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SOME NECESSARY AND SUFFICIENT CONDITIONS FOR NILPOTENT n-LIE SUPERALGEBRAS

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Abstract. The paper studies nilpotent n-Lie superalgebras over a field of characteristic zero. More specifically speaking, we prove Engel's theorem for n-Lie superalgebras which is a generalization of those for n-Lie algebras and Lie superalgebras. In addition, as an application of Engel's theorem, we give some properties of nilpotent n-Lie superalgebras and obtain several sufficient conditions for an n-Lie superalgebra to be nilpotent by using the notions of the maximal subalgebra, the weak ideal and the Jacobson radical.

Keywords: nilpotent n-Lie superalgebra; Engel's theorem; S^* algebra; Frattini subalgebra

MSC 2010: 17B45, 17B50

1. Introduction

The nilpotent theories of many algebras attract more and more attention. For example: In [5], [14], [15], the authors study nilpotent Leibniz n-algebras, nilpotent Lie and Leibniz algebras, nilpotent n-Lie algebras, respectively; D. W. Barnes discusses Engel subalgebras of Leibniz algebras in [3], and so on. In 1996, the concept of n-Lie superalgebras was first introduced by Yu. Daletskii and V. Kushnirevich in [11]. Moreover, N. Cantarini and V. G. Kac gave a more general concept of n-Lie superalgebras again in 2010 in [6]. n-Lie superalgebras are generalizations of n-Lie algebras and Lie superalgebras. As the structural properties of n-Lie superalgebras mostly remain unexplored and motivated by the investigation on Engel's theorem and nilpotency of n-Lie algebras [4], [8], [9], [13], [15] and Leibniz n-algebras [1], [5], [7], [12], it is natural to ask about the extension of these properties to the n-Lie

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superalgebras category. As is well known, for n-Lie algebras and Leibniz n-algebras, Engel's theorem and nilpotency play a predominant role in Lie theory. Analogously, Engel's theorem and nilpotency for n-Lie superalgebras will also play an important role in Lie theory.

The goal of the present paper is to study Engel's theorem and nilpotency for n-Lie superalgebras. We first prove Engel's theorem for n-Lie superalgebras, which will generalize Engel's theorems for n-Lie algebras and Lie superalgebras, then we research some properties of nilpotent n-Lie superalgebras, and moreover, we give several sufficient conditions for an n-Lie superalgebra to be nilpotent.

Definition 1.1 ([6]). An n-Lie superalgebra is an anti-commutative n-superalgebra A of parity α, such that all endomorphisms $D(a_1, \dots, a_{n-1})$ of $A(a_1, \dots, a_{n-1} \in A)$, defined by

$$D(a_1, \dots, a_{n-1})(a_n) = [a_1, \dots, a_{n-1}, a_n],$$

are derivations of A, i.e., the following Filippov-Jacobi identity holds:

$$\begin{aligned}[]
[a_1, \dots, a_{n-1}, [b_1, \dots, b_n]] &= (-1)^{\alpha(p(a_1)+\dots+p(a_{n-1}))}([[a_1, \dots, a_{n-1}, b_1], b_2, \dots, b_n] \\
&+ (-1)^{p(b_1)(p(a_1)+\dots+p(a_{n-1}))}[b_1, [a_1, \dots, a_{n-1}, b_2], b_3, \dots, b_n] + \dots \\
&+ (-1)^{(p(b_1)+\dots+p(b_{n-1}))(p(a_1)+\dots+p(a_{n-1}))}[b_1, \dots, b_{n-1}, [a_1, \dots, a_{n-1}, b_n]]).
\end{aligned}$$

From the above definition, we may see that $p([a_1, \dots, a_n]) = \alpha + \sum_{i=1}^{n} p(a_i)$ and $[a_1, \dots, a_i, a_{i+1}, \dots, a_n] = -(-1)^{p(a_i)p(a_{i+1})}[a_1, \dots, a_{i+1}, a_i, \dots, a_n]$ for all $a_i \in A$, $1 \leqslant i \leqslant n$, where $p([a_1, \dots, a_n])$ and $p(a_i)$ denote the degrees of $[a_1, \dots, a_n]$ and a_i, respectively. Moreover, since the n-Lie superalgebra A is related to α, it is also denoted by (A, α).

Analogously to the n-Lie algebras (see [13]), we have the following definition:

Definition 1.2. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an n-Lie superalgebra and I a subspace of A.

(i) I is called a vector superspace, if $I = I_{\bar{0}} \oplus I_{\bar{1}}$, where $I_{\bar{0}} = I \cap A_{\bar{0}}$, $I_{\bar{1}} = I \cap A_{\bar{1}}$.
(ii) A vector superspace $I \subseteq A$ is called a subalgebra, if $[I, I, \dots, I, I] \subseteq I$.
(iii) A vector superspace $I \subseteq A$ is called an ideal ($I \lhd A$), if $[A, A, \dots, A, I] \subseteq I$.
(iv) A vector superspace $I \subseteq A$ is called a weak ideal, if $[A, I, \dots, I, I] \subseteq I$.
(v) An ideal I is called abelian, if $[A, A, \dots, A, I, I] = 0$.
(vi) An ideal I of an algebra A is called nilpotent, if $I^v = 0$ for some $v \geqslant 0$, where $I^1 = I$, $I^{s+1} = [A, \dots, A, I, I^s]$.

In the sequel, let $\mathbb{F}$ be a field of characteristic zero and A a finite-dimensional n-Lie superalgebra over a field $\mathbb{F}$.

2. Engel's theorem of n-Lie superalgebras

Definition 2.1. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an n-Lie superalgebra over a field $\mathbb{F}$. A vector superspace V over $\mathbb{F}$ is called an A-module if on the direct sum of vector spaces $V \oplus A = B$ the structure of an n-Lie superalgebra is defined such that A is a subalgebra of B and V is an abelian ideal of B.

