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METHODS OF ANALYSIS OF THE CONDITION FOR CORRECT SOLVABILITY IN $L_p(\mathbb{R})$ OF GENERAL STURM-LIOUVILLE EQUATIONS

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Abstract. We consider the equation

$$(*)\qquad -(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R}$$

where $f \in L_p(\mathbb{R})$, $p \in (1,\infty)$ and

$$r > 0, \quad q \geqslant 0, \quad \frac{1}{r} \in L_1^{\rm loc}(\mathbb{R}), \quad q \in L_1^{\rm loc}(\mathbb{R}),$$
$$\lim_{|d|\to\infty} \int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-d}^{x} q(t)\,\mathrm{d}t = \infty.$$

In an earlier paper, we obtained a criterion for correct solvability of $(*)$ in $L_p(\mathbb{R})$, $p \in (1,\infty)$. In this criterion, we use values of some auxiliary implicit functions in the coefficients r and q of equation $(*)$. Unfortunately, it is usually impossible to compute values of these functions. In the present paper we obtain sharp by order, two-sided estimates (an estimate of a function $f(x)$ for $x \in (a,b)$ through a function $g(x)$ is sharp by order if $c^{-1}|g(x)| \leqslant |f(x)| \leqslant c|g(x)|$, $x \in (a,b)$, $c = \mathrm{const}$) of auxiliary functions, which guarantee efficient study of the problem of correct solvability of $(*)$ in $L_p(\mathbb{R})$, $p \in (1,\infty)$.

Keywords: correct solvability; Sturm-Liouville equation

MSC 2010: 34B24

1. Introduction

In the present paper, we consider the equation

$$(1.1)\qquad -(r(x)y'(x))' + q(x)y(x) = f(x), \quad x \in \mathbb{R}$$

where $f \in L_p(\mathbb{R})$ $(L_p(\mathbb{R}) := L_p)$, $p \in (1, \infty)$ and

$$r > 0, \quad q \geqslant 0, \quad \frac{1}{r} \in L_1^{\mathrm{loc}}(\mathbb{R}), \quad q \in L_1^{\mathrm{loc}}(\mathbb{R}), \tag{1.2}$$

$$\lim_{|d| \to \infty} \int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-d}^{x} q(t)\,\mathrm{d}t = \infty. \tag{1.3}$$

In the sequel, by a solution of (1.1) we mean any function y, absolutely continuous together with ry' and satisfying (1.1) almost everywhere on $\mathbb{R}$. In addition, we say that for a given $p \in (1, \infty)$, equation (1.1) is correctly solvable in L_p if the following hold:

I) for every function $f \in L_p$, there exists a unique solution $y \in L_p$ of (1.1);

II) there is an absolute (i.e., not dependent on $f \in L_p(\mathbb{R})$) constant $c(p) \in (0, \infty)$ such that the solution $y \in L_p$ of (1.1) satisfies the inequality

$$\|y\|_p \leqslant c(p)\|f\|_p, \quad \forall f \in L_p.$$

From now on, for brevity we say "problem I)–II)," "question on I)–II)".

The conditions for validity of I)–II) are given in [2]. To state them, we need the following lemma.

Lemma 1.1 ([5]). *Under conditions* (1.2)–(1.3), *for a given $x \in \mathbb{R}$ consider the equations in $d \geqslant 0$:*

$$\int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-d}^{x} q(t)\,\mathrm{d}t = 1, \qquad \int_{x}^{x+d} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x}^{x+d} q(t)\,\mathrm{d}t = 1.$$

Each equation has a unique finite positive solution. Denote these solutions by $d_1(x)$ and $d_2(x)$, respectively, and set

$$\varphi(x) = \int_{x-d_1(x)}^{x} \frac{\mathrm{d}t}{r(t)}, \quad \psi(x) = \int_{x}^{x+d_2(x)} \frac{\mathrm{d}t}{r(t)}, \quad x \in \mathbb{R}, \tag{1.4}$$

$$h(x) = \frac{\varphi(x)\psi(x)}{\varphi(x) + \psi(x)}, \quad x \in \mathbb{R}. \tag{1.5}$$

Further, consider the equation in $d \geqslant 0$

$$\int_{x-d}^{x+d} \frac{\mathrm{d}t}{r(t)h(t)} = 1. \tag{1.6}$$

Equation (1.6) *has a unique finite positive solution. Denote it by $d(x)$. The function $d(x)$ is continuous for $x \in \mathbb{R}$ and, in addition,*

$$\lim_{x \to -\infty} (x + d(x)) = -\infty, \quad \lim_{x \to \infty} (x - d(x)) = \infty.$$

We now present the main result of [2] (see [3] for the case $r \equiv 1$, $0 \leqslant q \in L^{\mathrm{loc}}(\mathbb{R})$).

Theorem 1.2 ([2]). *Under conditions* (1.2)–(1.3), *equation* (1.1) *is correctly solvable in* L_p, $p \in (1, \infty)$ *if and only if* $B < \infty$. *Here*

$$B = \sup_{x \in \mathbb{R}} (h(x) d(x)). \tag{1.7}$$

The following assertions allow us to answer the question on I)–II) without using the function d (see (1.7)).

Corollary 1.3 ([2]). *Under conditions* (1.2)–(1.3), *equation* (1.1) *is correctly solvable in* L_p, $p \in (1, \infty)$ *if any of the following conditions holds:*

$$1)\quad B_1 < \infty, \quad B_1 = \sup_{x \in \mathbb{R}} (r(x) h^2(x)); \tag{1.8}$$

$$2)\quad B_2 < \infty, \quad B_2 = \sup_{x \in \mathbb{R}} (|x| \cdot h(x));$$

$$3)\quad B_3 < \infty, \quad B_3 = \sup_{x \in \mathbb{R}} \left(|x| \cdot \int_{-\infty}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x}^{\infty} \frac{\mathrm{d}t}{r(t)} \right). \tag{1.9}$$

Note that the solution of (1.1), as well as the functions h and d, can be found only in special cases. Therefore, after obtaining the results in [2], the main problem in the investigation of I)–II) is checking the inequality $B < \infty$ for particular equations. In other words, we have to find technical tools and standard procedures that allow us to apply Theorem 1.2 to concrete equations (1.1) in an efficient way. Our general approach to this problem consists in finding, under additional requirements on the coefficients of (1.1), two-sided, sharp by order estimates for the functions h and d (see (1.7)), which, by Theorem 1.2, provide a complete answer to the question on I)–II). (We say an estimate of a function $f(x)$ for $x \in (a, b)$ through a function $g(x)$ is sharp by order if $c^{-1}|g(x)| \leqslant |f(x)| \leqslant c|g(x)|$, $x \in (a, b)$, $c = \mathrm{const}$. Note that in this paper some results from [4], [5] are strengthened. Moreover, in the special case $r \equiv 1$, $0 \leqslant q \in L^{\mathrm{loc}}(\mathbb{R})$, another more effective form for the solution of the considered problem is obtained (for details see [1], [3])).

The structure of the paper is as follows. Preliminaries are given in §2, results are presented in §3, proofs are collected in §4, and examples are given in §5.

2. Preliminaries

Lemma 2.1 ([2]). *Under conditions* (1.2)–(1.3), *for a given* $x \in \mathbb{R}$, *let us introduce functions* $F_i(\eta)$, $\overline{1,3}$ *for* $\eta \geqslant 0$:

$$(2.1) \qquad F_1(\eta) = \int_{x-\eta}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-\eta}^{x} q(t)\,\mathrm{d}t, \quad F_2(\eta) = \int_{x}^{x+\eta} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x}^{x+\eta} q(t)\,\mathrm{d}t,$$

$$F_3(\eta) = \int_{x-\eta}^{x+\eta} \frac{\mathrm{d}t}{r(t)h(t)}.$$

Then the following assertions hold:

1) *the inequality* $\eta \geqslant d_i(x)$ $(0 \leqslant \eta \leqslant d_i(x))$ *holds if and only if* $F_i(\eta) \geqslant 1$ $(F_i(\eta) \leqslant 1)$, $i = 1, 2$;
2) *the inequality* $\eta \geqslant d(x)$ $(0 \leqslant \eta \leqslant d(x))$ *holds if and only if* $F_3(\eta) \geqslant 1$ $(F_3(\eta) \leqslant 1)$.

Remark 2.2. Auxiliary functions similar to the functions from Lemma 1.1 were first introduced by M. Otelbaev (see [8]). Lemma 2.1 is nothing else than a formalization of a trick used by Otelbaev (see [8]).

3. Results

Below we give estimates for the functions h and d (see (1.5) and (1.6)) and some consequences.

Towards this end, we provide two definitions.

Definition 3.1. We say that a pair $\{r, q\}$ of functions, defined on $\mathbb{R}$, everywhere positive and absolutely continuous for all $|x| \gg 1$ (in the sequel "a pair $\{r, q\}$") belongs to the class $K(\mu)$ (denoted as $\{r, q\} \in K(\mu)$), $\mu \geqslant 0$ if

$$(3.1) \qquad \lim_{|x|\to\infty} \varkappa_i(x, \mu) = 0, \quad i = 1, 2,$$

$$(3.2) \qquad \lim_{|x|\to-\infty} (x + \mu\hat{d}(x)) = -\infty, \quad \lim_{x\to\infty} (x - \mu\hat{d}(x)) = \infty.$$

Here

$$(3.3) \qquad \varkappa_1(x, \mu) = r(x) \sup_{|t| \leqslant \mu\hat{d}(x)} \left| \int_{x}^{x+t} \frac{r'(\xi)}{r^2(\xi)}\,\mathrm{d}\xi \right|, \quad |x| \gg 1,$$

$$(3.4) \qquad \varkappa_2(x, \mu) = \frac{1}{q(x)} \sup_{|t| \leqslant \mu\hat{d}(x)} \left| \int_{x}^{x+t} q'(\xi)\,\mathrm{d}\xi \right|, \quad |x| \gg 1,$$

$$\hat{d}(x) = \sqrt{\frac{r(x)}{q(x)}}, \quad x \in \mathbb{R}.$$

Note that the following inclusion obviously holds:

$$K(\mu_2) \subseteq K(\mu_1), \quad 0 \leqslant \mu_1 \leqslant \mu_2.$$

Below we will see that for a sufficiently large μ the class $K(\mu)$ contains pairs for which the question on I)–II) for the corresponding equations (1.1) admits a particular simple answer (see Corollary 3.6 below).

Definition 3.2. Suppose that the functions r and q satisfy (1.2). We say that the pair of functions $\{r, q\}$ (in the sequel "pair") belongs to the class $S(\mu)$, $\mu \geqslant 0$ (and denote that as $\{r, q\} \in S(\mu)$) if there exists a pair $\{r_1, q_1\} \in K(\mu)$ such that the following relations hold:

$$\inf_{x\in\mathbb{R}} \frac{r(x)}{r_1(x)} \geqslant \delta, \quad \delta \in (0, 1], \tag{3.5}$$

$$\lim_{|x|\to\infty} \varkappa_i(x, \mu) = 0, \quad i = 3, 4. \tag{3.6}$$

Here

$$\varkappa_3(x, \mu) = \sqrt{r_1(x)q_1(x)} \sup_{|t|\leqslant\mu\hat{d}_1(x)} \left| \int_x^{x+t} \frac{r(\xi) - r_1(\xi)}{r_1^2(\xi)} \, \mathrm{d}\xi \right|, \quad |x| \gg 1, \tag{3.7}$$

$$\varkappa_4(x, \mu) = \frac{1}{\sqrt{r_1(x)q_1(x)}} \sup_{|t|\leqslant\mu\hat{d}_1(x)} \left| \int_x^{x+t} (q(\xi) - q_1(\xi)) \, \mathrm{d}\xi \right|, \quad |x| \gg 1, \tag{3.8}$$

$$\hat{d}_1(x) = \sqrt{\frac{r_1(x)}{q_1(x)}}, \quad x \in \mathbb{R}. \tag{3.9}$$

Note the obvious relations

$$K(\mu) \subseteq S(\mu), \quad S(\mu) \setminus K(\mu) \neq \emptyset, \quad \mu > 0,$$
$$S(\mu_2) \subseteq S(\mu_1), \quad 0 \leqslant \mu_1 \leqslant \mu_2.$$

Below we will see that if for a sufficiently large μ the pair $\{r, q\} \in S(\mu)$ can be "approximated" (in the sense of (3.6), (3.7), (3.8)) by a pair $\{r_1, q_1\} \in K(\mu)$, then the problems I)–II) for the equations (1.1) corresponding to these pairs are equivalent, i.e., these equations either are or are not correctly solvable together in L_p, $p \in (1, \infty)$. In other words, equations (1.1) with coefficients $\{r, q\} \in S(\mu)$ are perturbations of the corresponding (in the sense of Definition 3.2) equations (1.1) with coefficients $\{r_1, q_1\} \in K(\mu)$.

