Ioan Florin Bugariu
Uniform controllability for the beam equation with vanishing structural damping

Czechoslovak Mathematical Journal, Vol. 64 (2014), No. 4, 869–881

Persistent URL: http://dml.cz/dmlcz/144147

UNIFORM CONTROLLABILITY FOR THE BEAM EQUATION WITH VANISHING STRUCTURAL DAMPING

Ioan Florin Bugariu, Craiova

(Received October 29, 2012)

Abstract. This paper is devoted to studying the effects of a vanishing structural damping on the controllability properties of the one dimensional linear beam equation. The vanishing term depends on a small parameter $\varepsilon \in (0,1)$. We study the boundary controllability properties of this perturbed equation and the behavior of its boundary controls v_ε as ε goes to zero. It is shown that for any time T sufficiently large but independent of ε and for each initial data in a suitable space there exists a uniformly bounded family of controls $(v_\varepsilon)_\varepsilon$ in $L^2(0,T)$ acting on the extremity $x=\pi$. Any weak limit of this family is a control for the beam equation. This analysis is based on Fourier expansion and explicit construction and evaluation of biorthogonal sequences. This method allows us to measure the magnitude of the control needed for each eigenfrequency and to show their uniform boundedness when the structural damping tends to zero.

Keywords: beam equation; null-controllability; structural damping; moment problem; biorthogonals

MSC 2010: 93B05, 30E05, 58J45

1. Introduction

The starting point of this paper is a controllability problem for the one dimensional linear beam equation

This work was supported by the strategic grant POSDRU/CPP107/DMI1.5/S/78421, Project ID 78421 (2010), co-financed by the European Social Fund—Investing in People, within the Sectoral Operational Programme Human Resources Development 2007–2013 and by a grant of the Romanian National Authority for Scientific Research, CNCS—UEFISCDI, project number PN-II-ID-PCE-2011-3-0257.

$$
(1.1)\qquad \begin{cases} u_{tt}(t,x)+u_{xxxx}(t,x)=0, & (t,x)\in(0,T)\times(0,\pi),\\ u(t,0)=u_{xx}(t,0)=u_{xx}(t,\pi)=0, & t\in(0,T),\\ u(t,\pi)=v(t), & t\in(0,T),\\ u(0,x)=u^0(x), & x\in(0,\pi),\\ u_t(0,x)=u^1(x), & x\in(0,\pi). \end{cases}
$$

Given $T>0$ we say that equation (1.1) is *null-controllable in time T* if for every initial data $(u^0,u^1)\in\mathcal{H}$, there exists a control $v\in L^2(0,T)$ such that the corresponding solution of (1.1) verifies

$$
(1.2)\qquad u(T,x)=u_t(T,x)=0,\quad x\in(0,\pi),
$$

where

$$
(1.3)\qquad \mathcal{H}=H^{-1}(0,\pi)\times V'
$$

and $V=\{\varphi\in H^3(0,\pi);\ \varphi(0)=\varphi(\pi)=\varphi_{xx}(0)=\varphi_{xx}(\pi)=0\}$.

In the sequel, given any function $h\in L^2(0,\pi)$, we denote by $\widehat{h}_n$ the n-th Fourier coefficient of h,

$$
\widehat{h}_n=\int_0^\pi h(x)\sin(nx)\,\mathrm{d}x,\quad n\in\mathbb{N}^*.
$$

There exists a large literature concerning the controllability of both the linear and nonlinear beam equation. In our case (1.1), we use one of the oldest methods used to study the controllability problems. This consists in reducing it to a moment problem whose solution is given in terms of an explicit biorthogonal sequence to a family of exponential functions. The exponential functions are given by $(\mathrm{e}^{t\nu_n})_{n\in\mathbb{Z}^*}$, where $\nu_n=-\mathrm{i}\,\mathrm{sgn}(n)n^2$ are the eigenvalues of the operator $\begin{pmatrix}0 & -I\\ \partial^4_{xxxx} & 0\end{pmatrix}$.

