for all* $\theta \in \mathrm{WF}$*.*

P r o o f of Theorem 9. In order to show this we will use another result of [4], page 450. In that paper it is shown that if $c_0 \hookrightarrow X$, then $\mathscr{W}(X)$ is not complemented in $\mathscr{L}(X)$. Let $\varphi_{\mathscr{E},2}\colon \mathrm{Tr} \to \mathrm{SB}$, where $\mathscr{E}$ is the standard basis of c_0. Let us observe that $\varphi^{-1}_{\mathscr{E},2}(\mathscr{W}) = \mathrm{WF}$. Indeed, if $\theta \in \mathrm{IF}$ we saw that $c_0 \hookrightarrow \varphi_{\mathscr{E},2}(\theta)$, hence $\varphi_{\mathscr{E},2}(\theta) \notin \mathscr{W}$. If $\theta \in \mathrm{WF}$, then Lemma 10 implies that $\varphi_{\mathscr{E},2}(\theta)$ is reflexive, which implies $\varphi_{\mathscr{E},2}(\theta) \in \mathscr{W}$. Indeed, a Banach space is reflexive if and only if its unit ball is weakly compact, therefore $\mathscr{W}(X) = \mathscr{L}(X)$. $\square$

Problem 11. Is $\mathscr{W}$ coanalytic? If yes, we had shown that $\mathscr{W}$ is complete coanalytic.

5. Geometry of Banach spaces

5.1. Banach-Saks property. A Banach space X is said to have the *Banach-Saks property* if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that its Cesàro mean $n^{-1}\sum\limits_{k=1}^{n} x_{n_k}$ is norm convergent. We denote the subset of SB coding the separable Banach spaces with the Banach-Saks property by BS.

In [5], page 373, B. Beauzamy characterized not having the Banach-Saks property in terms of the existence of a sequence satisfying some geometrical inequality. Precisely:

Theorem 12. *An* $X \in \mathrm{SB}$ *does not have the Banach-Saks property if and only if there exist* $\varepsilon > 0$ *and a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* B_X *such that, for all subsequences* $(x_{n_k})_{k\in\mathbb{N}}$ *for all* $m \in \mathbb{N}$*, and for all* $l \in \{1, \ldots, m\}$*, the following holds:*

$$\left\| \frac{1}{m}\left(\sum_{k=1}^{l} x_{n_k} - \sum_{k=l+1}^{m} x_{n_k}\right)\right\| \geqslant \varepsilon.$$

Theorem 13. BS *is coanalytic in* SB.

P r o o f. This is just a matter of applying Theorem 12 and counting quantifiers. Indeed,

$$\begin{aligned} X \in \mathrm{BS} \Leftrightarrow \forall (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}},\ &\forall \varepsilon \in \mathbb{Q}_+, \\ &\exists m \in \mathbb{N},\ \exists l \in \{1, \ldots, m\},\ \exists k_1 < \ldots < k_m \in \mathbb{N}, \\ &\text{such that } \left\| \frac{1}{m}\left(\sum_{j=1}^{l} S_{n_{k_j}}(B_X) - \sum_{j=l+1}^{m} S_{n_{k_j}}(B_X)\right)\right\| < \varepsilon, \end{aligned}$$

where $\{S_n\}_{n\in\mathbb{N}}$ is the sequence of Borel functions in Lemma 1. As $X \mapsto B_X$ is a Borel function from SB into $\mathscr{F}(C(\Delta))$, we are done. $\square$

The previous theorem shows that BS is at least coanalytic in SB, but it does not say anything about BS being Borel or not. The next theorem takes care of this by showing that coanalyticity is the most we can get of BS in relation to its complexity.

Theorem 14. BS *is* Π^1_1*-hard. Moreover,* BS *is complete analytic.*

Proof. Let $\mathscr{E}$ be the standard ℓ_1 basis, and $p = 2$. Let us verify that $\varphi^{-1}_{\mathscr{E},p}(\mathrm{BS}) = \mathrm{WF}$.

If $\theta \in \mathrm{IF}$ we clearly have $\ell_1 \hookrightarrow \varphi_{\mathscr{E},p}(\theta)$. Indeed, if β is a branch of θ we have $\varphi_{\mathscr{E},p}(\beta) \cong \ell_1$. As $\ell_1 \hookrightarrow \varphi_{\mathscr{E},p}(\theta)$ and ℓ_1 is clearly not in BS (taking its standard basis for example, it clearly does not have a subsequence with norm converging Cesàro mean) we conclude that $\varphi_{\mathscr{E},p}(\theta) \notin \mathrm{BS}$.

Let us show that if $\theta \in \mathrm{WF}$, then $\varphi_{\mathscr{E},p}(\theta) \in \mathrm{BS}$. We proceed by transfinite induction on the order of $\theta \in \mathrm{WF}$. Say $o(\theta) = 1$. Then, for all basic sequences $\widetilde{\mathscr{E}}$, $\varphi_{\widetilde{\mathscr{E}},p}(\theta)$ is 1-dimensional and we are clearly done. Assume $\varphi_{\widetilde{\mathscr{E}},p}(\theta) \in \mathrm{BS}$ for all basic sequences $\widetilde{\mathscr{E}}$, and all $\theta \in \mathrm{WF}$ with $o(\theta) < \alpha$ for some $\alpha < \omega_1$. Pick $\theta \in \mathrm{WF}$ with $o(\theta) = \alpha$, a basic sequence $\widetilde{\mathscr{E}}$, and let us show that $\varphi_{\widetilde{\mathscr{E}},p}(\theta) \in \mathrm{BS}$.