We need some more notation. Let $\mu > 0$, and let $\{r, q\} \in S(\mu)$. Then we write

$$\{r, q\} \sim \{r_1, q_1\} \in K(\mu)$$

if the pair $\{r_1, q_1\}$ satisfies Definition 3.2. By c, $c(\cdot)$ we denote absolute (not dependent on the input data of the problem under consideration) positive constants which are not essential for exposition and may differ even within a single chain of computations.

Finally, let $f(x)$ and $g(x)$ be continuous, positive functions for $x \in (a, b)$ $(-\infty \leqslant a < b \leqslant \infty)$. Then we write $f(x) \asymp g(x)$, $x \in (a, b)$ if the following inequalities hold:

$$c^{-1} f(x) \leqslant g(x) \leqslant c f(x), \quad x \in (a, b).$$

Let us now formulate our main statements.

Theorem 3.3. *Let $\{r, q\} \in S(\mu)$, $\mu \geqslant 2$ and let $\{r, q\} \sim \{r_1, q_1\} \in K(\mu)$. Then* (1.3) *holds, and*

$$h(x)\sqrt{r_1(x) q_1(x)} \asymp 1, \quad x \in \mathbb{R}. \tag{3.10}$$

Corollary 3.4. *Suppose that under the hypotheses of Theorem* 3.3 *any of the following inequalities holds:*

$$B_4 < \infty, \quad B_4 = \sup_{x \in \mathbb{R}} \frac{|x|}{\sqrt{r_1(x) q_1(x)}}, \tag{3.11}$$

$$B_5 < \infty, \quad B_5 = \sup_{x \in \mathbb{R}} \frac{r(x)}{r_1(x) q_1(x)}. \tag{3.12}$$

Then equation (1.1) *is correctly solvable in L_p, $p \in (1, \infty)$.*

Theorem 3.5. *Let $\{r, q\} \in S(\mu)$ and let $\{r, q\} \sim \{r_1, q_1\} \in K(\mu)$. If $\mu \geqslant 100\delta^{-2}$* (*see* (3.5)), *then*

$$d(x) \asymp \hat{d}_1(x), \quad x \in \mathbb{R} \tag{3.13}$$

(*see* (1.6) *and* (3.9)).

Corollary 3.6. *Under the hypotheses of Theorem* 3.5, *equation* (1.1) *is correctly solvable in L_p, $p \in (1, \infty)$ if and only if $m > 0$. Here*

$$m = \inf_{x \in \mathbb{R}} q_1(x).$$

Remark 3.7. In the condition $\mu \geqslant 100\delta^{-2}$, the main obstacle for reducing μ (in our main Theorem 3.3 we have $\mu \geqslant 2$ and so it does not depend on δ) is the factor δ^{-2}.

Let us emphasize that in the framework of the method presented here it is impossible to make a choice of μ independent of δ. On the other hand, the factor 100 in the same condition is chosen for convenience in calculations. In the proof presented here, one can, of course, reduce it but such a reduction is not essential because it is majorated by reducing δ. Note that the constant 30 in Theorem 3.8 is the integer closest to the number $4\mathrm{e}^2$ from a priori inequalities (4.17) and (4.18). The choice of 30 instead of $4\mathrm{e}^2$ is necessary, as above, for the proof as well as for the convenience in applications of Theorem 3.8.

The next assertion is often more convenient than Theorem 3.5.

Theorem 3.8. *Suppose that under conditions* (1.2)–(1.3), *the function r is absolutely continuous together with r' for $|x| \gg 1$, and*

$$\lim_{|x|\to\infty} l(x) = 0, \quad l(x) = \sup_{|t|\leqslant 30 r(x)h(x)} r^2(x)h(x)\left|\int_{x-t}^{x+t}\left(\frac{1}{r(\xi)}\right)''\,\mathrm{d}\xi\right|.$$

Then

$$d(x) \asymp r(x)h(x), \quad x \in \mathbb{R}. \tag{3.14}$$

Corollary 3.9. *Under the hypotheses of Theorem* 3.8, *the inequality $B < \infty$ holds if and only if $B_1 < \infty$ (see* (1.8)*). In addition, equation* (1.1) *is correctly solvable in L_p, $p \in (1,\infty)$ if $B_6 < \infty$, where*

$$B_6 = \sup_{x\in\mathbb{R}} r(x)\left(\int_{-\infty}^{x}\frac{\mathrm{d}t}{r(t)}\cdot\int_{x}^{\infty}\frac{\mathrm{d}t}{r(t)}\right)^2.$$

Remark 3.10. The proofs of Theorems 3.5 and 3.8 rely on a priori inequalities (4.20) below. Here different goals naturally lead to different results. More precisely, in Theorem 3.5, by strengthening one of the requirements of Theorem 3.3 ($\mu \geqslant 100\delta^{-2}$ instead of $\mu \geqslant 2$), we get estimates for the function d (see (3.17)). In Theorem 3.8, under conditions (1.2) and (1.3), we only strengthen requirements to the function r and, therefore we have new possibilities and get an explicit relationship between the functions h and d (compare (4.20) and (3.17)). We want to emphasize that in this case the requirements of Theorem 3.3 may not hold. Note that from the point of view of Theorem 3.8, we need Theorem 3.3 only in the case where one has to obtain estimates for the function d relying only on (3.17). In this special case, we use (3.11). Thus, by obtaining new possibilities for estimating the function d under different assumptions of Theorems 3.5 and 3.8, we increase our chances for success in our investigation of (1.1).

The following assertion is applicable to equations (1.1) with an oscillating coefficient q.

Theorem 3.11. *Suppose that together with* (1.2)–(1.3) *the functions r and q satisfy the following additional conditions:*

1) *the function q is three times continuously differentiable, vanishes at the points $\{x_k\}_{k=1}^{\infty}$, and*

$$q(x_k) = q'(x_k) = 0, \quad q''(x_k) > 0, \quad k \gg 1; \tag{3.15}$$

2) *there exists an everywhere positive, absolutely continuous function $r_1(x)$ for $|x| \gg 1$ such that*

$$r(x) \geqslant \delta r_1(x), \quad |x| \gg 1, \quad \delta \in (0,1];$$

3) *the equalities*

$$\lim_{k\to\infty} \tau_i(x_k) = 0, \quad i = 1,2,3 \tag{3.16}$$

hold where for $k \geqslant 1$:

$$\tau_1(x_k) = r_1(x_k) \sup_{|s|\leqslant\eta_k} \left| \int_{x_k}^{x_k+s} \frac{r_1'(\xi)}{r_1^2(\xi)} \,\mathrm{d}\xi \right|, \tag{3.17}$$

$$\tau_2(x_k) = \sqrt[4]{r_1^3(x_k)q''(x_k)} \sup_{|s|\leqslant\eta_k} \left| \int_{x_k}^{x_k+s} \frac{\Delta r(\xi)}{r_1^2(\xi)} \,\mathrm{d}\xi \right|, \tag{3.18}$$

$$\tau_3(x_k) = \frac{1}{q''(x_k)} \sup_{|s|\leqslant\eta_k} \left| \int_{x_k}^{x_k+s} q'''(t) \,\mathrm{d}t \right|. \tag{3.19}$$

Here

$$\Delta r(t) = r(t) - r_1(t), \quad t \in \mathbb{R}; \quad \eta_k = \frac{100\hat{d}_k}{\delta^2}, \quad \hat{d}_k = \sqrt[4]{\frac{r_1(x_k)}{q''(x_k)}}.$$

Then the following relations hold:

$$h(x_k)d(x_k) \asymp (r(x_k)q''(x_k))^{-1/2}, \quad k \gg 1, \tag{3.20}$$

$$B^2 \geqslant c^{-1} \sup_{k\geqslant 1} (r(x_k)q''(x_k))^{-1}. \tag{3.21}$$

Remark 3.12. Note that we prefer not to integrate in (3.19) because it is often easier to estimate the value of the integral in (3.19) rather than the value of the difference $(q''(x_k+s) - q''(x_k))$. Our way keeps both the possibilities for estimating $\tau_3(x_k)$.

Let us return to Theorems 3.3 and 3.5. These statements may not be applicable, say, to pairs of functions $\{r, q\}$ such that for one of them the integral in the product (1.3) converges, whereas the integral of the other one diverges but grows sufficiently slowly. For example, if

$$r(x) = \begin{cases} 1, & |x| \leqslant 1, \\ x^2, & |x| \geqslant 1, \end{cases} \qquad q(x) = \exp(|x|), \quad x \in \mathbb{R}$$

then by Theorems 3.3 and 3.5 we have

$$|x| \exp\Big(\frac{1}{2}|x|\Big) h(x) \asymp 1, \quad \exp\Big(\frac{1}{2}|x|\Big) d(x) \asymp |x|, \quad x \gg 1.$$

However, the same theorems are not applicable to the pair

$$r(x) = \begin{cases} 1, & |x| \leqslant 1, \\ x^2, & |x| \geqslant 1, \end{cases} \qquad q(x) = \begin{cases} 1, & |x| \leqslant 1, \\ |x|^{-1/2}, & |x| \geqslant 1. \end{cases}$$

Let us list all such situations. We introduce the following notation:

$$J^{(-)} = \int_{-\infty}^{0} \frac{\mathrm{d}t}{r(t)}, \quad J^{(+)} = \int_{0}^{\infty} \frac{\mathrm{d}t}{r(t)}, \tag{3.22}$$

$$I^{(-)} = \int_{-\infty}^{0} q(t)\,\mathrm{d}t, \quad I^{(+)} = \int_{0}^{\infty} q(t)\,\mathrm{d}t. \tag{3.23}$$

Clearly, for the pairs $\alpha = \{J^{(-)}, I^{(-)}\}$ and $\beta = \{J^{(+)}, I^{(+)}\}$, only the following situations can arise:

α_1:	$J^{(-)} = \infty$,	$I^{(-)} = \infty$	β_1:	$J^{(+)} = \infty$,	$I^{(+)} = \infty$
α_2:	$J^{(-)} < \infty$,	$I^{(-)} = \infty$	β_2:	$J^{(+)} < \infty$,	$I^{(+)} = \infty$
α_3:	$J^{(-)} = \infty$,	$I^{(-)} < \infty$	β_3:	$J^{(+)} = \infty$,	$I^{(+)} < \infty$

Thus, there are altogether eight interesting combinations (α_i, β_i), $i, j = \overline{1,3}$ (the case (α_1, β_1) is naturally excluded). Below we only consider the case (α_2, β_2) because all the other ones can be treated similarly, using the theorems stated above. We also note that below we present only the statements related to estimates for the functions d_1 and d_2 (the proofs for d_1 and d_2 are similar, and therefore in Section 4 we only consider the function d_1). We believe that a deeper discussion of this topic would be superfluous here, because usually once one has estimates for d_1 and d_2 all the remaining parts of the investigation of problem I)–II) can be concluded with help of the results presented above, or by the methods of proofs (see §5).

Thus, until the end of this section, we combine (1.2) and (1.3) with the condition $1/r \in L_1$ and do not include these requirements in the statement, using them as standing assumptions. Note that the estimates for $d_1(x)$ and $d_2(x)$ as $x \to -\infty$ and $x \to \infty$ are given in a different way, and we have to study them separately.