We recall that a family $(\xi_m)_{m\in\mathbb{Z}^*}\subset L^2(-T/2,T/2)$ with the property

$$
(1.4)\qquad \int_{-T/2}^{T/2}\xi_m(t)\mathrm{e}^{\overline{\nu}_n t}\,\mathrm{d}t=\delta_{mn},\quad m,n\in\mathbb{Z}^*
$$

is called a biorthogonal sequence to $(\mathrm{e}^{\nu_n t})_{n\in\mathbb{Z}^*}$ in $L^2(-T/2,T/2)$. In (1.4), δ_{mn} stands for the *Kronecker* symbol.

Once a family of biorthogonal functions $(\xi_m)_{m\in\mathbb{Z}^*}$ verifying (1.4) is given, a control $v\in L^2(0,T)$ for (1.1) may be easily constructed. Indeed, for any initial data $(u^0,u^1)\in\mathcal{H}$ such that $\widehat{u}^0_n=\int_0^\pi u^0(x)\sin(nx)\,\mathrm{d}x$ and $\widehat{u}^1_n=\int_0^\pi u^1(x)\sin(nx)\,\mathrm{d}x$ the formula

$$
(1.5)\qquad v(t)=\sum_{m\in\mathbb{Z}^*}\frac{(-1)^{m+1}}{m^3}(-\widehat{u}^1_m+\nu_m\widehat{u}^0_m)\xi_m\Big(t-\frac{T}{2}\Big)\mathrm{e}^{-T/2\nu_m},\quad t\in(0,T)
$$

gives such a control leading the solution (u, u_t) of (1.1) to zero in time T, provided that the series in (1.5) converges in $L^2(0,T)$.

The existence of a biorthogonal sequence to $(\mathrm{e}^{t\nu_n})_{n\in\mathbb{Z}^*}$ is a consequence of Ingham's inequality

$$\sum_{n\in\mathbb{Z}^*} |a_n|^2 \leqslant C(T)\int_{-T/2}^{T/2}\Big|\sum_{n\in\mathbb{Z}^*} a_n \mathrm{e}^{t\nu_n}\Big|^2 \,\mathrm{d}t, \quad (a_n)_{n\in\mathbb{Z}^*}\subset l^2, \tag{1.6}$$

which holds for any $T > 0$ due to the fact that $\liminf_{n\to\infty} |\nu_{n+1} - \nu_n| = \infty$ (see [2], [10]).

One of the first articles using this method in the framework of partial differential equations is the one by Fattorini and Rusell [6], [7]. In their papers, a linear parabolic problem is shown to be null-controllable for a large class of initial data. Some of their ideas for the construction of a biorthogonal sequence will be used in our article, even if the properties of the corresponding exponential functions are quite different.

In this paper, we study the possibility of obtaining a control for (1.1) as the limit of controls of the perturbed equation

$$\begin{cases} u_{tt}(t,x) + u_{xxxx}(t,x) - \varepsilon u_{txx}(t,x) = 0, & (t,x)\in(0,T)\times(0,\pi),\\ u(t,0) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0, & t\in(0,T),\\ u(t,\pi) = v_\varepsilon(t), & t\in(0,T),\\ u(0,x) = u^0(x), & x\in(0,\pi),\\ u_t(0,x) = u^1(x), & x\in(0,\pi), \end{cases} \tag{1.7}$$

where ε is a small parameter which tends to zero. As for (1.1), $v_\varepsilon \in L^2(0,T)$ is a control for (1.7) in time T if the corresponding solution verifies (1.2). If, for any $(u^0, u^1) \in \mathcal{H}$, there exists a control $v_\varepsilon \in L^2(0,T)$ for (1.7) we say that (1.7) is *null-controllable in time T*. In (1.7), $-\varepsilon u_{txx}(t,x)$ represents the structural damping, supposed to vanish as ε goes to zero. The introduction of a vanishing term is a common tool in the study of Cauchy problems or in improving convergence of numerical schemes for hyperbolic conservation laws and shocks. For instance, in [8], [9], it is proved that, by adding an extra numerical viscosity term, the dispersive properties of the finite difference scheme for the nonlinear Schrödinger equation become uniform when the mesh-size tends to zero. On the other hand, a viscosity term is introduced in [3] to prove the existence of solutions of hyperbolic equations. In both the examples the viscosity is supposed to tend to zero in order to obtain the original system. Thus, a legitimate question is related to the behavior and the sensitivity of the controls during this process which is precisely the aim of this paper. The main result of this paper is given by the following theorem.