Definition 2.2. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a vector superspace over a field $\mathbb{F}$ and (A, α) an n-Lie superalgebra over $\mathbb{F}$. We define a multilinear mapping $\varrho\colon A^{\times(n-1)} = \underbrace{A \times A \times \ldots \times A}_{n-1} \to \operatorname{End} V$, $(x_1, x_2, \ldots, x_{n-1}) \mapsto \varrho(x_1, \ldots, x_{n-1})$. Then ϱ is called a representation and V is called an A-module, if the following relations are satisfied:

$$\varrho(a_1, \ldots, a_i, a_{i+1}, \ldots, a_{n-1}) = -(-1)^{p(a_i)p(a_{i+1})}\varrho(a_1, \ldots, a_{i+1}, a_i, \ldots, a_{n-1}), \quad a_i \in A. \tag{2.1}$$

$$\varrho(b)\varrho(a) = (-1)^{p(a)(p(b)+\alpha)}\varrho(a)\varrho(b) + \sum_{i=1}^{n-1}(-1)^{p(b)(\sum_{j=1}^{i-1}p(a_j)+\alpha)} \times \varrho(a_1, \ldots, D(b)(a_i), \ldots, a_{n-1}), \tag{2.2}$$

where $a = (a_1, \ldots, a_{n-1})$, $b = (b_1, \ldots, b_{n-1})$, $a_i, b_i \in A$.

$$\varrho(a_1, \ldots, a_{n-2}, [b_1, \ldots, b_n])(c) = \sum_{i=1}^{n} \lambda_i \varrho(b_1, \ldots, \hat{b}_i, \ldots, b_n)\varrho(a_1, \ldots, a_{n-2}, b_i)(c), \tag{2.3}$$

where
$\lambda_i = (-1)^{n-i}(-1)^{p(a)\sum_{j=1, j\neq i}^{n} p(b_j) + (p(b_i)+\alpha)\sum_{j=i+1}^{n} p(b_j)}(-1)^{\alpha(p(a_1)+p(a_2)+\ldots+p(a_{n-2}))}$, $p(a) = \sum\limits_{i=1}^{n-2} p(a_i)$, $\hat{b}_i$ denotes b_i is omitted, and $a_i, b_i, c \in A$.

$$\varrho(a)(V_\theta) \subseteq V_{\theta+\beta}, \tag{2.4}$$

where $a = (a_1, \ldots, a_{n-1})$, $\theta \in \mathbb{Z}_2$, $\beta = p(a) = \sum\limits_{i=1}^{n-1} p(a_i)$, $a_i \in A$.

Remark 2.3. Definition 2.2 is equivalent to Definition 2.1. Definition 2.2 can imply Definition 2.1. In fact, let ϱ be a representation of A and let V be an A-module. Then ϱ is a linear transformation on V. We can define on the direct sum of linear spaces $V \oplus A$ a skew-super-symmetric n-ary operator

$$[x_1, \ldots, x_{n-2}, v_1, v_2] := 0, \quad [x_1, \ldots, x_{n-1}, v] := \varrho(x_1, \ldots, x_{n-1})(v) \in V,$$

where $x_1, \ldots, x_{n-2} \in A$, $v_1, v_2, v \in V$. For $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1} \in A$, $v \in V$, by (2.1) we have

$$
\begin{aligned}
&[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_{n-1}, v]] = \varrho(x)\varrho(y)(v) = (-1)^{p(y)(p(x)+\alpha)}\varrho(y)\varrho(x)(v) \\
&\quad + \sum_{i=1}^{n-1}(-1)^{p(x)(\sum_{j=1}^{i-1} p(y_j)+\alpha)}\varrho(y_1, \ldots, D(x)(y_i), \ldots, y_{n-1})(v) \\
&= (-1)^{p(x)(p(y)+\alpha)}[y_1, \ldots, y_{n-1}, [x_1, \ldots, x_{n-1}, v]] \\
&\quad + \sum_{i=1}^{n-1}(-1)^{p(x)(\alpha+\sum_{j=1}^{i-1} p(y_j))}[y_1, \ldots, y_{i-1}, [x_1, \ldots, x_{n-1}, y_i], y_{i+1}, \ldots, y_{n-1}, v] \\
&= (-1)^{p(x)\alpha}\Bigg\{ \sum_{i=1}^{n-1}(-1)^{p(x)\sum_{j=1}^{i-1} p(y_j)}[y_1, \ldots, y_{i-1}, [x_1, \ldots, x_{n-1}, y_i], y_{i+1}, \ldots, y_{n-1}, v] \\
&\quad + (-1)^{p(x)p(y)}[y_1, \ldots, y_{n-1}, [x_1, \ldots, x_{n-1}, v]]\Bigg\},
\end{aligned}
$$

where $p(x) = \sum_{i=1}^{n-1} p(x_i)$, $p(y) = \sum_{i=1}^{n-1} p(y_i)$, that is, the above formula satisfies the Filippov-Jacobi identity. Hence $V \oplus A$ is an n-Lie superalgebra on the above operator such that A is a subalgebra of $V \oplus A$ and V is an abelian ideal of $V \oplus A$.

Definition 2.1 can also imply Definition 2.2. In fact, for any $a_1, \ldots, a_{n-1} \in A$, there is a corresponding linear transformation $\varrho(a_1, \ldots, a_{n-1})$ of V, where $\varrho(a_1, \ldots, a_{n-1})(v) = [a_1, \ldots, a_{n-1}, v]$. Then the operators $\varrho(a)$ satisfy the formulas (2.1), (2.2) and (2.3). It is clear that (2.1) holds. Further,

$$
\begin{aligned}
&\varrho(b)\varrho(a)(c) = [b_1, \ldots, b_{n-1}, [a_1, \ldots, a_{n-1}, c]] \\
&= (-1)^{\alpha p(b)}\Bigg\{ \sum_{i=1}^{n-1}(-1)^{p(b)\sum_{j=1}^{i-1} p(a_j)}[a_1, \ldots, a_{i-1}, [b_1, \ldots, b_{n-1}, a_i], a_{i+1}, \ldots, a_{n-1}, c] \\
&\quad + (-1)^{p(b)p(a)}[a_1, \ldots, a_{n-1}, [b_1, \ldots, b_{n-1}, c]]\Bigg\} \\
&= (-1)^{p(b)(p(a)+\alpha)}\varrho(a)\varrho(b)(c) \\
&\quad + \sum_{i=1}^{n-1}(-1)^{p(b)(\sum_{j=1}^{i-1} p(a_j)+\alpha)}\varrho(a_1, \ldots, D(b)(a_i), \ldots, a_{n-1})(c),
\end{aligned}
$$

where $D(b) = D(b_1, \ldots, b_{n-1})$, that is, (2.2) holds. Finally,

$$
\begin{aligned}
&(-1)^{\alpha(p(c)+\sum_{i=1}^{n-2} p(a_i))}\varrho(a_1, \ldots, a_{n-2}, [b_1, \ldots, b_n])(c) \\
&= (-1)^{\alpha(p(c)+\sum_{i=1}^{n-2} p(a_i))}[a_1, \ldots, a_{n-2}, [b_1, \ldots, b_n], c] \\
&= (-1)^{\alpha(p(c)+\sum_{i=1}^{n-2} p(a_i))}\big\{-(-1)^{p(c)(\alpha+\sum_{j=1}^{n} p(b_j))}[a_1, \ldots, a_{n-2}, c, [b_1, \ldots, b_n]]\big\}
\end{aligned}
$$