Let us start with the first case. For $x \ll -1$, $a \leqslant 0$ and $\nu \geqslant 1$, consider the equations in $d \geqslant 0$

$$\int_{-\infty}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-d}^{x} q(t)\,\mathrm{d}t = \nu, \tag{3.24}$$

$$\int_{x}^{x+d} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x}^{a} q(t)\,\mathrm{d}t = \nu. \tag{3.25}$$

Theorem 3.13. *For given $\nu \geqslant 1$ and $x \ll -1$, equation (3.24) has at least one finite positive solution. Set*

$$\begin{aligned} \alpha_1^{(-)}(x) &= \sup_{d>0}\left\{d\colon \int_{-\infty}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-d}^{x} q(t)\,\mathrm{d}t = 1\right\}, \\ \beta_1^{(-)}(x,\nu) &= \inf_{d>0}\left\{d\colon \int_{-\infty}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-d}^{x} q(t)\,\mathrm{d}t = \nu\right\}. \end{aligned} \tag{3.26}$$

We have the inequality

$$d_1(x) \geqslant \alpha_1^{(-)}(x), \quad x \ll -1. \tag{3.27}$$

In addition, if for some $\nu > 1$ there is $x_0 \ll -1$ such that

$$\int_{-\infty}^{x} \frac{\mathrm{d}t}{r(t)} \leqslant \nu \int_{x-\beta_1^{(-)}(x,\nu)}^{x} \frac{\mathrm{d}t}{r(t)} \quad \text{for} \quad x \leqslant x_0, \tag{3.28}$$

then we have the estimate

$$d_1(x) \leqslant \beta_1^{(-)}(x,\nu), \quad x \leqslant x_0. \tag{3.29}$$

Theorem 3.14. *For given $a \leqslant 0$, $\nu \geqslant 1$ and $x \ll -1$, equation (3.25) has a unique solution. Let $\alpha_2^{(-)}(x)$ and $\beta_2^{(-)}(x,\nu)$ be the solutions of (3.25) for $\nu = 1$ and $\nu > 1$, respectively. We have the relations*

$$\lim_{x\to-\infty}(x + \alpha_2^{(-)}(x)) = -\infty, \quad \lim_{x\to-\infty}(x + \beta_2^{(-)}(x,\nu)) = -\infty,$$
$$d_2(x) \geqslant \alpha_2^{(-)}(x), \quad \text{for } x + \alpha_2^{(-)}(x) \leqslant a.$$

In addition, if for some $\nu > 1$ *there is* $x_0 \ll -1$ *such that*

$$\int_x^a q(t)\,\mathrm{d}t \leqslant \nu \int_x^{x+\beta_2^{(-)}(x,\nu)} q(t)\,\mathrm{d}t \quad \text{for } x + \beta_2^{(-)}(x,\nu) \leqslant a,\ x \leqslant x_0,$$

then we have the estimate

$$d_2(x) \leqslant \beta_2^{(-)}(x,\nu) \quad \text{for } x + \beta_2^{(-)}(x,\nu) \leqslant a,\ x \leqslant x_0.$$

Let us now study $d_1(x)$ and $d_2(x)$ as $x \to \infty$. Towards this end, for given $a \geqslant 0$, $\nu \geqslant 1$ and $x \gg 1$, consider the equations in $d \geqslant 0$

$$\int_{x-d}^x \frac{\mathrm{d}t}{r(t)} \cdot \int_a^x q(t)\,\mathrm{d}t = \nu, \quad x \gg 1, \tag{3.30}$$

$$\int_x^\infty \frac{\mathrm{d}t}{r(t)} \cdot \int_x^{x+d} q(t)\,\mathrm{d}t = \nu, \quad x \gg 1. \tag{3.31}$$

Theorem 3.15. *For given* $a \geqslant 0$, $\nu \geqslant 1$ *and* $x \gg 1$, *equation* (3.30) *has a unique solution. Let* $\alpha_1^{(+)}(x)$ *and* $\beta_1^{(+)}(x,\nu)$ *be the solutions of* (3.30) *for* $\nu = 1$ *and* $\nu > 1$, *respectively. We have the following relations:*

$$\lim_{x\to\infty} (x - \alpha_1^{(+)}(x)) = \infty, \quad \lim_{x\to\infty} (x - \beta_1^{(+)}(x,\nu)) = \infty, \tag{3.32}$$

$$d_1(x) \geqslant \alpha_1^{(+)}(x), \quad x - \alpha_1^{(+)}(x) \geqslant a. \tag{3.33}$$

In addition, if for some $\nu > 1$ *there is* $x_0 \gg 1$ *such that*

$$\int_a^x q(t)\,\mathrm{d}t \leqslant \nu \int_{x-\beta_1^{(+)}(x,\nu)}^x q(t)\,\mathrm{d}t \quad \text{for } x - \beta_1^{(+)}(x,\nu) \geqslant a,\ x \geqslant x_0, \tag{3.34}$$

then we have the estimate

$$d_1(x) \leqslant \beta_1^{(+)}(x,\nu) \quad \text{for } x - \beta_1^{(+)}(x,\nu) \geqslant a,\ x \geqslant x_0. \tag{3.35}$$

Theorem 3.16. *For given $\nu \geqslant 1$ and $x \gg 1$, equation* (3.31) *has at least one finite positive solution. Set*

$$\alpha_2^{(+)}(x) = \sup_{d>0}\left\{d\colon \int_x^\infty \frac{\mathrm{d}t}{r(t)} \cdot \int_x^{x+d} q(t)\,\mathrm{d}t = 1\right\},$$
$$\beta_2^{(+)}(x,\nu) = \inf_{d>0}\left\{d\colon \int_x^\infty \frac{\mathrm{d}t}{r(t)} \cdot \int_x^{x+d} q(t)\,\mathrm{d}t = \nu\right\}.$$

Then we have the inequality

$$d_2(x) \geqslant \alpha_2^{(+)}(x), \quad x \gg 1.$$

In addition, if for some $\nu > 1$ there is $x_0 \gg 1$ such that

$$\int_x^\infty \frac{\mathrm{d}t}{r(t)} \leqslant \nu \int_x^{x+\beta_2^{(+)}(x,\nu)} \frac{\mathrm{d}t}{r(t)} \quad \text{for } x \geqslant x_0,$$

then we have the estimate

$$d_2(x) \leqslant \beta_2^{(+)}(x,\nu) \quad \text{for } x \geqslant x_0.$$

4. Proofs

P r o o f of Theorem 3.3. Below we assume that the hypotheses of the theorem are satisfied and do not include them in the statements.

Lemma 4.1. *Let $\delta \in (0,1]$ and $\theta + 1 \geqslant \delta$. Then*

$$1 - \frac{1+\theta}{\delta^2} \leqslant \frac{\theta}{\theta+1} \leqslant \theta. \tag{4.1}$$

P r o o f. The following relations imply (4.1):

$$\theta - \frac{\theta}{\theta+1} = \frac{\theta^2}{\theta+1} > 0, \quad \frac{\theta}{\theta+1} + \frac{\theta+1}{\delta^2} = \frac{\theta\delta^2 + (\theta+1)^2}{(\theta+1)\delta^2} \geqslant \frac{\theta\delta^2+\delta^2}{(\theta+1)\delta^2} = 1.$$

□

Denote for $t \in \mathbb{R}$

$$(\Delta r)(t) = r(t) - r_1(t), \quad \theta(t) = \frac{(\Delta r)(t)}{r_1(t)} \quad (\Delta q)(t) = q(t) - q_1(t).$$

Lemma 4.2. *Let $x \in \mathbb{R}$, $\eta \geqslant 0$. Then*

$$\int_x^{x+\eta} \frac{\mathrm{d}t}{r(t)} = \int_x^{x+\eta} \frac{\mathrm{d}t}{r_1(t)} - \int_x^{x+\eta} \frac{\theta(t)}{\theta(t)+1} \frac{\mathrm{d}t}{r_1(t)}. \tag{4.2}$$

P r o o f. The following relations are obvious:

$$\begin{aligned} \int_x^{x+\eta} \frac{\mathrm{d}t}{r(t)} &= \int_x^{x+\eta} \frac{\mathrm{d}t}{r_1(t)} + \int_x^{x+\eta} \Big[\frac{1}{r(t)} - \frac{1}{r_1(t)}\Big]\,\mathrm{d}t \\ &= \int_x^{x+\eta} \frac{\mathrm{d}t}{r_1(t)} - \int_x^{x+\eta} \frac{\theta(t)}{\theta(t)+1} \frac{\mathrm{d}t}{r_1(t)}. \end{aligned}$$

□

Lemma 4.3. *Let $x \in \mathbb{R}$, $\eta \geqslant 0$. Then*

$$\int_x^{x+\eta} \frac{\mathrm{d}t}{r(t)} \leqslant \frac{1}{\delta^2} \int_x^{x+\eta} \frac{\mathrm{d}t}{r_1(t)} + \frac{1}{\delta^2}\bigg|\int_x^{x+\eta} \frac{\theta(t)}{r_1(t)}\,\mathrm{d}t\bigg|, \tag{4.3}$$

$$\int_x^{x+\eta} \frac{\mathrm{d}t}{r(t)} \geqslant \int_x^{x+\eta} \frac{\mathrm{d}t}{r_1(t)} - \bigg|\int_x^{x+\eta} \frac{\theta(t)}{r_1(t)}\,\mathrm{d}t\bigg|. \tag{4.4}$$

P r o o f. Since $1+\theta(t) \geqslant \delta$, $t \in \mathbb{R}$ (see (3.5)), from (4.1) and (4.2) it follows that

$$\begin{aligned} \int_x^{x+\eta} \frac{\mathrm{d}t}{r(t)} &\leqslant \int_x^{x+\eta} \frac{\mathrm{d}t}{r_1(t)} + \int_x^{x+\eta} \Big[\frac{1+\theta(t)}{\delta^2} - 1\Big] \frac{\mathrm{d}t}{r_1(t)} \\ &\leqslant \frac{1}{\delta^2} \int_x^{x+\eta} \frac{\mathrm{d}t}{r_1(t)} + \frac{1}{\delta^2}\bigg|\int_x^{x+\eta} \frac{\theta(t)}{r_1(t)}\,\mathrm{d}t\bigg|. \end{aligned}$$

Inequality (4.4) is checked in a similar way. □

Lemma 4.4. *Let $x \in \mathbb{R}$, $\eta = \mu\hat{d}_1(x)$, $\mu > 0$. Then (see* (3.3), (3.7)*)*

$$\int_x^{x+\eta} \frac{\mathrm{d}t}{r(t)} \leqslant \frac{1}{\delta^2} \frac{\mu}{\sqrt{r_1(x)q_1(x)}} \Big(1 + \varkappa_1(x,\mu) + \frac{\varkappa_3(x,\mu)}{\mu}\Big), \tag{4.5}$$

$$\int_x^{x+\eta} \frac{\mathrm{d}t}{r(t)} \geqslant \frac{\mu}{\sqrt{r_1(x)q_1(x)}} \Big(1 - \varkappa_1(x,\mu) - \frac{\varkappa_3(x,\mu)}{\mu}\Big). \tag{4.6}$$

P r o o f. Below we use (4.3).

$$\begin{aligned}\int_x^{x+\eta}\frac{\mathrm{d}t}{r(t)} &\leqslant \frac{1}{\delta^2}\int_0^\eta\frac{\mathrm{d}\xi}{r_1(x+\xi)}+\frac{1}{\delta^2}\sup_{|t|\leqslant\mu\hat{d}_1(x)}\biggl|\int_x^{x+t}\frac{\theta(\xi)}{r_1(\xi)}\,\mathrm{d}\xi\biggr|\\ &=\frac{1}{\delta^2}\frac{\eta}{r_1(x)}+\frac{1}{\delta^2}\int_0^\eta\int_x^{x+\xi}\Bigl(\frac{1}{r_1(s)}\Bigr)'\,\mathrm{d}s\,\mathrm{d}\xi+\frac{1}{\delta^2}\frac{\varkappa_3(x,\mu)}{\sqrt{r_1(x)q_1(x)}}\\ &\leqslant\frac{1}{\delta^2}\frac{\mu}{\sqrt{r_1(x)q_1(x)}}+\frac{1}{\delta^2}\mu\hat{d}_1(x)\sup_{|t|\leqslant\mu\hat{d}_1(x)}\biggl|\int_x^{x+t}\frac{r_1'(\xi)}{r_1^2(\xi)}\,\mathrm{d}\xi\biggr|+\frac{1}{\delta^2}\frac{\varkappa_3(x,\mu)}{\sqrt{r_1(x)q_1(x)}}\\ &\leqslant\frac{\mu}{\delta^2}\frac{1}{\sqrt{r_1(x)q_1(x)}}\Bigl[1+\varkappa_1(x,\mu)+\frac{\varkappa_3(x,\mu)}{\mu}\Bigr].\end{aligned}$$