Theorem 1.1. *There exists $T > 0$ with the property that, for any $(u^0, u^1) \in \mathcal{H}$ and $\varepsilon \in (0,1)$, there exists a control $v_\varepsilon \in L^2(0,T)$ of (1.7) such that the family $(v_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T)$ and any weak limit v of it is a control in time T of (1.1).*

The method we use to prove Theorem 1.1 is the same as the one presented above to show the null-controllability of (1.1). More precisely, we use an explicit biorthogonal sequence to the family of exponential functions $(\mathrm{e}^{\mu_n t})_{n\in\mathbb{Z}^*}$ to construct a family of controls $(v_\varepsilon)_{\varepsilon>0}$ for (1.7) which is uniformly bounded in $L^2(0,T)$. Here $(\mu_n)_{n\in\mathbb{Z}^*}$ are the eigenvalues of the differential operator $\begin{pmatrix} 0 & -I \\ \partial^4_{xxxx} & \varepsilon\partial^2_{xx} \end{pmatrix}$.

We remark that, unlike ν_n, the eigenvalues μ_n are no longer purely imaginary complex numbers. Thus, the existence of a biorthogonal sequence to the family $(\mathrm{e}^{\mu_n t})_{n\in\mathbb{Z}^*}$ cannot be obtained as a consequence of an Ingham inequality similar to (1.6). Instead, we shall use the biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ to $(\mathrm{e}^{\mu_n t})_{n\in\mathbb{Z}^*}$ constructed and evaluated in [14]. The main difference between $(\xi_m)_{m\in\mathbb{Z}^*}$ and $(\theta_m)_{m\in\mathbb{Z}^*}$ is the fact that the norm of $(\theta_m)_{m\in\mathbb{Z}^*}$ is not bounded by a constant as in the case of $(\xi_m)_{m\in\mathbb{Z}^*}$. Indeed, as we shall see in the proof of Theorem 1.1, the sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ verifies

$$\|\theta_m\|_{L^2(-T/2,T/2)} \leqslant C(T)\mathrm{e}^{\omega|\Re(\mu_m)|}, \tag{1.8}$$

where $C(T)$ and ω are two positive constants independent of m and ε.

As in the case of equation (1.1), a control v_ε for (1.7) is given by

$$v_\varepsilon(t) = \sum_{m\in\mathbb{Z}^*} \frac{(-1)^{m+1}}{m^3 - \varepsilon m\mu_m}(-\widehat{u}^1_{|m|} + (\mu_m - \varepsilon m^2)\widehat{u}^0_{|m|})\theta_m\Big(t - \frac{T}{2}\Big)\mathrm{e}^{-T/2\mu_m}, \quad t \in (0,T). \tag{1.9}$$

The absolute convergence of the series from (1.9) in $L^2(0,T)$ will be a consequence of the estimate (1.8) and the exponential decay of the sequence $(\mathrm{e}^{-T/2\mu_n})_{n\in\mathbb{Z}^*}$, provided that T is sufficiently large.

The existence of a biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ with the property (1.8) implies that the following Ingham-type inequality holds, for any finite sequence $(\beta_m)_{m\in\mathbb{Z}^*}$:

$$\int_{-T}^{T}\Big|\sum_{m\in\mathbb{Z}^*}\beta_m \mathrm{e}^{-\mathrm{i}\,\mathrm{sgn}(m)m^2+\varepsilon m^2}\Big|^2 \,\mathrm{d}t \geqslant C(T)\sum_{m\in\mathbb{Z}^*}|\beta_m|^2\mathrm{e}^{-2\omega\varepsilon m^2}, \tag{1.10}$$