$$= -(-1)^{p(c)(\alpha+\sum_{i=1}^{n} p(b_i))} \sum_{j=1}^{n} (-1)^{(p(c)+\sum_{k=1}^{n-2} p(a_k)) \sum_{l=1}^{j-1} p(b_l)}$$
$$\times [b_1, \ldots, b_{j-1}, [a_1, \ldots, a_{n-2}, c, b_j], b_{j+1}, \ldots, b_n]$$
$$= (-1)^{n+1+\alpha p(c)+(p(a_1)+\ldots+p(a_{n-2}))(p(b_2)+\ldots+p(b_n))+(p(b_1)+\alpha)(p(b_2)+\ldots+p(b_n))}$$
$$\times [b_2, \ldots, b_n, [a_1, \ldots, a_{n-2}, b_1, c]]$$
$$+ (-1)^{n+\alpha p(c)+(p(a_1)+\ldots+p(a_{n-2}))(p(b_1)+p(b_3)+\ldots+p(b_n))+(p(b_2)+\alpha)(p(b_3)+\ldots+p(b_n))}$$
$$\times [b_1, b_3, \ldots, b_n, [a_1, \ldots, a_{n-2}, b_2, c]]$$
$$+ (-1)^{n-1+\alpha p(c)+(p(a_1)+\ldots+p(a_{n-2}))(p(b_1)+p(b_2)+p(b_4)+\ldots+p(b_n))}$$
$$\times (-1)^{(p(b_3)+\alpha)(p(b_4)+\ldots+p(b_n))} [b_1, b_2, b_4, \ldots, b_n, [a_1, \ldots, a_{n-2}, b_3, c]] + \ldots$$
$$+ (-1)^{2+\alpha p(c)+(p(a_1)+\ldots+p(a_{n-2}))(p(b_1)+\ldots+p(b_{n-1}))} [b_1, \ldots, b_{n-1}, [a_1, \ldots, a_{n-2}, b_n, c]]$$
$$= \sum_{i=1}^{n} (-1)^{n-i+\alpha p(c)} (-1)^{\sum_{j=1}^{n-2} p(a_j) \sum_{j=1, j\neq i}^{n} p(b_j) + (p(b_i)+\alpha) \sum_{j=i+1}^{n} p(b_j)}$$
$$\times \varrho(b_1, \ldots, \hat{b}_i, \ldots, b_n) \varrho(a_1, \ldots, a_{n-2}, b_i)(c),$$

that is, (2.3) holds.

A special case of the representation is the regular representation $a \mapsto D(a)$, where $D(a) = D(a_1, \ldots, a_{n-1})$, $D(a)(a_n) = [a_1, \ldots, a_{n-1}, a_n]$, $a_i \in A$. The subspace $\ker \varrho = \{x \in A;\ \varrho(A, \ldots, A, x) = 0\}$ is called the kernel of the representation ϱ. It follows from (2.1) that $\ker \varrho \lhd A$. If $\ker \varrho = 0$, then the representation ϱ is called faithful. A subset $S \subseteq A$ will be called homogeneous multiplicatively closed (h.m.c.), if for any $x, x_1, \ldots, x_n \in S$, $\lambda \in \mathbb{F}$, we have $\lambda x \in S$, $[x_1, \ldots, x_n] \in S$. We denote the linear span of a h.m.c. set S by $F(S)$, it is clear that $F(S)$ is equal to the subalgebra generated by the set S.

Theorem 2.4 (Engel's Theorem)**.** *Suppose that ϱ is a representation on an n-Lie superalgebra A in a finite-dimensional space V, S is a h.m.c. subset of A and the operators $\varrho(a_1, \ldots, a_{n-1})$ are nilpotent for any $a_1, \ldots, a_{n-1} \in S$. Then the algebra S_ϱ^* generated by these operators is nilpotent. In addition, if the representation ϱ is faithful, the algebra $F(S)$ is also nilpotent and acts nilpotently on A.*

P r o o f. By considering the quotient algebra $A/\ker \varrho$, we may assume with no loss of generality that ϱ is faithful. With any subset $X \subseteq S$ we associate the subalgebra $X_\varrho^* \leqslant A_\varrho^*$ generated by the operators $\varrho(a_1, \ldots, a_{n-1}), a_i \in X$. Suppose that X is a maximal h.m.c. subset of S and its corresponding algebra X_ϱ^* is nilpotent. Our aim is to prove that $X = S$.

Suppose $(X_\varrho^*)^s = 0$. Put $C = F(X)$, $C_0 = A$, $C_{i+1} = [C, \ldots, C, C_i]$ for $i \geqslant 0$. We introduce an abbreviated notation for certain subspaces of A_ϱ^*:

$$\varrho(A, \ldots, A, C_i) = \varrho(A, C_i), \quad \varrho(C, \ldots, C, A) = \varrho(C, A), \quad \varrho(C, \ldots, C) = \varrho(C),$$

etc. By induction on k, we will show that for any $k \geqslant 0$,

$$\varrho(C, C_k) \subseteq \sum_{i=0}^{k} \varrho^i(C)\varrho(C, A)\varrho^{k-i}(C). \tag{2.5}$$

In fact, it follows from (2.2) that

$$\varrho(C, C_{k+1}) = \varrho(C, [C, \ldots, C, C_k]) \subseteq \varrho(C, C_k)\varrho(C) + \varrho(C)\varrho(C, C_k).$$

This enables us to complete the inductive passage from k to $k + 1$ in relation (2.4), it is trivial for $k = 0$. It follows from (2.2) that

$$\varrho(A, C_{k+1}) = \varrho(A, [C, \ldots, C, C_k]) \subseteq \varrho(C, C_k)\varrho(A, C) + \varrho(C)\varrho(A, C_k). \tag{2.6}$$

Again using induction on k and (2.4), we see that for $k \geqslant 1$

$$\varrho(A, C_k) \subseteq \varrho^k(C)\varrho(A) + \sum_{i+j=k-1} \varrho^i(C)\varrho(C, A)\varrho^j(C)\varrho(A, C).$$

Since $\varrho^s(C) = 0$, we obtain $\varrho(A, C_k) = 0$ for $k \geqslant 2s$, i.e., $C_k \subseteq \ker \varrho$, hence $C_k = 0$. This means that C acts nilpotently on A by left multiplications, in particular, the algebra C is itself nilpotent.