Let $\Lambda = \{\lambda \in \mathbb{N}; (\lambda) \in \theta\}$. As $\theta \in \mathrm{WF}$, Proposition 3 gives us

$$o(\theta(\lambda)) < o(\theta) = \alpha, \quad \forall \lambda \in \Lambda.$$

Our induction hypothesis implies that $\varphi_{\widetilde{\mathscr{E}}^*,p}(\theta(\lambda)) \in \mathrm{BS}$ for all $\lambda \in \Lambda$. Now, notice that

$$\varphi_{\widetilde{\mathscr{E}},p}(\theta) \cong \mathbb{R} \oplus \left(\bigoplus_{\lambda \in \Lambda} \varphi_{\widetilde{\mathscr{E}}^*,p}(\theta(\lambda)) \right)_{\ell_2},$$

where we get the $\mathbb{R}$ above because of the coordinate related to $s = \emptyset \in \theta$. By J. R. Partington's result in [23], page 370, we have that the ℓ_2-sum of spaces in BS is also in BS. Hence, $\left(\bigoplus_{\lambda \in \Lambda} \varphi_{\widetilde{\mathscr{E}}^*,p}(\theta(\lambda)) \right)_{\ell_2}$ is in BS and we conclude that $\varphi_{\widetilde{\mathscr{E}},p}(\theta) \in \mathrm{BS}$. The transfinite induction is now over, and so is our proof. $\square$

5.2. Alternating Banach-Saks property. A Banach space X is said to have the *alternating Banach-Saks property* if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that its alternating-signs Cesàro mean $n^{-1} \sum_{k=1}^{n} (-1)^k x_{n_k}$ is norm convergent. We denote the set coding the separable Banach spaces with the alternating Banach-Saks property by ABS.

In [5], page 369, B. Beauzamy proves the following:

Theorem 15. *A $X \in \mathrm{SB}$ does not have the alternating Banach-Saks property if and only if there exist $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X such that for all $l \in \mathbb{N}$, if $l \leqslant n(1) < \ldots < n(2^l)$, where $n(i) \in \mathbb{N}$ for all $i \in \{1, \ldots, 2^l\}$, then*

$$\left\| \sum_{i=1}^{2^l} c_i x_{n(i)} \right\| \geqslant \varepsilon \sum_{i=1}^{2^l} |c_i|$$

for all $c_1, \ldots, c_{2^l} \in \mathbb{R}$.

Theorem 16. ABS *is coanalytic in* SB.

Proof. This is just a matter of applying Theorem 15 and counting quantifiers. Indeed,

$$\begin{aligned} X \in \mathrm{ABS} \Leftrightarrow\ & \forall (n_k)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}},\ \forall \varepsilon \in \mathbb{Q}_+, \\ & \exists l \in \mathbb{N},\ \exists l \leqslant k(1) < \ldots < k(2^l) \in \mathbb{N}, \\ & \text{such that } \exists c_1, \ldots, c_{2^l} \in \mathbb{Q},\ \left\| \sum_{j=1}^{2^l} c_k S_{n_{k(j)}}(B_X) \right\| < \varepsilon \sum_{j=1}^{2^l} |c_j|. \end{aligned}$$

□

Now we show that coanalyticity is the most we can get of ABS in relation to its complexity.

Theorem 17. ABS *is* Π_1^1*-hard. Moreover,* ABS *is complete coanalytic.*

Proof. Let $\mathscr{E}$ be the standard ℓ_1 basis, and $p = 2$. We will show that $\varphi_{\mathscr{E},p}^{-1}(\mathrm{ABS}) = \mathrm{WF}$.

If $\theta \in \mathrm{IF}$, we have $\ell_1 \hookrightarrow \varphi_{\mathscr{E},p}(\theta)$. As ℓ_1 is not in ABS (we can take its standard basis again, it clearly does not have a subsequence with norm converging alternating-signs Cesàro mean) we conclude that $\varphi_{\mathscr{E},p}(\theta) \notin \mathrm{ABS}$.

Let us show that if $\theta \in \mathrm{WF}$, then $\varphi_{\mathscr{E},p}(\theta) \in \mathrm{ABS}$. We proceed by transfinite induction on the order of $\theta \in \mathrm{WF}$. Say $o(\theta) = 1$. Then, for any basic sequence $\widetilde{\mathscr{E}}$, $\varphi_{\widetilde{\mathscr{E}},p}(\theta)$ is 1-dimensional and we are clearly done. Assume $\varphi_{\widetilde{\mathscr{E}},p}(\theta) \in \mathrm{ABS}$ for all basic sequences $\widetilde{\mathscr{E}}$, and all $\theta \in \mathrm{WF}$ with $o(\theta) < \alpha$ for some $\alpha < \omega_1$. Pick $\theta \in \mathrm{WF}$ with $o(\theta) = \alpha$.

Using the same notation as in the proof of Theorem 14, we have

$$\varphi_{\widetilde{\mathscr{E}},p}(\theta) \cong \mathbb{R} \oplus \left(\bigoplus_{\lambda\in\Lambda} \varphi_{\widetilde{\mathscr{E}}^*,p}(\theta(\lambda)) \right)_{\ell_2}.$$

By Lemma 5, $\ell_1 \not\hookrightarrow \varphi_{\widetilde{\mathscr{E}},p}(\theta)$. B. Beauzamy showed in [5], page 368, that a Banach space not containing ℓ_1 has the alternating Banach-Saks property if and only if it has the weak Banach-Saks property. So, we only need to show that $\varphi_{\widetilde{\mathscr{E}},p}(\theta)$ is in WBS. As $\varphi_{\widetilde{\mathscr{E}},p}(\theta(\lambda)) \in \mathrm{ABS}$ for all $\lambda \in \Lambda$, we have $\varphi_{\widetilde{\mathscr{E}},p}(\theta(\lambda)) \in \mathrm{WBS}$ for all $\lambda \in \Lambda$. By a corollary of J. R. Partington (see [23], page 373), $\Big(\bigoplus_{\lambda\in\Lambda} \varphi_{\widetilde{\mathscr{E}}^*,p}(\theta(\lambda))\Big)_{\ell_2}$ is also in WBS. Thus, we conclude that $\varphi_{\widetilde{\mathscr{E}},p}(\theta) \in \mathrm{WBS}$, and we are done. □

5.3. Weak Banach-Saks property. A Banach space is said to have the *weak Banach-Saks property* if every weakly null sequence has a subsequence such that its Cesàro mean is norm convergent to zero. We denote the set coding the separable Banach spaces with the weak Banach-Saks property by WBS. The weak Banach-Saks property is often called the Banach-Saks-Rosenthal property.