where T and ε are any positive numbers, ω is an absolute positive constant and C a positive constant depending only of T. From this point of view our article extends the results obtained in [4], where Ingham-type inequalities are obtained under the condition that the real parts of the exponents do not increase too much. As shown in [4], Theorem 3, a consequence of these results is the uniform controllability property of the perturbed beam equation

$$u_{tt}(t,x)+u_{xxxx}(t,x)+\varepsilon(-\partial_{xx})^{2\alpha}u_t(t,x) = v_\varepsilon(t)f(x), \quad (t,x)\in(0,T)\times(0,\pi), \tag{1.11}$$

where $\alpha \in (0,1/4]$. Our problem (1.7) deals with the case of a much stronger dissipation by considering $\alpha = 1/2$.

The paper is related to [5] where the null-controllability of a beam equation with hinged ends and structural damping depending on a positive parameter is studied. The main difference with respect to our paper consists in the fact that in [5] the control function is located on a subset of the domain. The proof of the uniform controllability as the viscosity goes to zero is done by analyzing separately the controls for the high and the low frequencies. For the high frequencies, the authors use a Lebeau-Zuazua estimate for finite combinations of functions $(\sin(n\pi x))_{n\geqslant 1}$ (see [12]) together with exponential decay, while for low frequencies a generalization of Ingham inequality (see [10]) for exponential functions with exponents in a bounded vertical strip is the main ingredient. Their technique cannot be applied to our case, since the control depends only on time and the high frequencies cannot be treated by using the results from [12]. Inequalities similar to (1.10) are also present in [15], where the main concern is to study the behavior of the controls as T goes to zero (and not ε).

The rest of this paper is organized as follows. Section 2 gives the equivalent characterization of the controllability property in terms of a moment problem. In Section 3 we construct a biorthogonal sequence to the family $(e^{\mu_n t})_{n\in\mathbb{Z}^*}$ and evaluate its L^2-norm. Finally, in Section 4 we prove the main result of the paper.

2. The moment problem

For the sake of completeness, we first present a result that concern the well-posedness of (1.7).

Theorem 2.1. *Given any $T>0$, $\varepsilon\geqslant 0$, $(u^0,u^1)\in\mathcal{H}$ and $h\in L^2(0,T)$, there exists a unique weak solution $(u,u_t)\in C([0,t],\mathcal{H})$ of the problem*

$$\begin{cases} u_{tt}(t,x)+u_{xxxx}(t,x)-\varepsilon u_{txx}(t,x)=0, & (t,x)\in(0,T)\times(0,\pi),\\ u(t,0)=u_{xx}(t,0)=u_{xx}(t,\pi)=0, & t\in(0,T),\\ u(t,\pi)=h(t), & t\in(0,T),\\ u(0,x)=u^0(x), & x\in(0,\pi),\\ u_t(0,x)=u^1(x), & x\in(0,\pi). \end{cases} \tag{2.1}$$

P r o o f. The proof of this theorem is based on transposition method and is similar to the proof of Theorem 4.2 from [13]. Therefore we omit it and refer the interested reader to [13]. $\square$

The following property concerns the spectrum of the differential operator corresponding to the adjoint system of (2.1) and will be useful for the characterization of its controllability properties.

Proposition 2.1. *The unbounded operator in* $H_0^1(0,\pi)\times H^{-1}(0,\pi)$, $(D(A),A)$ *defined by*

$$D(A)=\{u\in H^3(0,\pi);\ u(0)=u(\pi)=u_{xx}(0)=x_{xx}(\pi)=0\},\quad A=\begin{pmatrix}0 & -I\\ \partial_x^4 & \varepsilon\partial_x^2\end{pmatrix}$$

has a sequence of complex eigenvalues

$$\mu_n=\frac{\varepsilon}{2}n^2-\mathrm{i}\frac{\sqrt{4-\varepsilon^2}}{2}\operatorname{sgn}(n)n^2,\quad n\in\mathbb{Z}^*, \tag{2.2}$$

and the corresponding eigenvectors

$$\Phi^n=\frac{1}{\operatorname{sgn}(n)n}\begin{pmatrix}1\\ -\mu_n\end{pmatrix}\sin(nx),\quad n\in\mathbb{Z}^* \tag{2.3}$$

form a Schauder basis of $H_0^1(0,\pi)\times H^{-1}(0,\pi)$.