If $S \neq X$, it follows easily from the preceding that $S \setminus X$ contains an element b such that

$$[X, \ldots, X, b] \subseteq X. \tag{2.7}$$

Then $Y = \mathbb{F}b \cup X$ is a h.m.c. subset of S strictly containing X. We will show that the algebra Y^*_ϱ is nilpotent, which is contrary to the maximality of X. Any element of $\varrho(Y)$ lies either in $\varrho(X)$ or in $\varrho(X, b)$. Suppose $U \in \varrho(Y)^m$, $m > 0$. If in the word U the operators in $\varrho(X)$ occur at least s times, then in view of (2.1) and (2.6), U can be transformed into a sum of words in which the operators in $\varrho(X)$ appear consecutively and the number of them is at least s, therefore $U = 0$.

On the other hand, if in U the operators in $\varrho(X)$ occur $l \leqslant s - 1$ times, then U has the form $U_1\varrho_1U_2\varrho_2 \ldots U_l\varrho_lU_{l+1}$, where $\varrho_i \in \varrho(X)$, U_i are products of elements $\varrho(X, b)$, and some of the words U_i can be empty.

Let us view A as an $(n - 1)$-Lie superalgebra A_b with operation

$$[a_1, \ldots, a_{n-1}]_b = [a_1, \ldots, a_{n-1}, b]$$

and V as an A_b-module on which the representation $\tilde{\varrho}$ of the algebra

$$A_b\colon\ \tilde{\varrho}(a_1,\ldots,a_{n-2})=\varrho(a_1,\ldots,a_{n-2},b)$$

acts. It follows from (2.6) that X is a h.m.c. set in A_b. Since the operators in $\tilde{\varrho}(X)=\varrho(X,b)$ are nilpotent, the induction assumption with respect to n is applicable to the triple $(A_b,X,\tilde{\varrho})$ and the algebra $X^*_{\tilde{\varrho}}$ is nilpotent, suppose that $(X^*_{\tilde{\varrho}})^t=0$. When $n=2$, since the algebra $X^*_{\tilde{\varrho}}$ is generated by the nilpotent operator $\varrho(b)$, $X^*_{\tilde{\varrho}}$ is nilpotent, which provides the basis for the induction.

If the ϱ-length of U_i is greater than or equal to t, then $U_i=0$, $1\leqslant i\leqslant l+1$. Consequently, when $m\geqslant st$ all words $U\in\varrho(Y)^m$ are zero, i.e., $(Y^*_{\varrho})^{\mathrm{st}}=0$ as required. This contradiction shows that $X=S$. The second assertion of the theorem has already been proved, since $C=F(X)=F(S)$. □

Corollary 2.5. *Suppose A is a finite-dimensional n-Lie superalgebra in which all left multiplication operators $D(a)$ are nilpotent, where $D(a)=D(a_1,\ldots,a_{n-1})$, $a_i\in A$, $1\leqslant i\leqslant n-1$. Then A is nilpotent.*

Proof. Let ϱ be the regular representation and $A=V=S$. By Theorem 2.4, we obtain A is nilpotent. □

3. Nilpotency of n-Lie superalgebras

Definition 3.1. The Frattini subalgebra, $F(A)$, of A is the intersection of all maximal subalgebras of A. The maximal ideal of A contained in $F(A)$ is denoted by $\varphi(A)$.

The next proposition contains results analogous to the corresponding ones for n-Lie algebras, their proof is similar to those for n-Lie algebras (see [2], Proposition 2.1).

Proposition 3.2. *Let A be an n-Lie superalgebra over $\mathbb{F}$. Then the following statements hold:*

(1) *If B is a subalgebra of A such that $B+F(A)=A$, then $B=A$.*
(2) *If B is a subalgebra of A such that $B+\varphi(A)=A$, then $B=A$.*

Lemma 3.3. *Let A be an n-Lie superalgebra over $\mathbb{F}$. Then $F(A)\subseteq A^2$; in particular, if A is abelian, then $F(A)=0$.*

Proof. If $A=A^2=[A,\ldots,A]$, then $F(A)\subseteq A^2$; if $A\neq A^2$ and $F(A)\nsubseteq A^2$, then there exists $x\in F(A)$, $x\notin A^2$ and a subalgebra B of A such that $A^2\subseteq B$, $x\notin B$ and $\dim B=\dim A-1$. Hence B is a maximal subalgebras of A which does not contain x. This contradicts $x\in F(A)$. Therefore, $F(A)\subseteq A^2$. □

Lemma 3.4 ([10]). *Let f be an endomorphism of a finite-dimensional vector superspace V over $\mathbb{F}$ and let χ be a polynomial such that $\chi(f) = 0$. Then the following statements hold:*

(1) *If $\chi = q_1 q_2$ and q_1, q_2 are relatively prime, then V is decomposed into a direct sum of f-invariant subspaces $V = U \oplus W$ such that $q_1(f)(U) = 0 = q_2(f)(W)$.*
(2) *V is decomposed into a direct sum of f-invariant subspaces $V = V_0 \oplus V_1$, for which $f|_{V_0}$ is nilpotent and $f|_{V_1}$ is invertible.*

Remark 3.5. Note that, in the case where V is finite-dimensional, we may choose χ to be the characteristic polynomial of f. The decomposition (2.2) is called the Fitting decomposition with respect to f. Subspaces V_0, V_1 are referred to as the Fitting-0 and Fitting-1 components of V, respectively.

Definition 3.6. An n-Lie superalgebra A satisfies condition $(*)$ if the only subalgebra K of A with the property $K + A^2 = A$ is $K = A$, where $A^2 = [A, A, \ldots, A]$; an n-Lie superalgebra satisfies condition $(**)$ if $a_i \in A_0(D(a_1, \ldots, a_{n-1}))$ for some $1 \leqslant i \leqslant n-1$ for arbitrary $a_i \in A$, where $A_0(D(a_1, \ldots, a_{n-1})) = \{x \in A;\ D^r(a_1, \ldots, a_{n-1})(x) = 0$ for some $r\}$.