Theorem 18. WBS *is* Π^1_1*-hard. In particular,* WBS *is non Borel.*

P r o o f. First we notice that we cannot use the same $\mathscr{E}$ as in Theorem 14 because, as ℓ_1 has the Schur property, ℓ_1 is clearly in WBS. Let $\mathscr{E}$ be a basis for $C(\Delta)$, and $p = 2$. It is shown in [14] that $C(\Delta)$ is not in WBS. If we proceed exactly as in the proof of Theorem 17, and use the stability of the weak Banach-Saks property under ℓ_2-sums (see [23], page 373), we will be done. □

Remark. It is worth noticing that the same $\varphi_{\mathscr{E},p}$ as constructed above could be used to prove Theorem 14, and Theorem 17.

With that being said, let us try to obtain more information about the complexity of WBS. For this we use the following lemma.

Lemma 19. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Banach space X. A sequence $(x_n)_{n\in\mathbb{N}}$ is weakly null if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a convex block subsequence converging to zero in norm. In particular, if $(x_n)_{n\in\mathbb{N}}$ is a weakly null sequence in a Banach space X, and if X embeds into another Banach space Y, then $(x_n)_{n\in\mathbb{N}}$ is weakly null in Y.*

P r o o f. Say every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a convex block subsequence converging to zero in norm. First we show that $(x_n)_{n\in\mathbb{N}}$ has a weakly null subsequence. As $(x_n)_{n\in\mathbb{N}}$ is bounded, Rosenthal's ℓ_1-theorem (see [27]) says that we can find a subsequence that is either weak-Cauchy or equivalent to the usual ℓ_1-basis. As ℓ_1's usual basis has no subsequence with a convex block sequence converging to zero in norm, we conclude that $(x_n)_{n\in\mathbb{N}}$ must have a weak-Cauchy subsequence. By hypothesis, this sequence must have a convex block subsequence converging to zero in norm, say $\left(y_k = \sum_{i=l_k+1}^{l_{k+1}} a_i x_{n_i}\right)_{k\in\mathbb{N}}$, for some subsequence (n_k) of natural numbers.

Say $(x_{n_k})_{k\in\mathbb{N}}$ is not weakly null. Then pick $f \in X^*$ such that $f(x_{n_k}) \not\to 0$. As $(x_{n_k})_{k\in\mathbb{N}}$ is weak-Cauchy, there exists $\delta \neq 0$ such that $f(x_{n_k}) \to \delta$. Hence, $f(y_k) \to \delta$, absurd, because $(y_k)_{k\in\mathbb{N}}$ is norm convergent to zero.

Now assume $(x_n)_{n\in\mathbb{N}}$ is not weakly null. Then we can pick $f \in X^*$, a subsequence $(n_k)_{k\in\mathbb{N}}$, and $\delta \neq 0$, such that $f(x_{n_k}) \to \delta$. As the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ has the same property as $(x_n)_{n\in\mathbb{N}}$, we can pick a weakly null subsequence, say $(x_{n_{k_l}})_{l\in\mathbb{N}}$. Hence $f(x_{n_{k_l}}) \to 0$, absurd.

For the converse we only need to apply Mazur's theorem. □

For every $X \in \mathrm{SB}$, let

$$E(X) = \Big\{((x_k)_{k\in\mathbb{N}}, (n_k)_{k\in\mathbb{N}}) \in X^{\mathbb{N}} \times [\mathbb{N}];\ \exists r \in \mathbb{N},\ \forall j \in \mathbb{N},\ \|x_j\| < r\ \forall \varepsilon \in \mathbb{Q}_+, \\ \forall n \in \mathbb{N},\ \exists a_n, \dots, a_{n+l} \in \mathbb{Q}_+\left(\sum_{i=n}^{n+l} a_i = 1\right),\ \left\|\sum_{i=n}^{n+l} a_i x_{n_i}\right\| < \varepsilon\Big\},$$

where $[\mathbb{N}]$ stands for the subset of $\mathbb{N}^{\mathbb{N}}$ consisting of all increasing sequences of natural numbers. As $[\mathbb{N}]$ is easily seen to be Borel, we have that $E(X)$ is Borel in $X^{\mathbb{N}} \times [\mathbb{N}]$. Define $F(X)$ by

$$F(X)^c = \pi(E(X)^c),$$

where π denotes the projection into the first coordinate. Notice that $F(X)$ is coanalytic and that $F(X)$ consists of all the bounded sequences in $X^{\mathbb{N}}$ with the property that all of its subsequences have a convex block subsequence converging to zero in norm. By Lemma 19, $F(X)$ is the set of all weakly null sequences of X.

Theorem 20. *The set of weakly null sequences $F(X) \subset X^{\mathbb{N}}$ of X is coanalytic, for all $X \in \mathrm{SB}$.*

Say $F = F(C(\Delta))$. Let $A = \{(X, (x_n)_{n\in\mathbb{N}}) \in \mathrm{SB} \times F;\ \forall n \in \mathbb{N},\ x_n \in X\}$, and

$$G = \pi\Bigg(\Big\{(X, (x_n)_{n\in\mathbb{N}}) \in A;\ \exists \varepsilon \in \mathbb{Q}_+,\ \forall n_1 < \ldots < n_m,\ \forall l \in \{1, \dots, m\}, \\ \left\|\frac{1}{m}\left(\sum_{k=1}^{l} x_{n_k} - \sum_{k=l+1}^{m} x_{n_k}\right)\right\| \geqslant \varepsilon\Big\}\Bigg),$$

where π denotes the projection into SB. B. Beauzamy's paper (see [5]) implies that $\mathrm{WBS} = G^c$. We have just shown that WBS is the complement of a Borel image of a coanalytic set. If a subset of a standard Borel space X has this property we say that it belongs to $\Pi_2^1(X)$, see [21] or [29] for more details on the projective hierarchy $(\Sigma_n^1, \Pi_n^1)_{n\in\mathbb{N}}$.

Theorem 21. $\mathrm{WBS} \in \Pi_2^1(\mathrm{SB})$.

Problem 22. Is WBS coanalytic? If yes, we have shown that WBS is complete coanalytic.

Remark. We have just seen that the set of weakly null subsequences $F(X) \subset X^{\mathbb{N}}$ of a separable Banach space X is coanalytic in $X^{\mathbb{N}}$. It is easy to see that $F(X)$ is actually Borel if X^* is separable. Indeed, if $\{f_n\}_{n\in\mathbb{N}}$ is dense in X^*, we have

$$F(X) = \bigcap_{n\in\mathbb{N}} \bigcap_{\varepsilon\in\mathbb{Q}_+} \bigcup_{k\in\mathbb{N}} \bigcap_{m>k} \{(x_j)_{j\in\mathbb{N}} \in X^{\mathbb{N}};\ |f_n(x_m)| < \varepsilon\}.$$

Also, as ℓ_1 is a Schur space, $F(\ell_1)$ consists of the set of norm null sequences in ℓ_1, and it is easily seen to be Borel. Which means, X^* does not need to be separable in order to $F(X)$ to be Borel.