P r o o f. It is easy to see that μ is an eigenvalue of A with the corresponding eigenfunction $\binom{u}{v}$ if and only if $v=-\mu u$ and the pair (μ,u) verifies

$$\begin{cases}u_{xxxx}+\varepsilon\mu u_{xx}+\mu^2u=0\\ u(0)=u(\pi)=u_{xx}(0)=u_{xx}(\pi)=0.\end{cases} \tag{2.4}$$

Since $(\sin(nx))_{n\geqslant 1}$ is a complete set of solutions in $H_0^1(0,\pi)$ for (2.4), if follows that $(\mu_n)_{n\in\mathbb{Z}^*}$ are all the eigenvalues of the operator A and the corresponding eigenvectors orthonormalized in $H_0^1(0,\pi)\times H^{-1}(0,\pi)$ are given by (2.3). The fact that $(\Phi^n)_{n\in\mathbb{Z}^*}$ forms a basis of $H_0^1(0,\pi)\times H^{-1}(0,\pi)$ is a consequence of the fact that $\left(\binom{\sqrt{2}n^{-1}\sin(nx)}{0}\right)_{n\geqslant 1}$ is an orthonormal basis of $H_0^1(0,\pi)\times\{0\}$ whereas $\left(\binom{0}{\sqrt{2}n\sin(nx)}\right)_{n\geqslant 1}$ is an orthonormal basis of $\{0\}\times H^{-1}(0,\pi)$. $\square$

We can give now the characterization of the controllability property of (1.7) in terms of a moment problem. We recall that, based on Fourier expansion of the solution, the moment problems have been widely used in linear control theory. We refer to [1], [11], [17] for a quite complete discussion on the subject.

Theorem 2.2. *Let $T > 0$, $\varepsilon \geqslant 0$ and let initial data $(u^0, u^1) \in \mathcal{H}$ be given. Then there exists a control $v_\varepsilon \in L^2(0,T)$ such that the solution (u, u_t) of (1.7) verifies (1.2) if and only if $v_\varepsilon \in L^2(0,T)$ is a solution of*

$$(2.5)\qquad (-1)^{n+1}(n^3 - \varepsilon n \mu_n) \int_{-T/2}^{T/2} v_\varepsilon\Big(t + \frac{T}{2}\Big) \mathrm{e}^{t\mu_n}\,\mathrm{d}t \\ = \mathrm{e}^{-T/2\mu_n}(-\widehat{u}^1_{|n|} + (\mu_n - \varepsilon n^2)\widehat{u}^0_{|n|}), \quad n \in \mathbb{Z}^*,$$

where $(\mu_n)_{n\in\mathbb{Z}^*}$ are given by (2.2).

Proof. Let us introduce the "backward" equation

$$(2.6)\qquad \begin{cases} \varphi_{tt}(t,x) + \varphi_{xxxx}(t,x) + \varepsilon\varphi_{txx}(t,x) = 0, & (t,x) \in (0,T)\times(0,\pi),\\ \varphi(t,0) = \varphi(t,\pi) = \varphi_{xx}(t,0) = \varphi_{xx}(t,\pi) = 0, & t \in (0,T),\\ \varphi(T,x) = \varphi^0(x), & x \in (0,\pi),\\ \varphi_t(T,x) = \varphi^1(x), & x \in (0,\pi). \end{cases}$$

We multiply (1.7) by $\overline{\varphi}$ and integrate by parts over $(0,T)\times(0,\pi)$. It follows that $v_\varepsilon \in L^2(0,T)$ is a control for (1.7) if and only if it verifies

$$(2.7)\qquad \int_0^T v_\varepsilon(t)(\overline{\varphi}_{xxx}(t,\pi) + \varepsilon\overline{\varphi}_{tx}(t,\pi))\,\mathrm{d}t = -\int_0^\pi u^1(x)\overline{\varphi}(0,x)\,\mathrm{d}x \\ + \int_0^\pi u^0(x)(\overline{\varphi}_t(0,x) + \varepsilon\overline{\varphi}_{xx}(0,x))\,\mathrm{d}x - \varepsilon u(0,\pi)\overline{\varphi}_x(0,\pi),$$

for any solution φ of (2.6).