Theorem 3.7. *Let A be an n-Lie superalgebra over $\mathbb{F}$. Then the following statements hold:*

(i) *If A satisfies condition $(**)$ and any maximal subalgebra M of A is a weak ideal of A, then A is nilpotent.*
(ii) *If A is nilpotent, then every maximal subalgebra M of A is an ideal of A.*

Proof. (i) Assume that A is not nilpotent. Then there exists a non-nilpotent left multiplication operator $D(a_1, \ldots, a_{n-1})$. Put $D(a) := D(a_1, \ldots, a_{n-1})$. Since $D(a)$ is non-nilpotent, the Fitting-0 component satisfies $A_0(D(a)) \neq A$. Let M be a maximal subalgebra of A containing $A_0(D(a))$. Then $a_i \in A_0(D(a)) \subseteq M$ for some $1 \leqslant i \leqslant n-1$ by assumption. Since the maximal subalgebra M of A is a weak ideal of A, $D(a)(A) \subseteq M$. Since $D(a)$ is an automorphism on the Fitting-1 component $A_1(D(a))$, we obtain that $A_1 = D(a)(A_1) = A_1 \cap M$. Hence $A_1 \subseteq M$. Then $A = A_0 \oplus A_1 \subseteq M \neq A$. This is a contradiction. Thus all left multiplication operators are nilpotent. Therefore, by Corollary 2.5, A is nilpotent.

(ii) We assume that A is nilpotent and M is any maximal subalgebra of A. Then R also acts nilpotently on A for all $R \in D(A)$, where $D(A)$ is the vector space generated by all left multiplications of A. Thus R acts nilpotently on A/M for all $R \in D(A)$. Then there is a $v \neq 0 \in A/M$ such that $R(v) = 0$ for all $R \in D(A)$. This means $R(v) \in M$ and hence $v \in N_A(M)$, where $N_A(M) = \{x \in A;\ [x, M, A, \ldots, A] \in M\}$, but since $v \neq 0 \in A/M$, we have that v is not in M, hence $M \subset N_A(M)$. By the maximality of M, then $N_A(M) = A$, i.e., M is an ideal of A. □

Corollary 3.8. *Let A be an n-Lie algebra over $\mathbb{F}$. Then A is nilpotent if and only if every maximal subalgebra M of A is a weak ideal of A.*

Remark 3.9. An n-Lie superalgebra with condition $(**)$ does exist. For example, let (A, α) be an n-Lie superalgebra with basis $\{b, c\}$, $A = A_{\bar{0}} \oplus A_{\bar{1}}$, $A_{\bar{0}} = \mathbb{F}c$, $A_{\bar{1}} = \mathbb{F}b$, $\alpha = \bar{0}$, and let its multiplication be as follow: $[b, \ldots, b, c] = 0$, $[b, \ldots, b] = c$, then $b, c \in A_0(D(b, \ldots, b, c))$.

Definition 3.10. An ideal I of an n-Lie superalgebra A is called the Jacobson radical, if I is the intersection of all maximal ideals of A, denoted by $J(A)$.

Proposition 3.11. *For any n-Lie superalgebra A, $J(A) \subseteq A^2$.*

P r o o f. The proof is similar to that of Lemma 3.3. □

Definition 3.12. The ideal I of an n-Lie superalgebra A is called k-solvable $(2 \leqslant k \leqslant n)$ if $I^{(r)} = 0$ for some $r \geqslant 0$, where $I^{(0)} = I$,

$$I^{(s+1)} = [\underbrace{I^{(s)}, I^{(s)}, \ldots, I^{(s)}}_{k}, \underbrace{A, \ldots, A}_{n-k}]$$

for some $s \geqslant 0$. When $A = I$, A is called a k-solvable n-Lie superalgebra. Clearly, if A is nilpotent, then it is k-solvable $(k \geqslant 2)$.

Lemma 3.13. *Let an algebra A be a k-solvable n-Lie superalgebra $(k \geqslant 2)$, then $J(A) = A^{(1)}$.*

P r o o f. According to Proposition 3.11, $J(A) \subseteq A^{(1)}$. We merely need to verify $A^{(1)} \subseteq J(A)$. Let I be an ideal of A. As A is k-solvable, A/I is k-solvable and does not contain any proper ideal of A/I, hence $[A/I, \ldots, A/I] = 0$, thus $A^{(1)} \subseteq I$, and by the definition of the Jacobson radical, we have $A^{(1)} \subseteq J(A)$. Then we get $J(A) = A^{(1)}$. □

Theorem 3.14. *Let A be a nilpotent n-Lie superalgebra over $\mathbb{F}$. Then $F(A) = A^{(1)} = \varphi(A) = J(A)$.*

P r o o f. Since A is nilpotent, by Theorem 3.7 (ii), any maximal subalgebra T is an ideal of A, A/T is a nilpotent n-Lie superalgebra, and A/T has no proper ideal, thus $[A/T, \ldots, A/T] = 0$, $A^{(1)} \subseteq T$, and $A^{(1)} \subseteq F(A)$. By Lemma 3.3, $F(A) = A^{(1)}$. Since A is nilpotent, A is k-solvable, and by Lemma 3.13, $J(A) = A^{(1)}$. Therefore, $F(A) = \varphi(A) = J(A) = A^{(1)}$. The proof is complete. □

Theorem 3.15. *Let A be an n-Lie superalgebra over $\mathbb{F}$. Then the following statements hold:*

(1) *If A satisfies conditions $(**)$ and $(*)$, then A is nilpotent.*

(2) *If A is nilpotent, then the condition $(*)$ holds in A.*

Proof. (1) Suppose that the condition $(*)$ holds in A. Let M be any maximal subalgebra of A. Since $M + A^2 \neq A$, $A^2 \subseteq M$, and M is an ideal in A. It follows from Theorem 3.7 (i) that A is nilpotent.

(2) Suppose that A is nilpotent. By Theorem 3.14, we have $A^2 = F(A)$. Then $K + A^2 = K + F(A) = A$ implies $K = A$ by Proposition 3.2. □

Corollary 3.16. *Let A be an n-Lie algebra over $\mathbb{F}$. Then A is nilpotent if and only if the condition $(*)$ holds in A.*

Definition 3.17. A subalgebra T of an n-Lie superalgebra A is called subinvariant if there exist subalgebras T_i such that $A = T_0 \supset T_1 \supset T_2 \supset \ldots \supset T_{n-1} \supset T_n = T$ where T_i is an ideal in T_{i-1} for $i = 1, 2, \ldots, n$. It is also denoted by $T = T_n \lhd T_{n-1} \lhd T_{n-2} \lhd \ldots \lhd T_1 \lhd T_0 = A$.

An upper chain, C_k, of length k consists of subalgebras $U_0, U_1, \ldots, U_k$ in A such that $U_0 = A$ and each U_i is maximal in U_{i-1} for $i = 1, 2, \ldots, k$. The subinvariance number of C_k, $s(C_k)$, is defined to be the number of $U_i \neq U_0 = A$ which are subinvariant in A; the invariance number of C_k, $v(C_k)$, is defined as $k - s(C_k)$ if $s(C_k) \neq 0$, and as k otherwise. Then the invariance number of A, $v(A)$, is the maximum of $v(C_k)$ for all C_k of A.