On the other hand, if $\mathscr{E}$ is the ℓ_1-basis and $p = 2$, we have that the standard basis of $\varphi_{\mathscr{E},p}(\theta)$ is weakly null if and only if $\theta \in \mathrm{WF}$. Therefore, $F(\varphi_{\mathscr{E},p}(\mathbb{N}^{<\mathbb{N}}))$ is complete coanalytic. For the same reason, $F(C(\Delta))$ is complete coanalytic.

Problem 23. Under what conditions is $F(X)$ (coanalytic) non Borel?

6. Complementability of ideals of $\mathscr{L}(X)$, Part II

6.1. Banach-Saks operators. In the same spirit as Sections 3 and 4, we now take a look at operator ideals of $\mathscr{L}(X)$. Let X be a Banach space, we say $T \in \mathscr{L}(X)$ is a emphBanach-Saks operator if for each bounded sequence $(x_n)_{n\in\mathbb{N}}$ there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that the Cesàro mean $n^{-1}\sum\limits_{k=1}^{n} T(x_{n_k})$ is norm convergent. We denote the space of Banach-Saks operators from X to itself by $\mathscr{BS}(X)$.

Theorem 24. *The set* $\mathscr{BS} = \{X \in \mathrm{SB};\ \mathscr{BS}(X) \overset{\perp}{\hookrightarrow} \mathscr{L}(X)\}$ *is* Π_1^1*-hard. In particular,* $\mathscr{BS}$ *is non Borel.*

Proof. Let $\mathscr{E}$ be a basis for $C(\Delta)$, and $p = 2$. If $\theta \in \mathrm{WF}$, then $\varphi_{\mathscr{E},p}(\theta) \in \mathrm{BS}$. Hence, $\mathscr{BS}(\varphi_{\mathscr{E},p}(\theta)) = \mathscr{L}(\varphi_{\mathscr{E},p}(\theta))$, and we have $\varphi_{\mathscr{E},p}(\theta) \in \mathscr{BS}$ for all $\theta \in \mathrm{WF}$. Let us show that the same cannot be true if $\theta \in \mathrm{IF}$.

Say $\theta \in \mathrm{IF}$. Then $\varphi_{\mathscr{E},p}(\theta) \cong C(\Delta) \oplus Y$ for some $Y \in \mathrm{SB}$. Let $P_1\colon C(\Delta) \oplus Y \to C(\Delta)$ be the standard projection. Suppose there exists a bounded projection

$P\colon \mathscr{L}(C(\Delta)\oplus Y)\to \mathscr{BS}(C(\Delta)\oplus Y)$. Define $P_0\colon \mathscr{L}(C(\Delta))\to \mathscr{BS}(C(\Delta))$ as, for all $T\in\mathscr{L}(C(\Delta))$,

$$P_0(T)=P_1(P(\widetilde{T}))_{|C(\Delta)},$$

where $\widetilde{T}\colon C(\Delta)\oplus Y\to C(\Delta)\oplus Y$ is the natural extension, i.e., $\widetilde{T}(x,y)=(T(x),0)$ for all $(x,y)\in C(\Delta)\oplus Y$. Notice that $P_0(T)\in\mathscr{BS}(C(\Delta))$, so P_0 is well defined. Also, if $T\in\mathscr{BS}(C(\Delta))$, then $\widetilde{T}\in\mathscr{BS}(C(\Delta)\oplus Y)$, which implies $P(\widetilde{T})=\widetilde{T}$ (because P is a projection). Therefore, P_0 is a projection from $\mathscr{L}(C(\Delta))$ onto $\mathscr{BS}(C(\Delta))$. Let us observe that this gives us a contradiction.

It is known that $T\colon C(\Delta)\to C(\Delta)$ has the Banach-Saks property if and only if T is weakly compact (see [11], page 112). Hence, $\mathscr{BS}(C(\Delta))=\mathscr{W}(C(\Delta))$ and, as $c_0\hookrightarrow C(\Delta)$, we have that $\mathscr{BS}(C(\Delta))$ is not complemented in $\mathscr{L}(C(\Delta))$ [4]. Absurd. □

Problem 25. Is $\mathscr{BS}$ coanalytic? If yes, our previous proof would show that $\mathscr{BS}$ is complete coanalytic.

We have studied three classes of ideals of $\mathscr{L}(X)$ ($\mathscr{U}(X)$, $\mathscr{W}(X)$, and $\mathscr{BS}(X)$) and whether those ideals are complemented in $\mathscr{L}(X)$ or not. Another natural question would be to study the complexity of pairs $(X,Y)\in\mathrm{SB}^2$ such that their respective ideals ($\mathscr{U}(X,Y)$, $\mathscr{W}(X,Y)$, and $\mathscr{BS}(X,Y)$) are complemented in $\mathscr{L}(X,Y)$. As mentioned in the introduction, this problem had been solved for the ideal of compact operators $\mathscr{K}(X,Y)$ by D. Puglisi in [26].

Let $\varphi_{\mathscr{E},p}\colon \mathrm{Tr}\to\mathrm{SB}$ be as defined above and define $\varphi(\theta)=(\varphi_{\mathscr{E},p}(\theta),\varphi_{\mathscr{E},p}(\theta))\in\mathrm{SB}^2$ for all $\theta\in\mathrm{Tr}$. Clearly, we have that $\varphi^{-1}(\{(X,Y)\in\mathrm{SB}^2;\ \mathscr{BS}(X,Y)\overset{\perp}{\hookrightarrow}\mathscr{L}(X,Y)\})=\mathrm{WF}$. Conclusion:

Theorem 26. *The following sets are* Π^1_1*-hard (hence, non Borel) in the product* SB^2: $\{(X,Y)\in\mathrm{SB}^2;\ \mathscr{BS}(X,Y)\overset{\perp}{\hookrightarrow}\mathscr{L}(X,Y)\}$, $\{(X,Y)\in\mathrm{SB}^2;\ \mathscr{U}(X,Y)\overset{\perp}{\hookrightarrow}\mathscr{L}(X,Y)\}$, *and* $\{(X,Y)\in\mathrm{SB}^2;\ \mathscr{W}(X,Y)\overset{\perp}{\hookrightarrow}\mathscr{L}(X,Y)\}$.