In a similar way (using (4.4)), we obtain (4.6). $\square$

Lemma 4.5. *Let $x\in\mathbb{R}$, $\eta=\mu\hat{d}_1(x)$, $\mu>0$. Then*

$$\int_x^{x+\eta}q(t)\,\mathrm{d}t\leqslant\mu\sqrt{r_1(x)q_1(x)}\Bigl[1+\varkappa_2(x,\mu)+\frac{\varkappa_4(x,\mu)}{\mu}\Bigr],$$

$$\int_x^{x+\eta}q(t)\,\mathrm{d}t\geqslant\mu\sqrt{r_1(x)q_1(x)}\Bigl[1-\varkappa_2(x,\mu)-\frac{\varkappa_4(x,\mu)}{\mu}\Bigr]. \tag{4.7}$$

P r o o f. Below we use (3.4), (3.8) and (3.9):

$$\begin{aligned}\int_x^{x+\eta}q(t)\,\mathrm{d}t&=\int_x^{x+\eta}q_1(t)\,\mathrm{d}t+\int_x^{x+\eta}(\Delta q)(t)\,\mathrm{d}t\\ &\leqslant\int_0^\eta q_1(x+\xi)\,\mathrm{d}\xi+\sup_{|t|\leqslant\mu\hat{d}_1(x)}\biggl|\int_x^{x+\eta}(\Delta q)(\xi)\,\mathrm{d}\xi\biggr|\\ &=\eta q_1(x)+\int_0^\eta\biggl(\int_x^{x+t}q_1'(\xi)\,\mathrm{d}\xi\biggr)\,\mathrm{d}t+\sqrt{r_1(x)q_1(x)}\varkappa_4(x,\mu)\\ &\leqslant q_1(x)\mu\hat{d}_1(x)+\mu\hat{d}_1(x)\sup_{|t|\leqslant\mu\hat{d}_1(x)}\biggl|\int_x^{x+t}q_1'(\xi)\,\mathrm{d}\xi\biggr|+\sqrt{r_1(x)q(x)}\varkappa_4(x,\mu)\\ &=\mu\sqrt{r_1(x)q_1(x)}\Bigl[1+\varkappa_2(x,\mu)+\frac{\varkappa_4(x,\mu)}{\mu}\Bigr].\end{aligned}$$

Inequality (4.7) can be checked in a similar way. $\square$

Corollary 4.6. *Let* $x \in \mathbb{R}$, $\eta = \mu\hat{d}_1(x)$, $\mu > 0$. *Then* (*see* (2.1))

$$F_2(\eta) \geqslant \mu^2\Big(1 - \varkappa_1(x,\mu) - \frac{\varkappa_3(x,\mu)}{\mu}\Big)\Big(1 - \varkappa_2(x,\mu) - \frac{\varkappa_4(x,\mu)}{\mu}\Big), \tag{4.8}$$

$$F_2(\eta) \leqslant \Big(\frac{\mu}{\delta}\Big)^2\Big(1 + \varkappa_1(x,\mu) + \frac{\varkappa_3(x,\mu)}{\mu}\Big)\Big(1 + \varkappa_2(x,\mu) + \frac{\varkappa_4(x,\mu)}{\mu}\Big). \tag{4.9}$$

P r o o f. Both estimates follow from Lemmas 4.4 and 4.5. □

Lemma 4.7. *Equality* (1.3) *holds.*

P r o o f. Let us prove (see (3.22), (3.23)) that the relations

$$J^{(-)} + I^{(-)} = J^{(+)} + I^{(+)} = \infty, \tag{4.10}$$

$$\int_{-\infty}^{x} q(t)\,\mathrm{d}t > 0, \quad \int_{x}^{\infty} q(t)\,\mathrm{d}t > 0, \quad \forall x \in \mathbb{R} \tag{4.11}$$

are valid. Assume that $J^{(+)} + I^{(+)} < \infty$. Then for any $\varepsilon > 0$ there is $x_0 = x_0(\varepsilon) \gg 1$ such that

$$\int_x^\infty \frac{\mathrm{d}t}{r(t)} \leqslant \varepsilon, \quad \int_x^\infty q(t)\,\mathrm{d}t \leqslant \varepsilon \quad \text{for } x \geqslant x_0(\varepsilon).$$

On the other hand, if $\eta = \mu\hat{d}_1(x)$, $\mu > 0$, then for $x \gg x_0$ we obtain (see (3.1), (3.6), (4.8))

$$\begin{aligned}\varepsilon^2 &\geqslant \int_x^\infty \frac{\mathrm{d}t}{r(t)} \cdot \int_x^\infty q(t)\,\mathrm{d}t \geqslant \int_x^{x+\eta} \frac{\mathrm{d}t}{r(t)} \cdot \int_x^{x+\eta} q(t)\,\mathrm{d}t = F_2(\eta) \\ &\geqslant \mu^2\Big(1 - \varkappa_1(x,\mu) - \frac{\varkappa_3(x,\mu)}{\mu}\Big)\Big(1 - \varkappa_4(x,\mu) - \frac{\varkappa_4(x,\mu)}{\mu}\Big) \geqslant \Big(\frac{\mu}{4}\Big)^2.\end{aligned} \tag{4.12}$$

From (4.12), for $\varepsilon < \mu/4$, we get a contradiction. In addition, since $F_2(\eta) > 0$ for $x \gg 1$, the second inequality in (4.11) also holds. The remaining relations in (4.10) and (4.11) can be checked in a similar way. From (4.10) and (4.11) we obtain (1.3). □

Lemma 4.8. *There exists* $x_0 \gg 1$ *such that*

$$2^{-1}\delta\hat{d}_1(x) \leqslant d_2(x) \leqslant 2\hat{d}_1(x) \quad \text{for } |x| \geqslant x_0. \tag{4.13}$$

P r o o f. There is $x_1 \gg 1$ such that

$$\begin{aligned}\varkappa_1(x,2) + 2^{-1}\varkappa_3(x,2) &\leqslant 2^{-1} \quad \text{for } |x| \geqslant x_1, \\ \varkappa_2(x,2) + 2^{-1}\varkappa_4(x,2) &\leqslant 2^{-1} \quad \text{for } |x| \geqslant x_1.\end{aligned}$$

Then from (4.8), for $\eta = 2\hat{d}_1(x)$ and $|x| \geqslant x_1$, it follows that

$$F_2(\eta) \geqslant 4\Big(1 - \frac{1}{2}\Big)\Big(1 - \frac{1}{2}\Big) = 1.$$

Hence, for $|x| \geqslant x_1$, the upper estimate in (4.13) holds (see Lemma 2.1). In a similar way, using (4.9) one can show that there is $x_2 \gg 1$ such that for $|x| \geqslant x_2$ the lower estimate in (4.13) holds. Then for $|x| \geqslant x_0 = \max\{x_1, x_2\}$, (4.13) holds. $\square$

Let us turn to (3.10). Below, to estimate ψ (see (1.4)) for $|x| \gg 1$, we use (4.13), (4.5), (4.6), (3.1) and (3.6):

$$\psi(x) = \int_x^{x+d_2(x)} \frac{\mathrm{d}t}{r(t)} \leqslant \int_x^{x+2\hat{d}_1(x)} \frac{\mathrm{d}t}{r(t)} \leqslant \frac{2}{\delta^2} \frac{1 + \varkappa_1(x,2) + \varkappa_3(x,2)}{\sqrt{r_1(x)q_1(x)}} \leqslant \frac{c}{\sqrt{r_1(x)q_1(x)}},$$

$$\psi(x) = \int_x^{x+d_2(x)} \frac{\mathrm{d}t}{r(t)} \geqslant \int_x^{x+\frac{\delta}{2}\hat{d}_1(x)} \frac{\mathrm{d}t}{r(t)} \geqslant \frac{\delta}{2} \frac{1 - \varkappa_1(x,2) - \frac{2}{\delta}\varkappa_3(x,2)}{\sqrt{r_1(x)q_1(x)}} \geqslant \frac{c^{-1}}{\sqrt{r_1(x)q_1(x)}}.$$

Thus, $\psi(x)\sqrt{r_1(x)q_1(x)} \asymp 1$ for $|x| \gg 1$. In a similar way, we prove the relations

$$\frac{\delta}{2}\hat{d}_1(x) \leqslant d_1(x) \leqslant 2\hat{d}_1(x), \quad |x| \geqslant x_0; \quad \varphi(x)\sqrt{r_1(x)q_1(x)} \asymp 1, \quad |x| \gg 1.$$

Together with (1.5), this implies (3.10) for $|x| \gg 1$, and it remains to get (3.10) for any finite segment.

We need Otelbaev's inequalities (see [5])

$$2^{-1}h(x) \leqslant \varrho(x) \leqslant 2h(x), \quad x \in \mathbb{R}. \tag{4.14}$$

Here $\varrho(x)$, $x \in \mathbb{R}$ is a continuous positive function constructed from a special fundamental system of solutions of the equation

$$(r(x)z'(x))' = q(x)z(x), \quad x \in \mathbb{R}.$$

Since the functions $r_1(x)$ and $q_1(x)$ are also continuous for $x \in \mathbb{R}$, for any $a \in (0, \infty)$ it follows from (4.14) that

$$c^{-1}(a) \leqslant \varrho(x)\sqrt{r_1(x)q_1(x)} \leqslant c(a), \quad x \in [-a, a]. \tag{4.15}$$

Here $c(a)$ is a constant depending only on a.

Our assertion now follows from (4.14) and (4.15). $\square$

P r o o f of Corollary 3.4. This is an immediate consequence of Corollary 1.3 and Theorem 3.3. $\square$

P r o o f of Theorem 3.5. The estimates of the functions φ, ψ and h for $|x|\geqslant x_0$, which were obtained above, can be written in a different way as follows:

$$\begin{gathered}\frac{\delta}{4}\frac{1}{\sqrt{r_1(x)q_1(x)}}\leqslant\varphi(x),\quad \psi(x)\leqslant\frac{4}{\delta^2}\frac{1}{\sqrt{r_1(x)q_1(x)}},\quad |x|\geqslant x_0,\\ \frac{\delta}{8}\frac{1}{\sqrt{r_1(x)q_1(x)}}\leqslant h(x)\leqslant\frac{2}{\delta^2}\frac{1}{\sqrt{r_1(x)q_1(x)}},\quad |x|\geqslant x_0.\end{gathered}\tag{4.16}$$

Below we use the a priori inequalities from [5], which hold under conditions (1.2)–(1.3):

$$\frac{1}{4\mathrm{e}^2}\leqslant\frac{h(t)}{h(x)}\leqslant 4\mathrm{e}^2\quad\text{for } |t-x|\leqslant d(x),\ x\in\mathbb{R}.\tag{4.17}$$

From (1.6) and (4.17) we obtain the estimates

$$\frac{1}{4\mathrm{e}^2}\int_{x-d(x)}^{x+d(x)}\frac{\mathrm{d}t}{r(t)}\leqslant h(x)\leqslant 4\mathrm{e}^2\int_{x-d(x)}^{x+d(x)}\frac{\mathrm{d}t}{r(t)},\qquad x\in\mathbb{R}.\tag{4.18}$$

Assume that there exists x such that

$$d(x)\geqslant\mu\hat{d}_1(x),\quad |x|\geqslant x_0,\quad \mu=\Big(\frac{10}{\delta}\Big)^2.\tag{4.19}$$

Below, for $|x|\geqslant x_0$, we use (4.19), (4.18), (4.17) and (4.16):

$$\int_{x-\mu\hat{d}(x)}^{x+\mu\hat{d}(x)}\frac{\mathrm{d}t}{r(t)}\leqslant\int_{x-d(x)}^{x+d(x)}\frac{\mathrm{d}t}{r(t)}\leqslant 4\mathrm{e}^2h(x)\leqslant\frac{8\mathrm{e}^2}{\delta^2}\frac{1}{\sqrt{r_1(x)q_1(x)}}.\tag{4.20}$$

On the other hand, from (4.6), (3.1) and (3.6), we obtain

$$\begin{aligned}\int_{x-\mu\hat{d}_1(x)}^{x+\mu\hat{d}_1(x)}\frac{\mathrm{d}t}{r(t)}&\geqslant\frac{200}{\delta^2}\Big(1-\varkappa_1(x,\mu)-\frac{\varkappa_3(x,\mu)}{\mu}\Big)\frac{1}{\sqrt{r_1(x)q_1(x)}}\\&\geqslant\frac{100}{\delta^2}\frac{1}{\sqrt{r_1(x)q_1(x)}},\quad |x|\geqslant x_0.\end{aligned}\tag{4.21}$$