Lemma 3.18. *Let A be a nonzero n-Lie superalgebra and V a maximal subalgebra of A. If V is not an ideal in A, then $v(A) > v(V)$.*

Proof. Suppose $C_n\colon\ V = V_0 \supset V_1 \supset V_2 \supset \ldots \supset V_n$ is an upper chain of length n in V. Then $A \supset V = V_0 \supset V_1 \supset V_2 \supset \ldots \supset V_n$ is an upper chain C_{n+1} of length $n+1$ in V. If V_i, $1 \leqslant i \leqslant n$, is subinvariant in A, then we have

$$A = U_0 \supset U_1 \supset U_2 \supset \ldots \supset U_k = V_i,$$

where U_i is an ideal in U_{i-1} for $i = 1, 2, \ldots, k$. We also have

$$V = A \cap V = U_0 \cap V \supseteq U_1 \cap V \supseteq \ldots \supseteq U_k \cap V = V_i.$$

Since U_i is an ideal in U_{i-1}, $U_i \cap V$ is an ideal in $U_{i-1} \cap V$ and V_i is subinvariant in V. Hence, if V_i, $1 \leqslant i \leqslant n$, is subinvariant in A, then it is subinvariant in V. Since V is not an ideal in A, $s(C_{n+1}) \leqslant s(C_n)$. If $s(C_{n+1}) > 0$, then $v(C_{n+1}) = (n+1) - s(C_{n+1}) \geqslant (n+1) - s(C_n) > n - s(C_n) = v(C_n)$. If $s(C_{n+1}) = 0$, then $v(C_{n+1}) = n + 1 > n \geqslant v(C_n)$. Hence, $v(A) > v(V)$. □

Theorem 3.19. *Let A be an n-Lie superalgebra over $\mathbb{F}$. Then the following statements hold:*

(1) *If A satisfies condition $(**)$ and $v(A) = v(U)$ for every proper subalgebra U in A, then A is nilpotent.*

(2) *If A is nilpotent, then for every proper subalgebra U in A, $v(A) = v(U)$.*

Proof. (1) Suppose that $\dim(A) = n$. Let V be any maximal subalgebra of A such that $v(V) = v(A)$. Then by Lemma 3.18, V is an ideal in A. It follows from Theorem 3.7 (i) that A is nilpotent.

(2) If A is nilpotent, then every subalgebra of A is subinvariant. Hence $v(A) = 1$. Since every subalgebra of A is also nilpotent, $v(V) = 1$, hence $v(A) = v(V)$. □

Corollary 3.20. *Let A be an n-Lie algebra over $\mathbb{F}$. Then A is nilpotent if and only if $v(A) = v(U)$ for every proper subalgebra U in A.*

Theorem 3.21. *Let U be a subinvariant subalgebra of n-Lie superalgebra A and K an ideal of U such that $K \subseteq F(A)$. If U/K is nilpotent, then U is nilpotent.*

Proof. We have a chain of subalgebras $U = U_r \lhd U_{r-1} \lhd \ldots \lhd U_1 \lhd U_0 = A$. Let $a_i \in U$, $1 \leqslant i \leqslant n-1$, and $D(a) = D(a_1, \ldots, a_{n-1})$. Then $D(a)U_{i-1} \subseteq U_i$ since $U_i \lhd U_{i-1}$. Hence $D^r(a)A \subseteq U$. But U/K is nilpotent, so $D^s(a)U \subseteq K$ for some s. Thus, if $\dim(A) = t$, we have $D^t(a)A \subseteq K$. Moreover, $A = \Im(D^t(a)) \oplus \operatorname{Ker}(D^t(a))$. In fact, we set $I := \bigcap_{i=1}^{\infty} D^i(a)(A)$ and $B := \bigcup_{i=1}^{\infty} B_i$, where $\{B_i = x \in A;\ D^i(a)(x) = 0\}$. Since $D(a)$ is a linear transformation of A, we have

$$A \supseteq D(a)(A) \supseteq \ldots \supseteq D^m(a)(A) \supseteq \ldots.$$

As $\dim A < \infty$, there exists a positive integer s such that $D^s(a)(A) = D^{s+1}(a)(A)$, and one gets $I = \bigcap_{i=1}^{\infty} D^i(a)(A) = D^s(a)(A)$ and $I = D(a)(I)$. Similarly

$$0 \subseteq B_1 \subseteq \ldots \subseteq B_j \subseteq \ldots.$$

There exists a positive integer k such that $B_k = B_{k+1}$. Thus $B = B_k$. Let $m = \max\{s, k\}$. Then $I = D^m(a)(A)$, $B = B_m = \{x \in A;\ D^m(a)(x) = 0\}$. It is clear that $I \cap B = 0$, and for any $x \in A$, if $D^m(a)(x) = 0$, then $D^m(a)(x) \in I = D^{2m}(a)(A)$. There exists $y \in A$ such that $D^m(a)(x) = D^{2m}(a)(y)$, hence $D^m(a)(x - D^m(a)(y)) = 0$. Put $z := x - D^m(a)(y)$, then $z \in B$. Therefore $A = I \oplus B$. In particular, we may take $m = t$. We get $A = \Im(D^t(a)) \oplus \operatorname{Ker}(D^t(a))$.

So $A = K + E_A(D(a))$, where $E_A(D(a)) = \{x \in A;\ D^r(a)(x) = 0 \text{ for some } r\}$. But $K \subseteq F(A)$, so this implies that $E_A(D(a)) = A$. Thus every $D(a)$ for all $a_i \in U$, $1 \leqslant i \leqslant n-1$, is nilpotent and U is nilpotent by Corollary 2.5. □

Example 3.22. Let (A, α) be an n-Lie superalgebra with basis $\{b, c\}$, $A = A_{\bar{0}} \oplus A_{\bar{1}}$, $A_{\bar{0}} = \mathbb{F}c$, $A_{\bar{1}} = \mathbb{F}b$, $\alpha = \bar{0}$, and let its multiplication be as follow: $[b, \ldots, b, c] = 0$, $[b, \ldots, b] = c$. Then A is nilpotent, however $\dim(A/A^2) = 1$.

The above example shows the definition of the S^* algebra for an n-Lie superalgebra is analogous to the case of a Leibniz algebra, thus we give the following definition:

Definition 3.23. An n-Lie superalgebra A is called an S^* algebra if every proper non-abelian subalgebra H of A either has $\dim(H/H^2) \geqslant 2$ or is nilpotent and generated by one element.