7. Geometry of Banach spaces, Part II

7.1. Schur property. We say that a Banach space X has the *Schur property* if every weakly convergent sequence of X is norm convergent.

Theorem 27. *Let* $\mathrm{S} = \{X \in \mathrm{SB};\ X$ *has the Schur property*$\}$, *then* S *is* Π_1^1*-hard. In particular, S is non Borel.*

P r o o f. Let $\mathscr{E}$ be the standard basis for c_0, and $p = 1$. As $c_0 \hookrightarrow \varphi_{\mathscr{E},p}(\theta)$ if $\theta \in \mathrm{IF}$, we have $\varphi_{\mathscr{E},p}(\theta) \notin \mathrm{S}$ for all $\theta \in \mathrm{IF}$. Mimicking the proof of Theorem 14 we have that

$$\varphi_{\mathscr{E},p}(\theta) \cong \mathbb{R} \oplus \left(\bigoplus_{\lambda \in \Lambda} \varphi_{\mathscr{E}^*,p}(\theta(\lambda)) \right)_{\ell_1},$$

where $\Lambda = \{\lambda \in \mathbb{N}; (\lambda) \in \theta\}$. Proceeding by transfinite induction and using B. Tanbay's result about the stability of the Schur property under ℓ_1-sums (see [30], page 350), we conclude that $\varphi_{\mathscr{E},p}(\theta) \in \mathrm{S}$ for all $\theta \in \mathrm{WF}$. $\square$

Let us try to obtain more information about the complexity of S. For this, notice that a Banach space X does not have the Schur property if and only if it has a weakly null sequence $(x_n)_{n\in\mathbb{N}}$ in S_X.

Let $F = F(C(\Delta))$ be defined as in Section 5, i.e., F is the set of all weakly null subsequences of $C(\Delta)$. Let $E = F \cap S_{C(\Delta)}^{\mathbb{N}}$, so E is coanalytic in $S_{C(\Delta)}^{\mathbb{N}}$, and define

$$G = \pi(\{(X, (x_n)_{n\in\mathbb{N}}) \in \mathrm{SB} \times E;\ \forall n \in \mathbb{N},\ x_n \in X\}),$$

where π denotes the projection into SB. We can easily see that $\mathrm{S} = G^c$. We have just shown that S is the complement of a Borel image of a coanalytic set.

Theorem 28. $\mathrm{S} \in \Pi_2^1(\mathrm{SB})$.

Remark. Notice that, if $F = F(C(\Delta))$ is Borel, then we have actually shown that S is coanalytic.

Problem 29. Is S coanalytic? If yes, our previous proof would show that S is complete coanalytic.

7.2. Dunford-Pettis property. A Banach space X is said to have the *Dunford-Pettis property* if every weakly compact operator $T\colon X \to Y$ from X into another Banach space Y takes weakly compact sets into norm-compact sets. In other words, X has the Dunford-Pettis property if every weakly compact operator from X into another Banach space Y is completely continuous. We have the following (see [28], and [13]):

Theorem 30. X^* *has the Schur property if and only if* X *has the Dunford-Pettis property and* X *does not contain* ℓ_1.

Theorem 31. *Let* $\mathrm{DP} = \{X \in \mathrm{SB};\ X$ *has the Dunford-Pettis property*$\}$. DP *is* Π^1_1*-hard. In particular,* DP *is non Borel.*

Proof. Let $\mathscr{E}$ be the standard basis for ℓ_2, and $p = 0$. We show that $\varphi^{-1}_{\mathscr{E},0}(\mathrm{DP}) = \mathrm{WF}$.

If $\theta \in \mathrm{IF}$ we have $\varphi_{\mathscr{E},0}(\theta) \cong \ell_2 \oplus Y$ for some Banach space Y. Hence, as ℓ_2 is reflexive, it is clear that $T(x, y) = (x, 0)$ is a weakly compact operator from $\ell_2 \oplus Y$ to itself which is not completely continuous. Therefore, $\varphi_{\mathscr{E},0}(\theta) \notin \mathrm{DP}$ for all $\theta \in \mathrm{IF}$.

Say $\theta \in \mathrm{WF}$. By Theorem 30, in order to show that $\varphi_{\mathscr{E},0}(\theta) \in \mathrm{DP}$ it is enough to show that $\varphi_{\mathscr{E},0}(\theta)^*$ has the Schur property. With the same notation as in the proofs of the previous theorems, we have

$$\varphi_{\mathscr{E},0}(\theta) \cong \mathbb{R} \oplus \Bigl(\bigoplus_{\lambda\in\Lambda} \varphi_{\mathscr{E}^*,0}(\theta(\lambda))\Bigr)_{c_0},$$

where $\Lambda = \{\lambda \in \mathbb{N};\ (\lambda) \in \theta\}$. Hence, we have

$$\varphi_{\mathscr{E},0}(\theta)^* \cong \mathbb{R} \oplus \Bigl(\bigoplus_{\lambda\in\Lambda} \varphi_{\mathscr{E}^*,0}(\theta(\lambda))^*\Bigr)_{\ell_1}.$$

Therefore, if we proceed by transfinite induction and use the stability of the Schur property under ℓ_1-sums (exactly as we did in the proof of Theorem 27), we will be done. □

Problem 32. Is DP coanalytic? If yes, our previous proof would show that DP is complete coanalytic.

An operator $T\colon X \to Y$ is said to be *completely continuous* if T maps weakly compact sets into norm-compact sets. For a given $X \in \mathrm{SB}$, let $\mathscr{C}\mathscr{C}(X)$ be the set of completely continuous operators from X to itself.

Problem 33. Let $\mathscr{C}\mathscr{C} = \{X \in \mathrm{SB};\ \mathscr{C}\mathscr{C}(X) \overset{\perp}{\hookrightarrow} \mathscr{L}(X)\}$. Is $\mathscr{C}\mathscr{C}$ non Borel? If yes, is it coanalytic?