Now from (4.20) and (4.21), it follows that

$$\frac{100}{\delta^2}\leqslant\sqrt{r_1(x)q_1(x)}\int_{x-\mu\hat{d}_1(x)}^{x+\mu\hat{d}_1(x)}\frac{\mathrm{d}t}{r(t)}\leqslant\frac{8\mathrm{e}^2}{\delta^2},\quad |x|\geqslant x_0,$$

which is a contradiction. Hence we have the inequality

$$d(x)\leqslant 100\delta^{-2}\hat{d}_1(x),\quad |x|\geqslant x_0.\tag{4.22}$$

Let us now assume that there exists x such that

$$d(x) \leqslant \mu \hat{d}_1(x), \quad |x| \geqslant x_0, \quad \mu = 16^{-2}\mathrm{e}^{-2}\delta^3. \tag{4.23}$$

Then, enlarging x_0 if needed and using (4.16), (4.18), (4.23), (4.5), (3.6) and (3.1), we obtain successively

$$\begin{aligned}\frac{\delta}{8}\frac{1}{\sqrt{r_1(x)q_1(x)}} &\leqslant h(x) \leqslant 4\mathrm{e}^2 \int_{x-d(x)}^{x+d(x)} \frac{\mathrm{d}(t)}{r(t)} \leqslant 4\mathrm{e}^2 \int_{x-\mu\hat{d}_1(x)}^{x+\mu\hat{d}_1(x)} \frac{\mathrm{d}t}{r(t)} \\ &\leqslant \frac{8\mu}{\delta^2}\frac{1+\varkappa_1(x,\mu)+\mu^{-1}\varkappa_3(x,\mu)}{\sqrt{r_1(x)q_1(x)}} \leqslant \frac{16}{\delta^2}\frac{\mu}{\sqrt{r_1(x)q_1(x)}} = \frac{16^{-1}\delta}{\sqrt{r_1(x)q_1(x)}},\end{aligned}$$

which is a contradiction. Hence we have the inequality

$$d(x) \geqslant (16\mathrm{e})^{-2}\delta^3, \quad |x| \geqslant x_0. \tag{4.24}$$

Further, by Lemma 1.1, the function

$$f(x) = d(x)(d_1(x))^{-1}, \quad |x| \leqslant x_0$$

is continuous and positive for $|x| \leqslant x_0$. Hence

$$c^{-1} \leqslant f(x) \leqslant c, \quad |x| \geqslant x_0. \tag{4.25}$$

The theorem follows from (4.22), (4.24) and (4.25). □

P r o o f of Corollary 3.6. This is a consequence of Theorems 1.2, 3.3 and 3.5. □

P r o o f of Theorem 3.8. We need the following assertion.

Lemma 4.9. *If a function f is defined on $\mathbb{R}$ and is absolutely continuous together with f', then for $x \in \mathbb{R}$ and $d \geqslant 0$, we have*

$$\int_{x-d}^{x+d} f(t)\,\mathrm{d}t = 2f(x)d + \int_0^d \int_0^t \int_{x-\xi}^{x+\xi} f''(s)\,\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}t. \tag{4.26}$$

P r o o f. The following transformations lead to (4.26):

$$\begin{aligned}\int_{x-d}^{x+d} f(t)\,\mathrm{d}t &= \int_0^d [f(x+t)+f(x-t)]\,\mathrm{d}t \\ &= 2f(x) + \int_0^d [f(x+t)-f(x)]\,\mathrm{d}t - \int_0^d [f(x)-f(x-t)]\,\mathrm{d}t \\ &= 2f(x) + \int_0^d\int_0^t [f(x+\xi)]'\,\mathrm{d}\xi\,\mathrm{d}t - \int_0^d\int_0^t [f(x-\xi)]'\,\mathrm{d}\xi\,\mathrm{d}t \\ &= 2f(x) + \int_0^d\int_0^t\int_{x-\xi}^{x+\xi} f''(s)\,\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}t.\end{aligned}$$

□

Let

$$\delta = \frac{119}{16}\frac{1}{\mathrm{e}^2} - 1 > 0, \quad \mu = \frac{1}{60},$$

and let $x_0 \gg 1$ be such that $l(x) \leqslant \delta$ for $|x| \geqslant x_0$. Suppose that there exists $x \in \mathbb{R}$ such that

$$d(x) \leqslant \mu r(x)h(x), \quad |x| \geqslant x_0.$$

Below we successively apply (4.16) and (4.26):

$$\begin{aligned} h(x) &\leqslant 4\mathrm{e}^2 \int_{x-d(x)}^{x+d(x)} \frac{\mathrm{d}t}{r(t)} \leqslant 4\mathrm{e}^2 \int_{x-\mu r(x)h(x)}^{x+\mu r(x)h(x)} \frac{\mathrm{d}t}{r(t)} \\ &= 8\mathrm{e}^2\mu h(x) + 4\mathrm{e}^2 \int_0^{\mu r(x)h(x)} \int_0^t \int_{x-\xi}^{x+\xi} \Big(\frac{1}{r(s)}\Big)'' \,\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}t \\ &\leqslant 8\mathrm{e}^2\mu h(x) + \frac{1}{2}\mu^2 h(x) l(x) = 8\mathrm{e}^2\Big(1 + \frac{\mu}{16\mathrm{e}^2} l(x)\Big) \\ &\leqslant 8\mathrm{e}^2(1+\delta)\frac{h(x)}{60} < h(x), \end{aligned}$$

and we get a contradiction. Hence our assumption is wrong, and therefore

$$d(x) \geqslant 60^{-1} r(x)h(x), \quad |x| \geqslant x_0.$$

We keep the values of δ and x_0 but set $\mu = 30$ and assume that there exists $x \in \mathbb{R}$ such that

$$d(x) \geqslant \mu r(x)h(x), \quad |x| \geqslant x_0.$$

Below we use (4.16) and (4.26):

$$\begin{aligned} h(x) &\geqslant \frac{1}{4\mathrm{e}^2} \int_{x-d(x)}^{x+d(x)} \frac{\mathrm{d}t}{r(t)} \geqslant \frac{1}{4\mathrm{e}^2} \int_{x-\mu r(x)h(x)}^{x+\mu r(x)h(x)} \frac{\mathrm{d}t}{r(t)} \\ &= \frac{1}{4\mathrm{e}^2}\Big[2\mu h(x) + \int_0^{\mu r(x)h(x)} \int_0^t \int_{x-\xi}^{x+\xi} \Big(\frac{1}{r(s)}\Big)'' \,\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}t\Big] \\ &\geqslant \frac{1}{4\mathrm{e}^2}\Big[2\mu h(x) - \Big(\int_0^{\mu r(x)h(x)} \int_0^t \mathrm{d}s\,\mathrm{d}t\Big) \cdot \sup_{|\xi| \leqslant \mu r(x)h(x)} \Big| \int_{x-\xi}^{x+\xi} \Big(\frac{1}{r(s)}\Big)'' \,\mathrm{d}s \Big|\Big] \\ &= \frac{1}{4\mathrm{e}^2}\Big[2\mu h(x) - \frac{\mu^2}{2} h(x) l(x)\Big] \geqslant \frac{\mu h(x)}{4\mathrm{e}^2} = \frac{30}{4\mathrm{e}^2} h(x) > h(x), \end{aligned}$$

which is a contradiction. Hence

$$d(x) \leqslant 30 r(x)h(x) \quad \text{for } |x| \geqslant x_0.$$

Further, since the functions d and r are continuous and positive (see Lemma 1.1), using (4.14), one can easily show (see, e.g., the proof of (4.15)) that

$$c^{-1} r(x)h(x) \leqslant d(x) \leqslant c r(x)h(x), \quad |x| \leqslant x_0, \quad c = c(x_0).$$

These estimates imply (3.14). $\square$

P r o o f of Corollary 3.9. The first statement follows from (1.8) and (3.14), and the other one is a consequence of Corollary 1.3 and the following estimate (see [2]):

$$h(x) \leqslant \tau \int_{-\infty}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x}^{\infty} \frac{\mathrm{d}t}{r(t)}, \quad x \in \mathbb{R},\ \tau^{-1} = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{r(t)}.$$

$\square$

P r o o f of Theorem 3.11. The proofs of (3.10), (3.13) and (3.20) are similar. Therefore, below we only present the details of the proof of (3.20) that are different from those that are known.

Let $\alpha, \beta \in (0, 100\delta^{-2})$. The following inequalities can be checked similarly to (4.5) and (4.6):

$$\int_{x_k}^{x_k+\alpha\hat{d}_k} \frac{\mathrm{d}t}{r(t)} \leqslant \frac{\alpha}{\delta^2}\Big(1 + \tau_1(x_k) + \frac{\tau_2(x_k)}{\alpha}\Big) \frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}}, \quad k \geqslant 1, \tag{4.27}$$

$$\int_{x_k}^{x_k+\beta\hat{d}_k} \frac{\mathrm{d}t}{r(t)} \geqslant \beta\Big(1 - \tau_1(x_k) - \frac{\tau_2(x_k)}{\beta}\Big) \frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}}, \quad k \geqslant 1. \tag{4.28}$$

$\square$

Lemma 4.10. *Let $\eta \geqslant 0$, $k \geqslant 1$. Then*

$$\int_{x_k}^{x_k+\eta} q(t)\,\mathrm{d}t = \frac{\eta^3}{6} q''(x_k)\Big[1 + \frac{1}{q''(x_k)\eta^3} \int_{x_k}^{x_k+\eta} q'''(t)(x_k+\eta-t)^3\,\mathrm{d}t\Big]. \tag{4.29}$$

P r o o f. Below we use integration by parts and (3.15):

$$\begin{aligned}
\int_{x_k}^{x_k+\eta} q(t)\,\mathrm{d}t &= q(t)(t-x_k-\eta)\Big|_{x_k}^{x_k+\eta} - \int_{x_k}^{x_k+\eta} q'(t)(t-x_k-\eta)\,\mathrm{d}t \\
&= -\frac{(t-x_k-\eta)^2}{2} q'(t)\Big|_{x_k}^{x_k+\eta} + \frac{1}{2}\int_{x_k}^{x_k+\eta} q''(t)(t-x_k-\eta)^2\,\mathrm{d}t \\
&= \frac{(t-x_k-\eta)^3}{6} q''(t)\Big|_{x_k}^{x_k+\eta} - \frac{1}{6}\int_{x_k}^{x_k+\eta} q'''(t)(t-x_k-\eta)^3\,\mathrm{d}t \\
&= \frac{\eta^3}{6} q''(x_k) + \frac{1}{6}\int_{x_k}^{x_k+\eta} q'''(t)(x_k+\eta-t)^3\,\mathrm{d}t \ \Rightarrow\ (4.29).
\end{aligned}$$

$\square$

Lemma 4.11. *Let* $\alpha, \beta \in (0, 100\delta^{-2})$, $k \geqslant 1$. *Then*

$$\int_{x_k}^{x_k+\alpha\hat{d}_k} q(t)\,\mathrm{d}t \leqslant \frac{\alpha^3}{6}\sqrt[4]{r_1^3(x_k)q''(x_k)}(1+\tau_3(x_k)), \tag{4.30}$$

$$\int_{x_k}^{x_k+\beta\hat{d}_k} q(t)\,\mathrm{d}t \geqslant \frac{\beta^3}{6}\sqrt[4]{r_1^3(x_k)q''(x_k)}(1-\tau_3(x_k)). \tag{4.31}$$

P r o o f. Both the inequalities are checked in the same way. Consider, say, (4.30). Below we use (4.29) and the second mean-value theorem ([9]):

$$\begin{aligned}\int_{x_k}^{x_k+\alpha\hat{d}_k} q'''(t)(x_k+\alpha\hat{d}_k-t)^3\,\mathrm{d}t &= \frac{(\alpha\hat{d}_k)^3}{6}\int_{x_k}^{x_k+s} q'''(t)\,\mathrm{d}t, \quad s\in(x_k, x_k+\alpha\hat{d}_k)\\ \Rightarrow \int_{x_k}^{x_k+\alpha\hat{d}_k} q(t)\,\mathrm{d}t &= \frac{(\alpha d_k)^3}{6}\,q''(x_k)\Big[1+\frac{1}{q''(x_k)}\int_{x_k}^{x_k+s} q'''(t)\,\mathrm{d}t\Big]\\ &\leqslant \frac{\alpha^3}{6}\sqrt[4]{r_1^3(x_k)q''(x_k)}(1+\tau_3(x_k)).\end{aligned}$$

□

Let us establish the inequalities

$$\frac{3}{2}\sqrt{\delta}\hat{d}_k \leqslant d_1(x_k), d_2(x_k) \leqslant 2\hat{d}_k, \quad k \gg 1. \tag{4.32}$$

Below we only consider $d_2(x_k)$ because (4.32) for $d_1(x_k)$ can be proved similarly. Let $\alpha = 3\cdot 2^{-1}\sqrt{\delta}$.