Lemma 3.24. *Let A be a non-abelian nilpotent n-Lie superalgebra. Then we have either $\dim(A/A^2) \geqslant 2$ or A is generated by one element.*

Proof. Since A is nilpotent, by Theorem 3.14 one gets $A^2 = F(A)$. It is clear that $\dim(A/A^2) \neq 0$ since A is nilpotent. If $\dim(A/A^2) = 1$, then A is generated by one element. Otherwise $\dim(A/A^2) \geqslant 2$. □

Lemma 3.25. *Let A be a non-nilpotent n-Lie superalgebra. If all proper subalgebras of A are nilpotent, then $\dim(A/A^2) \leqslant 1$.*

Proof. Suppose that $\dim(A/A^2) \geqslant 2$. Then there exist distinct maximal subalgebras M and N which contain A^2. Hence M and N are nilpotent ideals, $A = M + N$ is nilpotent, which is a contradiction. □

Theorem 3.26. *An n-Lie superalgebra A is an S^* algebra if and only if it is nilpotent.*

Proof. If A is nilpotent, then every subalgebra of A is nilpotent, so A is an S^* algebra by Lemma 3.24. Conversely, suppose that there exists an S^* algebra that is not nilpotent. Let A be the smallest dimensional and non-nilpotent. All proper subalgebras of A are S^* algebras, hence they are nilpotent. Thus $\dim(A/A^2) \leqslant 1$ by Lemma 3.25. Since A is an S^* algebra, it is generated by one element and it is nilpotent, which is a contradiction. □

Theorem 3.27. *Let (A, α) be an n-Lie superalgebra and D a derivation of A. For $x_1, \ldots, x_n \in A$, then $D^k[x_1, \ldots, x_n] = \sum\limits_{i_1+\ldots+i_n=k} a^{(k)}_{i_1,\ldots,i_n}[D^{i_1}(x_1), \ldots, D^{i_n}(x_n)]$, where $a^{(k)}_{i_1,\ldots,i_n} \in \mathbb{F}$.*

Proof. We proceed by induction on k. If $k = 1$, then

$$\begin{aligned} &D[x_1, x_2, \ldots, x_n] \\ &\quad = (-1)^{p(D)\alpha}[D(x_1), x_2, \ldots, x_n] + (-1)^{p(D)(p(x_1)+\alpha)}[x_1, D(x_2), x_3, \ldots, x_n] \\ &\qquad + \ldots + (-1)^{p(D)(p(x_1)+\ldots+p(x_n)+\alpha)}[x_1, x_2, \ldots, x_{n-1}, D(x_n)] \end{aligned}$$

and the base case is satisfied. We now assume that the result holds for k and consider $k+1$. Then

$$\begin{aligned}
&D^{k+1}[x_1,\dots,x_n]\\
&\quad = D\Bigg(\sum_{i_1+\ldots+i_n=k} a^{(k)}_{i_1,\ldots,i_n}[D^{i_1}(x_1),\dots,D^{i_n}(x_n)]\Bigg)\\
&\quad = \sum_{i_1+\ldots+i_n=k} a^{(k)}_{i_1,\ldots,i_n}\big\{(-1)^{p(D)\alpha}[D^{i_1+1}(x_1),\dots,D^{i_n}(x_n)]\\
&\qquad + \ldots + (-1)^{p(D)\{p(x_1)+\ldots+p(x_n)+\alpha+(i_1+\ldots+i_{n-1})p(D)\}}[D^{i_1}(x_1),\dots,D^{i_n+1}(x_n)]\big\}\\
&\quad = \sum_{j_1+\ldots+j_n=k+1} a^{(k+1)}_{j_1,\ldots,j_n}[D^{j_1}(x_1),\dots,D^{j_n}(x_n)].
\end{aligned}$$

The last equality holds because if we suppose that the array $(j_1,\dots,j_n)$ satisfies $j_1+\ldots+j_n = k+1$, then there must exist an array $(i_1,\dots,i_n)$ such that $i_1+\ldots+i_n = k$ and for $m\in\{1,\dots,n\}$ it satisfies $i_1 = j_1$, …, $i_{m-1} = j_{m-1}$, $i_m + 1 = j_m$, $i_{m+1} = j_{m+1}$, …, $i_n = j_n$, that is, $(i_1,\dots,i_{m-1},i_m+1,i_{m+1},\dots,i_n) = (j_1,\dots,j_{m-1},j_m,j_{m+1},\dots,j_n)$. This proves the theorem. □

Theorem 3.28. *Let A be an n-Lie superalgebra over $\mathbb{F}$. Suppose that B is an ideal of A and C is an ideal of B such that $C\subseteq B\cap F(A)$. If B/C is nilpotent, then B is nilpotent.*

Proof. Take any element x_i of B, $1\leqslant i\leqslant n-1$. By Remark 3.5, $A = A_0 + A_1$ is the Fitting decomposition relative to $D(x)$, where $D(x) = D(x_1,\dots,x_{n-1})$ is nilpotent in A_0 and $D(x)$ is an isomorphism of A_1. So $A_1\subset B$. Since B/C is nilpotent, there exists an integer n such that $A_1 = D^n(x)(A_1)\subset C$. Then $A = A_0 + F(A)$. If A_0 is a subalgebra of A, by Proposition 3.2 it implies that $A = A_0$. Hence, $D(x)$ is nilpotent for any element $x_i\in B$, $1\leqslant i\leqslant n-1$. Therefore, B is nilpotent by virtue of Corollary 2.5.

It remains to show that A_0 is a subalgebra of A. For $x_1,\dots,x_n\in A$, by Theorem 3.27 we have

$$D(x)^k[x_1,\dots,x_n] = \sum_{i_1+\ldots+i_n=k} a^{(k)}_{i_1,\ldots,i_n}[D(x)^{i_1}(x_1),\dots,D(x)^{i_n}(x_n)].$$

If $x_1,\dots,x_n\in A_0$, then $D(x)^k[x_1,\dots,x_n] = 0$ for an integer k big enough, hence $[x_1,\dots,x_n]\in A_0$. □

Corollary 3.29. *Let A be an n-Lie superalgebra with $B \lhd A$ such that $B \subseteq F(A)$. Then B is nilpotent. In particular, $\varphi(A)$ is a nilpotent ideal of A.*

Definition 3.30. A nilpotent n-Lie superalgebra A is said to be of class t if $A^{t+1} = 0$ and $A^t \neq 0$. We also denote $\mathrm{cl}(A) = t$.