Since $(\sin(nx))_{n\geqslant 1}$ is a basis for $L^2(0,\pi)$ we have to check (2.7) only for the initial data of the form $(\varphi^0,\varphi^1) = (\sin(nx), 0)$ and $(\varphi^0,\varphi^1) = (0, \sin(nx))$. In the former case the solution of (2.6) has the form

$$(2.8)\qquad \varphi(t,x) = \Big(\frac{\overline{\mu}_n}{\overline{\mu}_n - \mu_n}\mathrm{e}^{(t-T)\mu_n} + \frac{\mu_n}{\mu_n - \overline{\mu}_n}\mathrm{e}^{(t-T)\overline{\mu}_n}\Big)\sin(nx), \quad n \in \mathbb{Z}^*$$

and in the latter the solution of (2.6) has the form

$$(2.9)\qquad \varphi(t,x) = \Big(\frac{1}{\mu_n - \overline{\mu}_n}\mathrm{e}^{(t-T)\mu_n} + \frac{1}{\overline{\mu}_n - \mu_n}\mathrm{e}^{(t-T)\overline{\mu}_n}\Big)\sin(nx), \quad n \in \mathbb{Z}^*.$$

By taking in (2.7) φ of the form (2.8) and (2.9) we obtain that $v_\varepsilon \in L^2(0,T)$ is a control if and only if it verifies (2.5). □

It is easy to see from (2.5) that, if $(\theta_m)_{m\in\mathbb{Z}^*}$ is a biorthogonal sequence to the family $(\mathrm{e}^{t\mu_n})_{n\in\mathbb{Z}^*}$ in $L^2(-T/2,T/2)$, then a control v_ε of (1.7) is given by (1.9) provided that the series from the right hand side converges in $L^2(0,T)$. Now, the main task is to show that there exists a biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ and to evaluate its norm in order to prove the convergence of the series.

3. A biorthogonal sequence

The aim of this section is to construct and evaluate a biorthogonal sequence to the family of exponential functions $(\mathrm{e}^{\mu_n t})_{n\in\mathbb{Z}^*}$ in $L^2(-T/2,T/2)$.

Theorem 3.1. *There exists $\widetilde{T}>0$ such that, for any $\varepsilon\in(0,1]$, we find a biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ to the family $(\mathrm{e}^{\mu_n t})_{n\in\mathbb{Z}^*}$ in $L^2(-\widetilde{T}/2,\widetilde{T}/2)$ with the property*

$$\|\theta_m\|_{L^2(-\widetilde{T}/2,\widetilde{T}/2)}\leqslant C\exp(\alpha|\Re(\mu_m)|),\quad m\in\mathbb{Z}^*, \tag{3.1}$$

where C and α are two positive constants independent of m and ε.

Proof. For any $p\in\mathbb{Z}^*$, we define the function

$$R_p(z)=\prod_{\substack{k\in\mathbb{Z}^*\\|k|\neq p}}\Big(1-\frac{z}{\mathrm{i}\lambda_k}\Big)\frac{\lambda_k}{\lambda_k-\lambda_p}\,\frac{\sin\big(\pi\delta(4\Im(\lambda_p))^{-1}(\mathrm{i}\overline{\lambda}_p-z)\big)}{\sin(\pi\delta/2)}, \tag{3.2}$$

where $\delta>0$ is an arbitrary small constant and $(\lambda_k)_{k\in\mathbb{Z}^*}$ are given by

$$\lambda_k=\begin{cases}\overline{\mu}_{\sqrt{k}}, & k=q^2,\ q\in\mathbb{N}^*,\\ \mathrm{i}\dfrac{\sqrt{(1+\varepsilon^2)(4-\varepsilon^2)}}{2}k, & k\neq q^2,\ k>0,\ q\in\mathbb{N}^*,\\ \overline{\lambda}_{-k}, & k<0.\end{cases} \tag{3.3}$$