7.3. Complete continuous property. A Banach space X is said to have the *complete continuous property* (or just to have the CCP) if every operator from $L_1[0, 1]$ to X is completely continuous (i.e., if it carries weakly compact sets into norm-compact sets). It is well known that $L_1[0, 1]$ does not have this property.

Theorem 34. *Let* CCP $= \{X \in \mathrm{SB};\ X$ *has the* CCP$\}$. CCP *is* Π^1_1*-hard. In particular,* CCP *is non Borel.*

Proof. Let $\mathscr{E}$ be a basis of $L_1[0,1]$, and $p = 2$.

By Lemma 10, if $\theta \in \mathrm{WF}$, then $\varphi(\theta)$ is reflexive, which implies $\varphi(\theta) = \varphi(\theta)^{**}$ is a separable dual. As separable duals have the Radon-Nikodym property (Dunford-Pettis theorem, see [10]) and RNP implies CCP (see [15], page 61), we conclude that $\varphi(\theta) \in \mathrm{CCP}$, for all $\theta \in \mathrm{WF}$.

On the other hand, if $\theta \in \mathrm{IF}$ we have that $L_1[0,1] \hookrightarrow \varphi_{\mathscr{E},p}(\theta)$. As $L_1[0,1]$ does not have CCP, this clearly implies $\varphi_{\mathscr{E},p}(\theta) \notin \mathrm{CCP}$ for all $\theta \in \mathrm{IF}$. $\square$

M. Girardi had shown (see [15], page 70) that a Banach space X has the CCP if and only if X has no bounded δ-Rademacher bush on it (the original terminology used by M. Girardi was δ-Rademacher *tree*, but in order to be coherent with our terminology we chose to call it a bush). A *δ-Rademacher bush* on X is a set of the form $\{x^l_k \in X;\ k \in \mathbb{N},\ l \in \{1,\dots,2^k\}\}$ satisfying

(i) $x^l_{k-1} = \frac{1}{2}(x^{2l-1}_k + x^{2l}_k)$ for all $k \in \mathbb{N}$, and $l \in \{1,\dots,2^{k-1}\}$;

(ii) $\Big\|\sum\limits_{l=1}^{2^{k-1}} (x^{2l-1}_k - x^{2l}_k)\Big\| > 2^k\delta$ for all $k \in \mathbb{N}$.

Theorem 35. *A Banach space X has the* CCP *if and only if there exists no bounded δ-Rademacher bush on X.*

Theorem 36. CCP *is coanalytic. Moreover,* CCP *is complete coanalytic.*

Proof. We use M. Girardi's characterization of the complete continuous property to show that CCP is coanalytic. To simplify the notation below we denote by $(n^l_k)_{k\in\mathbb{N},l\in\{1,\dots,2^k\}} \in \mathbb{N}^{\mathbb{N}}$ the sequence $n^1_1, n^2_1, n^1_2, \dots, n^4_2, n^1_3, \dots$, etc.

$$X \in \mathrm{CCP} \Leftrightarrow \forall (n^l_k) \in \mathbb{N}^{\mathbb{N}} (\exists M \in \mathbb{N},\ \forall k \in \mathbb{N},\ \forall l \in \{1,\dots,2^k\},\ \|S_{n^l_k}(X)\| < M)$$
$$\wedge \Big(S_{n^l_{k-1}}(X) = \frac{S_{n^{2l-1}_k}(X) + S_{n^{2l}_k}(X)}{2},\ \forall k \in \mathbb{N},\ \forall l \in \{1,\dots,2^{k-1}\}\Big)$$
$$\Rightarrow \Big(\forall \delta \in \mathbb{Q}_+,\ \exists k \in \mathbb{N},\ \Big\|\sum_{l=1}^{2^{k-1}} (S_{n^{2l-1}_k}(X) - S_{n^{2l}_k}(X))\Big\| \leqslant 2^k\delta\Big).$$

The statement above holds because we assume $\{S_n\}_{n\in\mathbb{N}}$ to be closed under rational linear combinations. $\square$

7.4. Analytic Radon-Nikodym property. It was shown in [6] that RNP $= \{X \in \mathrm{SB};\ X$ has the Radon Nikodym property$\}$ is complete coanalytic. Here we deal with the analytic Radon Nikodym property and find a lower bound for its complexity.

A complex Banach space X has the *analytic Radon-Nikodym property* if every X-valued measure of bounded variation, defined on the Borel subsets of $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$ whose negative Fourier coefficients vanish, has a Radon-Nikodym derivative with respect to the Lebesgue measure on $\mathbb{T}$.

So far, we have only been working with real Banach spaces. But, as $C_{\mathbb{C}}(\Delta)$ (the space of the complexed valued continuous functions endowed with the supremum norm) is universal for the class of serapable complex Banach spaces, we can code the class of separable complex Banach spaces in an analogous way. Precisely, we let $\mathrm{SB}_{\mathbb{C}} = \{X \subset C_{\mathbb{C}}(\Delta);\ X \text{ is a closed linear subspace}\}$. Analogously as before, $\mathrm{SB}_{\mathbb{C}}$ endowed with the Effros-Borel structure is a Polish space and it makes sense to wonder whether classes of separable complex Banach spaces with specific properties are Borel or not in this coding. With this in mind we, have:

Theorem 37. *Let* a-RNP $= \{X \in \mathrm{SB}_{\mathbb{C}};\ X$ *has the analytic Radon-Nikodym property*$\}$. *Then* a-RNP *is* Π^1_1*-hard. In particular,* a-RNP *is non Borel.*

For the proof of this result two well known theorems will do the work (see [16]).

Theorem 38. *If X has the Radon-Nikodym property, then X has the analytic Radon-Nikodym property.*

Theorem 39. *If X has the analytic Radon-Nikodym property, then X does not contain c_0.*

Proof of Theorem 37. Let $\varphi\colon \mathrm{Tr} \to \mathrm{SB}_{\mathbb{C}}$ be defined as in the proof of Theorem 8. Say $\theta \in \mathrm{WF}$. Then $\varphi(\theta)$ is reflexive, hence $\varphi(\theta) = \varphi(\theta)^{**}$ is a separable dual, therefore it has the RNP. By Theorem 38, $\varphi(\theta) \in$ a-RNP for all $\theta \in \mathrm{WF}$.