From (3.16) it follows that

$$\Big(1+\tau_1(x_k)+\frac{\tau_2(x_k)}{\alpha}\Big)(1+\tau_3(x_k)) \leqslant \frac{11}{10}, \quad k \gg 1.$$

Therefore (see (4.27), (4.30) and (2.1)),

$$F_2(\alpha\hat{d}_k) \leqslant \frac{1}{6}\Big(\frac{3}{2}\Big)^4\frac{11}{10} < 1.$$

Hence (see Lemma 2.1), the lower estimate in (4.32) holds. Similarly, if $\beta := \sqrt[4]{12}$, then (3.16) yields that

$$\Big(1-\tau_1(x_k)-\frac{\tau_2(x_k)}{\beta}\Big)(1-\tau_3(x_k)) \geqslant \frac{1}{2},$$

and then (see (4.27), (4.31) and (2.1))

$$F_2(\beta\hat{d}_k) > 1.$$

Hence the upper estimate in (4.32) holds by Lemma 2.1. Let $k \gg 1$. Below we estimate $\psi(x_k)$ using (4.32), (4.27) and (4.28), (3.16), (3.17) and (3.18):

$$\begin{aligned}
\psi(x_k) &= \int_{x_k}^{x_k+d_2(x_k)} \frac{\mathrm{d}t}{r(t)} \leqslant \int_{x_k}^{x_k+2\hat{d}_k} \frac{\mathrm{d}t}{r(t)} \\
&\leqslant \frac{2}{\delta^2}\frac{1+\tau_1(x_k)+1/2\tau_2(x_k)}{\sqrt[4]{r_1^3(x_k)q''(x_k)}} \leqslant \left(\frac{2}{\delta}\right)^2 \frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}};\\
\psi(x_k) &= \int_{x_k}^{x_k+d_2(x_k)} \frac{\mathrm{d}t}{r(t)} \geqslant \int_{x_k}^{x_k+3\delta\hat{d}_k/2} \frac{\mathrm{d}t}{r(t)} \geqslant \frac{3}{2}\sqrt{\delta}\frac{1-\tau_1(x_k)-\frac{2}{3\sqrt{\delta}}\tau_2(x_k)}{\sqrt[4]{r_1^3(x_k)q''(x_k)}}\\
&\geqslant \frac{4}{3}\sqrt{\delta}\frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}}
\end{aligned}$$

$$\Rightarrow \frac{4\sqrt{\delta}}{3}\frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}} \leqslant \varphi(x_k),\ \psi(x_k) \leqslant \left(\frac{2}{\delta}\right)^2 \frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}}, \quad k \gg 1,$$

$$\frac{2\sqrt{\delta}}{3}\frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}} \leqslant h(x_k) \leqslant \left(\frac{2}{\delta}\right)^2 \frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}}, \quad k \gg 1. \tag{4.33}$$

(Estimates (4.33) follow from (1.5) and the estimates proved earlier.) Assume now that there is $k \gg 1$ such that

$$d(x_k) \geqslant \mu\hat{d}_k, \quad \mu = 100\delta^{-2}.$$

Then (see the proof of (3.13)), using (4.33), (4.18), (4.28) and (3.16), we get

$$\begin{aligned}
\frac{2}{\delta^2}\frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}} &\geqslant h(x_k) \geqslant \frac{1}{30}\int_{x_k-d(x_k)}^{x_k+d(x_k)} \frac{\mathrm{d}t}{r(t)} \geqslant \frac{1}{30}\int_{x_k-\mu\hat{d}_k}^{x_k+\mu\hat{d}_k} \frac{\mathrm{d}t}{r(t)}\\
&\geqslant \frac{\mu}{15}\frac{1-\tau_1(x_k)-\tau_2(x_k)/\mu}{\sqrt[4]{r_1^3(x_k)q''(x_k)}} \geqslant \frac{100}{30\delta^2}\frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}},
\end{aligned}$$

which is a contradiction. Hence

$$d(x_k) \leqslant 100\delta^{-2}\hat{d}_k, \quad k \gg 1.$$

Similarly, assume that there is $k \gg 1$ such that

$$d(x_k) \leqslant \mu\hat{d}_k, \quad \mu = 100^{-1}\delta^{5/2}.$$

Then (see (4.33), (4.18), (4.27) and (3.16))

$$\begin{aligned} \frac{2}{3}\frac{\sqrt{\delta}}{\sqrt[4]{r_1^3(x_k)q''(x_k)}} &\leqslant h(x_k) \leqslant 30\int_{x_k-d(x_k)}^{x_k+d(x_k)}\frac{\mathrm{d}t}{r(t)} \leqslant 30\int_{x_k-\mu\hat{d}_k}^{x_k+\mu\hat{d}_k}\frac{\mathrm{d}t}{r(t)} \\ &\leqslant \frac{60\mu}{\delta^2}\frac{1+\tau_1(x_k)+\tau_2(x_k)/\mu}{\sqrt[4]{r_1^3(x_k)q''(x_k)}} \leqslant \frac{60\mu}{\delta^2}\frac{11}{10}\frac{1}{\sqrt[4]{r_1^3(x_k)q''(x_k)}} \\ &\Rightarrow \frac{2}{3} \leqslant \frac{33}{50}, \end{aligned}$$

which is a contradiction. Hence

$$d(x_k) \geqslant \frac{\delta^{5/2}}{100}\hat{d}_k, \quad k \gg 1.$$

The estimates of $h(x_k)$, $d(x_k)$, $k \gg 1$, imply (3.20) and (3.21). □

P r o o f of Theorem 3.13. Denote

$$\Phi(d) = \int_{-\infty}^{x}\frac{\mathrm{d}t}{r(t)}\cdot\int_{x-d}^{x}q(t)\,\mathrm{d}t, \quad d \geqslant 0.$$

Then $\Phi(0) = 0$, $\Phi(\infty) = \infty$, and

$$\Phi'(d) = q(x-d)\int_{-\infty}^{x}\frac{\mathrm{d}t}{r(t)} \geqslant 0.$$

This implies that equation (3.24) has at least one finite positive solution. Further (see (3.24), (3.26)),

$$1 = \int_{-\infty}^{x}\frac{\mathrm{d}t}{r(t)}\cdot\int_{x-\alpha_1^{(-)}(x)}^{x}q(t)\,\mathrm{d}t \geqslant \int_{x-\alpha_1^{(-)}(x)}^{x}\frac{\mathrm{d}t}{r(t)}\cdot\int_{x-\alpha_1^{(-)}(x)}^{x}q(t)\,\mathrm{d}t.$$

This, in view of Lemma 2.1, implies (3.27). Let us now consider (3.29). From (3.24) and (3.28), it follows that

$$\begin{gathered} \nu = \int_{-\infty}^{x}\frac{\mathrm{d}t}{r(t)}\cdot\int_{x-\beta_1^{(-)}(x,\nu)}^{x}q(t)\,\mathrm{d}t \leqslant \nu\int_{x-\beta_1^{(-)}(x,\nu)}^{x}\frac{\mathrm{d}t}{r(t)}\cdot\int_{x-\beta_1^{(-)}(x,\nu)}^{x}q(t)\,\mathrm{d}t \\ \Rightarrow F_1(\beta_1^{(-)}(x,\nu)) \geqslant 1. \end{gathered}$$

Hence (see Lemma 2.1) estimate (3.29) holds. □

P r o o f of Theorem 3.15. For a given $x \gg a$, introduce the function

$$\Phi(d) = \int_{x-d}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{a}^{x} q(t)\,\mathrm{d}t, \quad d \geqslant 0.$$

From the obvious relations $\Phi(0) = 0$, $\Phi(x) > \nu$, for $x \gg a$ and

$$\Phi'(d) = \frac{1}{r(x-d)} \int_{a}^{x} q(t)\,\mathrm{d}t > 0,$$

it follows that equation (3.30) has a unique finite solution. Both equalities in (3.32) are checked in the same way. Consider, say, the first. If it does not hold, then there exist $c \in (0, \infty)$ and a sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$x_n - \alpha_1^{(+)}(x_n) \leqslant c, \quad x_n \to \infty \quad \text{as } n \to \infty$$
$$\Rightarrow 1 = \int_{x_n - \alpha_1^{(+)}(x_n)}^{x_n} \frac{\mathrm{d}t}{r(t)} \cdot \int_{a}^{x_n} q(t)\,\mathrm{d}t \geqslant \int_{c}^{x_n} \frac{\mathrm{d}t}{r(t)} \cdot \int_{a}^{x_n} q(t)\,\mathrm{d}t \to \infty, \quad n \to \infty$$

(see (1.3)). This is a contradiction, i.e., (3.32) holds. Therefore, for $x - \alpha_1^{(+)}(x) \geqslant a$, it follows from (3.30) that

$$1 = \int_{x-\alpha_1^{(+)}(x)}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{a}^{x} q(t)\,\mathrm{d}t \geqslant \int_{x-\alpha_1^{(+)}(x)}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-\alpha_1^{(+)}(x)}^{x} q(t)\,\mathrm{d}t.$$

By Lemma 2.1, this implies (3.33). Further, for $x - \beta_1^{(+)}(x) \geqslant a$, from (3.30) and (3.34), we obtain that

$$\nu = \int_{x-\beta_1^{(+)}(x)}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{a}^{x} q(t)\,\mathrm{d}t \leqslant \nu \int_{x-\beta_1^{(+)}(x)}^{x} \frac{\mathrm{d}t}{r(t)} \cdot \int_{x-\beta_1^{(+)}(x)}^{x} q(t)\,\mathrm{d}t$$
$$\Rightarrow F_1(\beta_1^{(+)}(x)) \geqslant 1.$$

Hence (see Lemma 2.1), estimate (3.35) holds. □

5. Examples

Below we give three examples of applications of our statements to concrete equations of the form (1.1). Since in all the examples the coefficients r and q of these equations are even functions, in the first two examples we estimate the functions h and d only for $x \geqslant 1$. In the third example, the cases $x \leqslant -1$ and $x \geqslant 1$ will be studied separately because the corresponding auxiliary functions depend on the sign of the argument (see, e.g., Theorems 3.13 and 3.15).