Put $AN^i = [A, \ldots, A, N^i]$ and $A^j N^i = [A, \ldots, A, A^{j-1}N^i]$ for some $j > 1$.

Lemma 3.31. *Let A be an n-Lie superalgebra with $N \lhd A$ and let A/N^2 be nilpotent. If $A^{m+1} \subset N^2$ for some minimal m, then $A^u N^r \subset N^{r+1}$ for $r > 0$ where $u = (r-1)(n-1)(m-1) + m$.*

Proof. We proceed by induction on r. If $r = 1$, then $A^{(1-1)(n-1)(m-1)+m} N^1 = A^m N \subseteq A^{m+1} \subset N^2$ and the base case is satisfied. We now assume that the result holds for r and consider $r+1$.

Let $s = r(n-1)(m-1)+m$ and $u = (r-1)(n-1)(m-1)+m$. By Theorem 3.27, we obtain

$$A^s N^{r+1} = A^s[N^r, N, A, \ldots, A] = \sum_{s_1+\ldots+s_n=s} [A^{s_1}N^r, A^{s_2}N, A^{s_3}A, \ldots, A^{s_n}A].$$

Suppose that $s_1 \geqslant u$. Then by the induction hypothesis, $A^{s_1}N^r \subset N^{r+1}$ and

$$\sum_{s_1+\ldots+s_n=s} [A^{s_1}N^r, A^{s_2}N, A^{s_3}A, \ldots, A^{s_n}A] \subset [N^{r+1}, N, A, \ldots, A] \subset N^{r+2}.$$

Suppose that $s_1 < u$. We claim there exists $s_k \geqslant m$. Assume that $s_j < m$ for all j. We obtain $s = (s_1) + (s_2 + \ldots + s_n) < u + (n-1)(m-1) = (r-1)(n-1)(m-1) + m + (n-1)(m-1) = r(n-1)(m-1) + m = s$. But this is impossible. Hence there exists $s_k \geqslant m$ for some k. As a result $A^{s_k}N \subset N^2$ and using the Filippov-Jacobi identity and skew super-symmetry, we obtain

$$\begin{aligned}
&[A^{s_1}N^r, A^{s_2}N, A^{s_3}A, \ldots, A^{s_k}A, \ldots, A^{s_n}A] \\
&\quad = [N^r, N, A, \ldots, A, N^2, A, \ldots, A] \\
&\quad = [N^r, N, A, \ldots, A, A, \ldots, A, N^2] \\
&\quad = [N^r, N, A, \ldots, A, A, \ldots, A, [N, \ldots, N]] \\
&\quad = [[N^r, N, A, \ldots, A, N,], N, \ldots, N] + [N, [N^r, N, A, \ldots, A, N,], N, \ldots, N] \\
&\qquad + \ldots + [N, \ldots, N, [N^r, N, A, \ldots, A, N,]] \\
&\quad \subseteq [N^{r+1}, N, N, \ldots, N] \\
&\quad \subseteq [N^{r+1}, N, A, \ldots, A] \\
&\quad = N^{r+2}.
\end{aligned}$$

This proves the lemma. □

Theorem 3.32. *Let A be an n-Lie superalgebra with $N \lhd A$. If $N^{t+1} = 0$ and $(A/N^2)^{m+1} = 0$, then $cl(A) \leqslant tm + \frac{1}{2}t(t-1)(m-1)(n-1)$.*

Proof. Using Lemma 3.31, we observe that $A^{m+1} \subset N^2$, $A^{m+(n-1)(m-1)}N^2 \subset N^3$, ..., $A^{m+(t-1)(n-1)(m-1)}N^t \subset N^{t+1} = 0$. By summing the exponents on the left-hand side, we see that $A^{\omega} = 0$, where $\omega = tm + \frac{1}{2}t(t-1)(m-1)(n-1) + 1$. The proof is complete. □

Definition 3.33. Let A be a nonzero n-Lie superalgebra and S a subset of A such that $S \supseteq \{0\}$. The normal closure of S in A, S^A, is the smallest ideal in A containing S.

Theorem 3.34. *Let A be a nonzero n-Lie superalgebra over $\mathbb{F}$. Then:*

(i) *If A satisfies condition $(**)$, then there exists a nonzero nilpotent subalgebra N in A such that $N^A = A$.*
(ii) *A is nilpotent if and only if the subalgebra N in* (i) *is A.*

Proof. (i) If A is nilpotent, then we may take $N = A$ and $N^A = A^A = A$. Consider the case that A is not nilpotent. We use induction on the dimension of A. A non-nilpotent n-Lie superalgebra of lowest dimension is two-dimensional, namely, $A = A_{\bar{0}} \oplus A_{\bar{1}}$, $A_{\bar{0}} = \mathbb{F}x$, $A_{\bar{1}} = \mathbb{F}y$, with a bilinear skew super-symmetric bracket multiplication $[x, x, y] = y$ defined on A. The normal closure of the one dimensional subalgebra $\mathbb{F}x$ is L. Assume that the theorem holds for all non-nilpotent n-Lie superalgebras whose dimension is less than n. Consider the case that A is an n-dimensional non-nilpotent n-Lie superalgebra. Then by Theorem 3.7 (i), there exists a maximal subalgebra M in A such that M is not an ideal in A. Since the dimension of M is less than n, by our inductive hypothesis there exists a nilpotent subalgebra N in M such that $N^M = M$. We claim that $N^A \supseteq M$. Since N^A is an ideal in A, $[A, \ldots, A, N^A] \subseteq N^A$. In particular, $[M, \ldots, M, N^A] \subseteq N^A$. Since M is a subalgebra, $[M, \ldots, M, N^A \cap M] \subseteq N^A \cap M$ and $N^A \cap M$ is an ideal in M containing N. Since N^M is the smallest ideal in M containing N, we have $N^A \cap M \supseteq N^M$, i.e., we have $N^A \supseteq N^A \cap M \supseteq N^M = M$. Since M is not an ideal of A and N^A is an ideal of A, $N^A \supset M$. Now $N^A = A$ follows from the fact that M is a maximal subalgebra in A.

(ii) If $A = N$ and N is nilpotent, A is nilpotent. Conversely, suppose that $\{0\} \neq N \neq A$. Then either N is a maximal subalgebra of nilpotent n-Lie superalgebra A or N is contained in a maximal subalgebra M of A. By Theorem 3.7 (ii), every maximal subalgebra in A is an ideal, $N^A \subseteq M \neq A$. This is a contradiction. Hence $N = A$. The proof is complete. □

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