On the other hand, if $\theta \in \mathrm{IF}$, then $c_0 \hookrightarrow \varphi(\theta)$, hence, by Theorem 39, $\varphi(\theta) \notin$ a-RNP. $\square$

8. Local structure of Banach spaces

8.1. Local unconditional structure. A Banach space X is said to have *local unconditional structure* (or *l.u.st.*) if there exists $\lambda > 0$ such that for each finite dimensional Banach space $E \subset X$ there exists a finite dimensional space F with an unconditional basis and operators $u\colon E \to F$ and $w\colon F \to X$ such that $w \circ u = \mathrm{Id}_{|E}$, and $ub(F)\|u\|\|w\| \leqslant \lambda$, where $ub(F)$ is an unconditional constant for F.

Theorem 40. *Let* $\mathrm{LUST} = \{X \in \mathrm{SB}\,;\ X$ *has l.u.st.*$\}$. LUST *is Borel.*

P r o o f. In order to make the idea behind the notation below clear, let us remember some simple facts about linear algebra. Let X be a Banach space and $x_1, \ldots, x_l \in X\backslash\{0\}$. Then $\mathrm{span}\{x_1, \ldots, x_l\}$ has dimension l if and only if there exists $K \in \mathbb{Q}_+$ such that $\left\|\sum_{i=1}^{k} a_i x_i\right\| \leqslant K\left\|\sum_{i=1}^{l} a_i x_l\right\|$ for all $k \leqslant l$, and all $a_1, \ldots, a_l \in \mathbb{Q}$. Also, if $x_1, \ldots, x_l \in X$ are linear independent, then $x_1, \ldots, x_l$ are M-unconditional if and only if $\left\|\sum_{i=1}^{l} a_i x_i\right\| \leqslant M\left\|\sum_{i=1}^{l} b_i x_i\right\|$ for all $a_1, \ldots, a_l, b_1, \ldots, b_l \in \mathbb{Q}$ such that $|a_i| \leqslant |b_i|$ for all $i \in \{1, \ldots, l\}$.

Remember the functions $\{S_n\}_{n\in\mathbb{N}}$ were chosen to be linearly closed under rational linear combinations. Say $X, Y \in \mathrm{SB}$, $n_1, \ldots, n_k \in \mathbb{N}$, and $n'_1, \ldots, n'_k \in \mathbb{N}$. If $(S_{n_i}(X))_{i=1}^{k}$ is linearly independent, we denote by $P(X, Y, (n_i), (n'_i))$ the linear function from $\mathrm{span}\{S_{n_1}(X), \ldots, S_{n_k}(X)\}$ to $\mathrm{span}\{S_{n'_1}(Y), \ldots, S_{n'_k}(Y)\}$ such that $S_{n_i}(X) \mapsto S_{n'_i}(Y)$ for all $i \in \{1, \ldots, k\}$. Now notice that

$$\begin{aligned}
\mathrm{LUST} = \bigcup_{\lambda\in\mathbb{Q}_+} \ \bigcap_{\substack{k\in\mathbb{N}\\ n_1,\ldots,n_k\in\mathbb{N}}} \ \bigcup_{\substack{n'_1,\ldots,n'_k\in\mathbb{N}\\ l\geqslant k,\ M\in\mathbb{Q}_+\\ n''_1,\ldots,n''_l\in\mathbb{N}\\ n'''_1,\ldots,n'''_l\in\mathbb{N}}} \ \bigcap_{\substack{a_1,\ldots,a_l\in\mathbb{Q}_+\\ b_1,\ldots,b_l\in\mathbb{Q}_+\\ (|a_i|\leqslant|b_i|,\ \forall i)\\ d_1,\ldots,d_k\in\mathbb{Q}}} \ \bigcup_{\substack{e_1,\ldots,e_l\in\mathbb{Q}\\ A,B\in\mathbb{Q}_+\\ MAB<\lambda}} \ \bigcap_{w_1,\ldots,w_l\in\mathbb{Q}_+} \\
\Bigg\{X \in \mathrm{SB}\,;\ \Bigg(\exists K \in \mathbb{N} \text{ such that } \ \forall m \leqslant k,\ \forall c_1, \ldots, c_k \in \mathbb{Q}, \\
\left\|\sum_{i=1}^{m} c_i S_{n_i}(X)\right\| \leqslant K\left\|\sum_{i=1}^{k} c_i S_{n_i}(X)\right\|\Bigg) \\
\Rightarrow \Bigg(\sum_{i=1}^{k} d_i S_{n'_i}(C(\Delta)) = \sum_{i=1}^{l} e_i S_{n''_i}(C(\Delta)) \\
\ \&\ \left\|\sum_{i=1}^{l} a_i S_{n''_i}(C(\Delta))\right\| \leqslant M\left\|\sum_{i=1}^{l} b_i S_{n''_i}(C(\Delta))\right\| \\
\ \&\ \left\|\sum_{i=1}^{k} w_i S_{n'_i}(C(\Delta))\right\| \leqslant A\left\|\sum_{i=1}^{k} w_i S_{n_i}(X)\right\| \\
\ \&\ \left\|\sum_{i=1}^{l} w_i S_{n'''_i}(X)\right\| \leqslant B\left\|\sum_{i=1}^{l} w_i S_{n''_i}(C(\Delta))\right\| \\
\ \&\ P(C(\Delta), X, (n''_i), (n'''_i))(S_{n'_i}(C(\Delta))) = S_{n_i}(X)\Bigg)\Bigg\}.
\end{aligned}$$

There are a couple of comments about the equality above that should be made. First, notice that the restrictions $\sum_{i=1}^{k} d_i S_{n'_i}(C(\Delta)) = \sum_{i=1}^{l} e_i S_{n''_i}(C(\Delta))$ and

$\left\|\sum_{i=1}^{l} a_i S_{n_i''}(C(\Delta))\right\| \leqslant M\left\|\sum_{i=1}^{l} b_i S_{n_i''}(C(\Delta))\right\|$ do not depend on X, i.e., these restrictions should actually be incorporated in the unions and intersections preceding the set. We believe this would only make the notation harder, so we take the liberty of writing it as above. Also, the only thing in the equality above that is not clearly Borel is $X \mapsto P(C(\Delta), X, (n_i''), (n_i'''))(S_{n_i'}(C(\Delta)))$. But $P(C(\Delta), X, (n_i''), (n_i'''))$ is nothing more than a matrix with coordinates depending on the Borel functions $X \mapsto S_{n_i'''}(X)$. So we are done. □