Example 5.1. Consider equation (1.1) with coefficients

$$r(x) = (1 + x^2) + \frac{1 + x^2}{2}\sin(\mathrm{e}^{|x|}), \quad x \in \mathbb{R}, \tag{5.1}$$

$$q(x) = \mathrm{e}^{|x|} + \mathrm{e}^{|x|}\cos(\mathrm{e}^{\alpha|x|}), \quad x \in \mathbb{R},\ \alpha > 0. \tag{5.2}$$

Conditions (1.2) in the case (5.1)–(5.2) obviously hold. Let us show that

$$\int_{-\infty}^{0} q(t)\,\mathrm{d}t = \int_{0}^{\infty} q(t)\,\mathrm{d}t = \infty. \tag{5.3}$$

Consider, say, the latter equality in (5.3). Let

$$\hat{x}_k = \frac{1}{\alpha}\ln\Bigl(2k + \frac{1}{2}\Bigr)\pi, \quad \tilde{x}_k = \frac{1}{\alpha}\ln(2k\pi), \quad k \geqslant 1$$
$$\Rightarrow \int_0^\infty q(t)\,\mathrm{d}t \geqslant \sum_{k=1}^{\infty}\int_{\tilde{x}_k}^{\hat{x}_k}(\mathrm{e}^t + \mathrm{e}^t\cos\mathrm{e}^{\alpha t})\,\mathrm{d}t \geqslant \sum_{k=1}^{\infty}\int_{\tilde{x}_k}^{\hat{x}_k}\mathrm{e}^t\,\mathrm{d}t$$
$$\geqslant \frac{1}{\alpha}\sum_{k=1}^{\infty}(2k\pi)^{1/\alpha}\ln\Bigl(1 + \frac{1}{4k}\Bigr) \geqslant c^{-1}\sum_{k=1}^{\infty}\frac{1}{k} = \infty.$$

Let us go to I)–II). From (5.1), it follows that

$$\frac{1 + x^2}{2} \leqslant r(x) \leqslant \frac{3}{2}(1 + x^2), \quad x \in \mathbb{R} \Rightarrow \frac{1}{r} \in L_1.$$

Hence, for all $\alpha > 0$, equation (1.1) is correctly solvable in L_p, $p \in (1,\infty)$ by Corollary 1.3 (since $B_3 < \infty$, see (1.9)). Using the same example, let us consider Theorems 3.3, 3.5, 3.8 and their corollaries. Set

$$r_1(x) = 1 + x^2, \quad q_1(x) = \mathrm{e}^{|x|}, \quad x \in \mathbb{R} \Rightarrow \hat{d}_1(x) = \sqrt{\frac{1 + x^2}{\mathrm{e}^{|x|}}}, \quad x \in \mathbb{R}. \tag{5.4}$$

Let us show that $\{r_1, q_1\} \in \mathcal{K}(\mu)$, $\mu \geqslant 2$. Checking (3.2) is obvious. Further, we use the elementary estimates

$$\frac{1}{2} \leqslant \frac{\xi}{x} \leqslant 2 \quad \text{for } |\xi - x| \leqslant \mu\hat{d}_1(x),\ |x| \gg 1. \tag{5.5}$$

Below, to check (3.1) we use (5.5) (see (3.3), (3.4)):

$$\varkappa_1(x,\mu) = (1 + x^2)\sup_{|t|\leqslant\mu\hat{d}_1(x)}\left|\int_x^{x+t}\frac{2\xi}{(1 + \xi^2)^2}\,\mathrm{d}\xi\right|$$
$$\leqslant c\mu\frac{|x|}{1 + x^2}\hat{d}_1(x) \to 0 \quad \text{as } |x| \to \infty,$$
$$\varkappa_2(x,\mu) = \frac{1}{\mathrm{e}^{|x|}}\sup_{|t|\leqslant\mu\hat{d}_1(x)}\left|\int_x^{x+t}\mathrm{e}^{|\xi|}\,\mathrm{d}\xi\right| \leqslant c\mu\hat{d}_1(x) \to 0 \quad \text{as } |x| \to \infty.$$

Thus, $\{r_1, q_1\} \in \mathcal{K}(\mu)$, $\mu \geqslant 2$. Let us show that $\{r, q\} \in S(\mu)$, $\mu \geqslant 2$. From (5.4) and (5.1), we obtain that $\delta = 1/2$. To check (3.6) for $i = 3, 4$ and $x \gg 1$, we use the second mean-value theorem (see [9]) and (5.5):

$$\begin{aligned}
\varkappa_3(x,\mu) &= \sqrt{(1+x^2)\mathrm{e}^x} \sup_{|t|\leqslant \mu \hat{d}_1(x)} \left| \int_x^{x+t} \frac{\mathrm{e}^\xi \cos \mathrm{e}^\xi}{\mathrm{e}^\xi(1+\xi^2)} \,\mathrm{d}\xi \right| \\
&\leqslant c\mu \frac{\sqrt{(1+x^2)\mathrm{e}^x}}{\mathrm{e}^x(1+x^2)} \sup_{|t|\leqslant \mu \hat{d}_1(x)} \left| \int_x^{x+t} \mathrm{e}^\xi \sin \mathrm{e}^\xi \,\mathrm{d}\xi \right| = \frac{c\mu}{\sqrt{(1+x^2)\mathrm{e}^x}} \to 0, \quad x \to \infty.
\end{aligned}$$

We estimate $\varkappa_4(x,\mu)$ for $\alpha \geqslant 1/2$, $x \gg 1$:

$$\begin{aligned}
\varkappa_4(x,\mu) &= \frac{1}{\sqrt{(1+x^2)\mathrm{e}^x}} \sup_{|t|\leqslant \mu \hat{d}_1(x)} \left| \int_x^{x+t} \mathrm{e}^\xi \cos(\mathrm{e}^{\alpha\xi}) \,\mathrm{d}\xi \right| \\
&= \frac{1}{\sqrt{(1+x^2)\mathrm{e}^x}} \sup_{|t|\leqslant \mu \hat{d}_1(x)} \left| \int_x^{x+t} \frac{\mathrm{e}^{(1-\alpha)\xi}}{\alpha} [\alpha \mathrm{e}^{\alpha\xi} \cos(\mathrm{e}^{\alpha\xi})] \,\mathrm{d}\xi \right| \\
&\leqslant c \frac{\mathrm{e}^{(1-2\alpha)x/2}}{\sqrt{1+x^2}} \sup_{|t|\leqslant \mu \hat{d}_1(x)} \left| \int_x^{x+t} \alpha \mathrm{e}^{\alpha\xi} \cos(\mathrm{e}^{\alpha\xi}) \,\mathrm{d}\xi \right| \\
&\leqslant c \frac{\mathrm{e}^{(1-2\alpha)x/2}}{\sqrt{1+x^2}} \to 0 \quad \text{as } x \to \infty.
\end{aligned}$$

Since in all the estimates given above the number $\mu > 0$ can be as large as we wish, by Theorems 3.3 and 3.5 we have

$$h(x) \asymp ((1+x^2)\mathrm{e}^{|x|})^{-1/2}, \quad x \in \mathbb{R}, \ \alpha \geqslant \frac{1}{2}, \tag{5.6}$$

$$d(x) \asymp ((1+x^2)\mathrm{e}^{-|x|})^{1/2}, \quad x \in \mathbb{R}, \ \alpha \geqslant \frac{1}{2} \tag{5.7}$$

((5.7) also follows from (5.6) and Theorem 3.8). Hence, for $\alpha \geqslant 1/2$, by Corollary 3.6 (or Corollary 3.4) equation (1.1) is correctly solvable in L_p, $p \in (1,\infty)$. Thus, problem I)–II) in the case (5.1)–(5.2) can be studied with help of Theorems 3.3, 3.5 and 3.8 only for $\alpha \geqslant 1/2$. In this example, the combination of these theorems is weaker than Corollary 1.3 by itself, which encompasses all the cases $\alpha > 0$ (see (1.9)). On the other hand, from Corollary 1.3 we can only extract information on problem I)–II) whereas the potential of Corollary 3.3, Theorem 3.5 and Theorem 3.8 is far beyond that.

Relations (5.6)–(5.7) allow one (for $\alpha \geqslant 1/2$) to study properties of solutions of problem I)–II) in more detail, while the mere existence of these solutions can be proved in a simpler way, using Corollary 1.3 (see (1.9)). In particular, using (5.6)–(5.7), one can obtain precise information on the solvability of the Dirichlet and Neumann problems for this particular equation (1.1) (see [6], [7]).

Example 5.2. Consider equation (1.1) with coefficients

$$r(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ |x|^{-1/2} & \text{if } |x| \geqslant 1, \end{cases} \tag{5.8}$$

$$q(x) = 1 + \cos(|x|^\alpha), \quad x \in \mathbb{R},\ \alpha > 0. \tag{5.9}$$

Below, we show that equation (1.1) with coefficients given by (5.8)–(5.9) is correctly solvable in L_p, $p \in (1, \infty)$ if and only if $\alpha \geqslant 5/4$.

Let us start the proof. To apply Theorem 3.3 for $|x| \gg 1$, set

$$r_1(x) := r(x), \quad q_1(x) := 1, \quad \hat{d}_1(x) = \sqrt{\frac{r_1(x)}{q_1(x)}} = \frac{1}{\sqrt[4]{|x|}}.$$

Then $\{r_1, q_1\} \in \mathcal{K}(\mu)$, $\mu \geqslant 2$ (we do not present an elementary proof of this inclusion which is straightforward). It is easy to see that $\{r, q\} \in S(\mu)$, $\mu \geqslant 2$, in the case (5.8)–(5.9). Indeed, the identity $\varkappa_3(x, \mu) \equiv 0$, $x \in \mathbb{R}$ is obvious, and (3.6) for $i = 4$ and $\alpha > 5/4$ follows from the second mean-value theorem (see [9]):

$$\begin{aligned} \varkappa_4(x, \mu) &= \sqrt[4]{x} \sup_{|t| \leqslant \mu \hat{d}_1(x)} \left| \int_x^{x+t} \cos(t^\alpha)\, \mathrm{d}t \right| \\ &= \sqrt[4]{x} \sup_{|t| \leqslant \mu \hat{d}_1(x)} \left| \int_x^{x+t} \frac{1}{\alpha t^{\alpha-1}} [\alpha t^{\alpha-1} \cos(t^\alpha)]\, \mathrm{d}t \right| \leqslant \frac{c}{x^{\alpha-5/4}} \to 0, \quad x \to \infty. \end{aligned}$$

Then (see (3.10))

$$h(x) \asymp (r_1(x) q_1(x))^{-1/2} = \sqrt[4]{x}, \quad x \gg 1.$$

Since $\{r, q\} \in S(\mu)$ for any $\mu \geqslant 2$, by Theorem 3.5 we have

$$d(x) \asymp \hat{d}_1(x) = \frac{1}{\sqrt[4]{|x|}}, \quad |x| \gg 1.$$

(Theorem 3.8 gives an analogous result.) Hence $B < \infty$ (see (1.7)), and therefore for $\alpha > 5/4$ equation (1.1) is correctly solvable in L_p, $p \in (1, \infty)$. Let us now consider the case $\alpha \in (0, 5/4)$.

Below we apply Theorem 3.11. Towards this end, set

$$\begin{gathered} r_1 := r \Rightarrow \delta = 1, \\ q(x) = 1 + \cos(|x|^\alpha) := 0 \Rightarrow |x_k| = [(2k+1)\pi]^{1/\alpha}, \quad k \geqslant 1. \end{gathered} \tag{5.10}$$

It is easy to check the following relations:

$$q''(x_k) \asymp x_k^{2\alpha-2}, \quad x_k \geqslant 1,\ k \geqslant 1, \tag{5.11}$$

$$|q'''(x)| \leqslant c(x^{\alpha-3} + x^{2\alpha-3} + x^{3\alpha-3}), \quad x \geqslant 1, \tag{5.12}$$

$$\hat{d}_k = \sqrt[4]{\frac{r(x_k)}{q''(x_k)}} \asymp \frac{1}{x_k^{(4\alpha-3)/8}}, \quad k \geqslant 1. \tag{5.13}$$

Equality (3.16) for $i = 1$ can be established in a straightforward way, and for $i = 2$ it is trivial. Consider $\tau_3(x_k)$, $k \gg 1$. From (5.10), (5.12), (5.13), (5.11) and (3.19), it follows that

$$\tau_3(x_k) \leqslant C \frac{x_k^{\alpha-3} + x_k^{2\alpha-4} + x_k^{3\alpha-3}}{x_k^{2\alpha-2}} x_k^{-(4\alpha-3)/8} \leqslant \frac{C}{x_k^{(5/4-\alpha)/2}} \to 0, \quad k \to \infty.$$

Then (3.20) and (3.21) imply that $B = \infty$, i.e., by Theorem 1.2 for $\alpha \in (0, 5/4)$ equation (1.1) is not correctly solvable in L_p, $p \in (1, \infty)$. It remains to consider the case $\alpha = 5/4$. Below we use Lemma 2.1 and show that

$$d_1(x) \asymp d_2(x) \asymp |x|^{1/4} \quad \text{for } \alpha = \frac{5}{4},\ |x| \gg 1. \tag{5.14}$$

We need the following obvious inequalities:

$$\begin{aligned} 1 + \frac{3}{2}\nu &\leqslant (1+\nu)^{3/2} \leqslant 1 + 2\nu, \quad \nu \in [0, 1], \\ 1 - \frac{3}{2}\nu &\leqslant (1-\nu)^{3/2} \leqslant 1 - \nu, \quad \nu \in \Big[0, \frac{3}{4}\Big]. \end{aligned} \tag{5.15}$$

Let $\eta(x) = \nu x^{-1/4}$, $x \gg 1$, $\nu > 0$. If we are able to choose $\nu \in (0, \infty)$ so that for all $x \gg 1$ the inequality $F(\eta(x)) \leqslant 1$ holds, then by Lemma 2.1 we get that with such a choice of $\eta(x)$ the estimate $d_2(x) \geqslant \eta(x)$ holds for $x \gg 1$.

Thus (see (2.1) and (5.15)):

$$\begin{aligned} F_2(\eta) &= \int_x^{x+\eta} \sqrt{t}\,dt \cdot \int_x^{x+\eta} (1 + \cos t^{5/4})\,dt \\ &\leqslant \frac{2}{3}[(x+\eta)^{3/2} - x^{3/2}]2\eta = \frac{4}{3}\eta x^{3/2}\Big[\Big(1 + \frac{\eta}{x}\Big)^{3/2} - 1\Big] \\ &\leqslant \frac{4}{3}\nu \frac{x^{3/2}}{x^{1/4}}\Big[1 + \frac{2\nu}{x^{5/4}} - 1\Big] = \frac{8}{3}\nu^2 = 1. \end{aligned}$$

Hence we have

$$\nu = \sqrt{\frac{3}{8}}, \quad d_2(x) \geqslant \frac{1}{2\sqrt[4]{x}}, \quad x \gg 1. \tag{5.16}$$

Further, $\eta := \beta x^{-1/4}$, $x \gg 1$, $\beta \geqslant 0$. Then (see (2.1), (5.15))

$$\begin{aligned}
(5.17)\qquad F_2(\eta) &= \int_x^{x+\eta} \sqrt{t}\,\mathrm{d}t \cdot \int_x^{x+\eta} (1+\cos t^{5/4})\,\mathrm{d}t \\
&= \frac{2}{3}x^{3/2}\Big[\Big(1+\frac{\eta}{x}\Big)^{3/2} - 1\Big]\Big[\eta - \frac{4}{5}\int_x^{x+\eta} \frac{(\cos t^{5/4})'}{t^{1/4}}\,\mathrm{d}t\Big] \\
&\geqslant \frac{2}{3}x^{3/2}\Big[1+\frac{3}{2}\frac{\eta}{x} - 1\Big]\Big[\eta - \frac{4}{5}\frac{\cos t^{5/4}}{t^{1/4}}\Big|_x^{x+\eta} - \frac{1}{5}\int_x^{x+\eta}\frac{\cos t^{5/4}}{t^{5/4}}\,\mathrm{d}t\Big] \\
&\geqslant \beta x^{1/4}\Big[\eta - \frac{8}{5}\frac{1}{x^{1/4}} - \frac{1}{5}\int_x^{x+\eta}\frac{\mathrm{d}t}{t^{1/4}}\Big] \\
&\geqslant \beta x^{1/4}\Big[\frac{\beta}{x^{1/4}} - \frac{8}{5}\frac{1}{x^{1/4}} - \frac{\eta}{5x^{1/4}}\Big] \geqslant \beta(\beta-2) \geqslant 1 \\
&\Rightarrow \beta := \frac{5}{2} \Rightarrow d_2(x) \leqslant \frac{5}{2}\frac{1}{\sqrt[4]{x}}, \quad x \gg 1.
\end{aligned}$$

From (5.16) and (5.17), we obtain (5.14) for $d_2(x)$, $|x| \gg 1$ and similarly for $d_1(x)$, $|x| \gg 1$.

Now, from (1.4) and (5.14), it follows that

$$(5.18)\qquad c^{-1}|x|^{1/4} \leqslant \varphi(x), \psi(x) \leqslant c|x|^{1/4}, \quad |x| \gg 1,$$

and from (5.18) and (1.5), we get

$$(5.19)\qquad h(x) \asymp |x|^{1/4}, \quad |x| \gg 1.$$

Finally, from (5.19) and (4.14) we obtain that (see (1.8)) $B_1 < \infty$, and it remains to refer to Corollary 1.3.

Example 5.3. Consider equation (1.1) with coefficients

$$(5.20)\qquad r(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ x^2 & \text{if } |x| \geqslant 1; \end{cases} \qquad q(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ |x|^{-1/2} & \text{if } |x| \geqslant 1. \end{cases}$$

In [2], using test functions and Lemma 2.1, we have shown that in the case (5.20) equation (1.1) is correctly solvable in L_p, $p \in (1,\infty)$. Below, we get the same result using standardized methods proposed in (3.9)–(3.12) and Theorem 1.2 or Corollary 1.3.

Thus, according to (5.20), equation (1.1) is of type (α_2, β_2) (see §3). Our first step consists in getting estimates for $d_1(x)$ and $d_2(x)$ for $|x| \gg 1$ (see Lemma 1.1):

$$\frac{x^2}{4} \leqslant d_1(x) \leqslant 2x^2 \qquad \text{for } x \ll -1, \tag{5.21}$$

$$x - 2\sqrt{x} \leqslant d_1(x) \leqslant x - \frac{\sqrt{x}}{2} \qquad \text{for } x \gg 1, \tag{5.22}$$

$$|x| - 2\sqrt{|x|} \leqslant d_2(x) \leqslant |x| - \frac{\sqrt{|x|}}{2} \qquad \text{for } x \ll -1, \tag{5.23}$$

$$\frac{x^2}{4} \leqslant d_2(x) \leqslant 2x^2 \qquad \text{for } x \gg 1. \tag{5.24}$$

The estimates for d_1 and d_2 are proved in the same way, and therefore below we only consider d_1. To prove (5.21), we use Theorem 3.13. In this case, we have $q > 0$ (see (5.20)), and therefore equation (3.24) has a unique solution for $x \ll -1$. For $\nu \geqslant 1$ and $x \ll -1$, we get

$$\begin{gathered}\nu = \int_{-\infty}^{x} \frac{\mathrm{d}t}{t^2} \cdot \int_{x-d}^{x} \frac{\mathrm{d}t}{\sqrt{-t}} = \frac{2}{|x|}[\sqrt{|x|+d} - \sqrt{|x|}] \\ \Rightarrow \beta_1^{(-)}(x,\nu) = \frac{\nu^2 x^2}{4} + \nu|x|^{3/2} + |x| \geqslant \frac{\nu^2 x^2}{4}.\end{gathered} \tag{5.25}$$

For $\nu = 1$, from (5.25) we obtain the lower estimate in (5.21). To get the upper estimate in (5.21), we check (3.28). For $x \ll -1$, we have

$$\begin{gathered}\int_{-\infty}^{x} \frac{\mathrm{d}t}{t^2} \leqslant \nu \int_{x-\beta_1^{(-)}(x,\nu)}^{x} \frac{\mathrm{d}t}{t^2} \Leftrightarrow \frac{1}{|x|} \leqslant \nu\Big[\frac{1}{|x|} - \frac{1}{|x| + \beta_1^{(-)}(x,\nu)}\Big] \\ \Leftrightarrow \frac{1}{\nu} \leqslant 1 - \frac{|x|}{|x| + \beta_1^{(-)}(x,\nu)}, \quad x \ll -1.\end{gathered} \tag{5.26}$$

From (5.25) and (5.26), it follows that (3.28) holds for any $\nu > 1$. Let $\nu = 2$. Then (see (5.25))

$$d_1(x) \leqslant \beta_1^{(+)}(x,\nu) \leqslant 2x^2, \quad x \ll -1 \Rightarrow (5.21).$$

Let us go to (5.22). These estimates are proved with help of Theorem 3.15. For $a = 1$ we have (see (3.30))

$$\begin{gathered}\nu = \int_{x-\beta_1^{(+)}(x,\nu)}^{x} \frac{\mathrm{d}t}{t^2} \cdot \int_{1}^{x} \frac{\mathrm{d}t}{\sqrt{t}} = \Big(\frac{1}{x - \beta_1^{(+)}(x,\nu)} - \frac{1}{x}\Big)(2\sqrt{x} - 2) \\ \Rightarrow \beta_1^{(+)}(x,\nu) = \frac{\nu x^2}{\nu x + 2\sqrt{x} - 2}.\end{gathered}$$

Hence for $x \gg 1$, we obtain

$$
\begin{aligned}
(5.27) \qquad \beta_1^{(+)}(x,\nu) &= \frac{x}{1+(2\sqrt{x}-2)/(\nu x)} \\
&= x\Big[1 - \frac{2\sqrt{x}-2}{\nu x} + \Big(\frac{2\sqrt{x}-2}{\nu x}\Big)^2 + O\Big(\frac{1}{x\sqrt{x}}\Big)\Big] \\
&= x\Big[1 - \frac{2}{\nu}\frac{1}{\sqrt{x}} + \frac{2}{\nu x} + \frac{4}{\nu^2}\Big(\frac{1}{x} - \frac{2}{x\sqrt{x}} + \frac{1}{x^2}\Big) + O\Big(\frac{1}{x\sqrt{x}}\Big)\Big] \\
&= x - \frac{2}{\nu}\sqrt{x} + \Big(\frac{2}{\nu} + \frac{4}{\nu^2}\Big) + O\Big(\frac{1}{\sqrt{x}}\Big).
\end{aligned}
$$

Here the constant in the symbol $O(\cdot)$ is absolute, i.e., it does not depend on $\nu \geqslant 1$ for all $x \gg 1$.

In (5.27), set $\nu = 1$. Then

$$
\begin{aligned}
\alpha_1^{(+)}(x) = x - 2\sqrt{x} + 6 + O\Big(\frac{1}{\sqrt{x}}\Big) \geqslant x - 2\sqrt{x}, \quad x \gg 1 \\
\Rightarrow d_1(x) \geqslant \alpha_1^{(+)}(x) \geqslant x - 2\sqrt{x}, \quad x \gg 1.
\end{aligned}
$$

Let us check (3.34). Let $\nu > 1$, $x \gg 1$. Then

$$
2(\sqrt{x}-1) = \int_1^x \frac{\mathrm{d}t}{\sqrt{t}} \leqslant \nu \int_{x-\beta_1^{(+)}(x,\nu)}^x \frac{\mathrm{d}t}{\sqrt{t}} = 2\nu\Big(\sqrt{x} - \sqrt{x - \beta_1^{(+)}(x,\nu)}\Big).
$$

The following implication is obvious:

$$
\begin{aligned}
2(\sqrt{x}-1) \leqslant 2\sqrt{x} \leqslant 2\nu\Big(\sqrt{x} - \sqrt{x-\beta_1^{(+)}(x,\nu)}\Big) \\
\Rightarrow \frac{1}{\nu} \leqslant 1 - \sqrt{1 - \frac{\beta_1^{(+)}(x,\nu)}{x}} = 1 - \sqrt{\frac{2}{\nu}\frac{1}{\sqrt{x}} + O\Big(\frac{1}{\sqrt{x}}\Big)}, \quad x \gg 1.
\end{aligned}
$$

The latter inequality holds for any $\nu > 1$ and $x \gg 1$. In particular, for $\nu = 2$, we get

$$
\beta_1^{(+)}(x,2) = x - \sqrt{x} + 2 + O\Big(\frac{1}{\sqrt{x}}\Big) \leqslant x - \frac{1}{2}\sqrt{x}, \quad x \gg 1.
$$

Hence estimate (5.22) holds. From inequalities (5.21)–(5.24), (1.4) and (1.5), it easily follows (see [2]) that

$$
\begin{gathered}
\varphi(x) \asymp \begin{cases} |x|^{-1}, & x \ll -1, \\ |x|^{-1/2}, & x \gg 1, \end{cases} \qquad \psi(x) \asymp \begin{cases} |x|^{-1/2}, & x \ll -1, \\ |x|^{-1}, & x \gg 1, \end{cases} \\
h(x) \asymp |x|^{-1}, \quad |x| \gg 1.
\end{gathered} \tag{5.28}
$$

Now, to prove the correct solvability of (1.1) in the case (5.20), we can use any of the assertions of Corollary 1.3 (in this case we have $B_1 < \infty$, $B_2 < \infty$). Finally, by (5.28), using the scheme of the proof of Theorem 3.5, one can show that

$$d(x) \asymp |x|, \quad |x| \gg 1$$

and then refer to Theorem 1.2. The rest of the proof is simple and so we leave it to the reader.

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