9. Non-universality results

In this section we use ideas that can be found in [29] (Chapter 6) to show the non existence of universal spaces for some specific classes of Banach spaces. Precisely, say $\mathscr{P}$ is a property of separable Banach spaces, i.e., $\mathscr{P} \subset \mathrm{SB}$ and $Y \cong X \in \mathscr{P}$ implies $Y \in \mathscr{P}$, can we find a Banach space X with property $\mathscr{P}$ such that all Banach spaces with property $\mathscr{P}$ can be isomorphically embedded in X? If yes, we say X is a $\mathscr{P}$-universal element of $\mathscr{P}$. Analogously, we say that $X \in \mathscr{P}$ is a complementedly $\mathscr{P}$-universal element of $\mathscr{P} \subset \mathrm{SB}$ if every element of $\mathscr{P}$ can be complementedly isomorphically embedded in X. We say a property $\mathscr{P}$ is *pure* if $Y \hookrightarrow X \in \mathscr{P}$ implies $Y \in \mathscr{P}$ and complementedly pure if $Y \overset{\perp}{\hookrightarrow} X \in \mathscr{P}$ implies $Y \in \mathscr{P}$. We have the following easy lemma.

Lemma 41. *Let $\mathscr{P} \subset \mathrm{SB}$ be a pure property and assume $\mathscr{P}$ is non analytic. Then $\mathscr{P}$ has no $\mathscr{P}$-universal element. If $\mathscr{P}$ is assumed to be complementedly pure then we have that $\mathscr{P}$ has no complementedly $\mathscr{P}$-universal element.*

P r o o f. Say $X \in \mathscr{P}$ is $\mathscr{P}$-universal. Let $A = \{Y \in \mathrm{SB};\ Y \hookrightarrow X\}$. It is well known that A is analytic, for all $X \in \mathrm{SB}$ (see [29], Theorem 3.5, page 80). Clearly $\mathscr{P} = A$, contradicting our hypothesis that $\mathscr{P}$ is not analytic. For the complementedly universal case we let $A = \{Y \in \mathrm{SB};\ Y \overset{\perp}{\hookrightarrow} X\}$ and, as A is also well known to be analytic, we are done. □

This lemma together with our previous results easily give us some interesting corollaries.

Corollary 42. *Let $\mathscr{U}$ and $\mathscr{W}$ be as in the previous sections. There is no complementedly universal space $X \in \mathscr{U}$ for the class $\mathscr{U}$. The same is true for $\mathscr{W}$.*

P r o o f. First notice that we have actually shown that both these classes are not only non Borel but non analytic. Now, we only need to notice that if $X \cong$

$X_1 \oplus X_2$ and $P\colon \mathscr{L}(X) \to \mathscr{U}(X)$ is a projection then $\widetilde{P}(T) = P_1 \circ P(T)_{|X_1}$, where $P_1\colon X_1 \oplus X_2 \to X_1$ is the standard projection, is a projection from $\mathscr{L}(X_1)$ to $\mathscr{U}(X_1)$ (the same works for the class $\mathscr{W}$). □

Corollary 43. *There is no* $X \in \mathrm{BS}$ *universal for the class* BS. *The same holds for* ABS *and* WBS.

Proof. One way of noticing WBS is pure is Lemma 19. □

Corollary 44. *There is no* $X \in \mathscr{BS}$ *complementedly universal for the class* $\mathscr{BS}$.

Corollary 45. *There is no* $X \in \mathrm{S}$ *universal for the class* S.

Corollary 46. *There is no* $X \in \mathrm{DP}$ *complementedly universal for the class* DP.

Corollary 47. *There is no* $X \in \mathrm{RNP}$ *universal for the class* RNP. *The same holds for* CCP *and* a-RNP.

The first claim of the corollary above can be obtained by results in [6] or by letting $\varphi_{\mathscr{E},p}$ be as in the proof of Theorem 37. After getting this corollary, we discovered that its first claim had already been discovered by M. Talagrand by completely different methods. Talagrand's proof remains unpublished though.

Let us take a look at other easy (but profitable) lemma.

Lemma 48. *Say* $\mathscr{P}_1, \mathscr{P}_2 \subset \mathrm{SB}$. *Assume there exists a Borel* $\varphi\colon \mathrm{Tr} \to \mathrm{SB}$ *such that* $\varphi(\mathrm{WF}) \subset \mathscr{P}_1$ *and* $\varphi(\mathrm{IF}) \subset \mathscr{P}_2$. *Let* $A \subset \mathrm{SB}$ *be an analytic subset containing* $\mathscr{P}_1$. *Then* $A \cap \mathscr{P}_2 \neq \emptyset$. *In particular, if* $\mathscr{P}_2 \subset \{X \in \mathrm{SB};\ X \text{ is universal for } \mathrm{SB}\}$, *we have that if* X *is universal for* $\mathscr{P}_1$, *then* X *is universal for* SB.

Proof. As $\mathrm{WF} \subset \varphi^{-1}(A)$ and WF is non analytic we cannot have equality. Hence, there exists $\theta \in \mathrm{IF}$ such that $\varphi(\theta) \in A$. As $\varphi(\theta) \in \mathscr{P}_2$ we are done. For the second claim, let X be universal for $\mathscr{P}_1$, define $A = \{Y \in \mathrm{SB};\ Y \hookrightarrow X\}$, and apply the first claim. □

The proofs of the following corollaries are either contained in the previous sections or are just slight modifications of them.

Corollary 49. *If* $X \in \mathrm{SB}$ *is universal for either* $\mathscr{U}$ *or* $\mathscr{W}$, *then* X *is universal for* SB. *In particular, these classes admit no element universal for themselves.*

Corollary 50. *If* $X \in \mathrm{SB}$ *is universal for the class BS, then* X *is universal for* SB. *The same holds for* ABS *and* WBS.

Corollary 51. *If* $X \in \mathrm{SB}$ *is universal for the class* S, *then* X *is universal for* SB.

Corollary 52. *If* $X \in \mathrm{SB}$ *is universal for the class* RNP, *then* X *is universal for* SB. *The same holds for* CCP *and* a-RNP.

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