Like in [14], Lemma 3.1, we can prove that for any $m\in\mathbb{Z}^*$, R_{m^2} is an entire function of exponential type independent of m and ε with the property

$$R_{m^2}(\mathrm{i}\,\overline{\mu}_n)=\delta_{mn},\quad n\in\mathbb{Z}^*. \tag{3.4}$$

Moreover, by using the estimate from [14], Lemma 3.2, we deduce that there exist two constants C_1 and α_1 independent of m and ε such that

$$|R_{m^2}(x)|\leqslant C_1\exp(\alpha_1\varepsilon(\sqrt{|x|}+|\Re(\mu_m)|)),\quad x\in\mathbb{R}. \tag{3.5}$$

Finally, from [14], Lemma 3.3, it follows that there exists an entire function $M_{m^2,\varepsilon}$ and two positive constants C_2 and α_2, independent of m and ε, such that

$$M_{m^2,\varepsilon}(\mathrm{i}\,\overline{\mu}_m) = 1 \tag{3.6}$$

and

$$|M_{m^2,\varepsilon}(x)| \leqslant C_2 \exp(-\alpha_1\varepsilon\sqrt{|x|} + \alpha_2|\Re(\mu_m)|). \tag{3.7}$$

For each $m \in \mathbb{Z}^*$ we define the function

$$\Psi_m(z) = R_{m^2}(z) M_{m^2,\varepsilon}(z) \frac{\sin(\delta(z - \mathrm{i}\,\overline{\mu}_m))}{\delta(z - \mathrm{i}\,\overline{\mu}_m)} \tag{3.8}$$

and deduce that there exists $\widetilde{T} > 0$ independent of m and ε such that Ψ_m is an entire function of exponential type $\widetilde{T}/2$. Moreover, from (3.5), (3.7) and Plancherel-Polya's Theorem [16], Theorem 16, Chapter 2, Section 3, we have that

$$\begin{aligned}
\|\Psi_m\|^2_{L^2(\mathbb{R})} &\leqslant \frac{C_1C_2}{\sin(\pi\delta/2)} \mathrm{e}^{2(\alpha_2+\alpha_1\varepsilon)|\Re(\mu_m)|} \\
&\quad\times \int_{\mathbb{R}} \left| \frac{\sin(\pi\delta(4\Im(\lambda_m))^{-1}(\mathrm{i}\,\overline{\mu}_m - x))\sin(\delta(x - \mathrm{i}\,\overline{\mu}_m))}{\delta(x - \mathrm{i}\,\overline{\mu}_m)} \right|^2 \mathrm{d}x \\
&\leqslant \frac{C_1C_2\pi^2}{\sin(\pi\delta/2)} \mathrm{e}^{2(\alpha_2+\alpha_1\varepsilon+2\delta)|\Re(\mu_m)|} \\
&\quad\times \int_{\mathbb{R}} \left| \frac{\sin(\pi\delta(4\Im(\lambda_m))^{-1}x)\sin(\delta x)}{\delta x} \right|^2 \mathrm{d}x \leqslant C\mathrm{e}^{2\alpha|\Re(\mu_m)|},
\end{aligned}$$

where α is a number greater than $\alpha_2 + \alpha_1\varepsilon + 2\delta$.

Now, we have all the ingredients needed to construct the biorthogonal sequence. For each $m \in \mathbb{Z}^*$, we define the function

$$\theta_m(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \Psi_m(x)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}x. \tag{3.9}$$

By taking into account the properties of Ψ_m and by applying the Paley-Wiener Theorem we deduce that $\theta_m \in L^2(-\widetilde{T}/2, \widetilde{T}/2)$. Moreover, from the inverse Fourier transform property we obtain that $(\theta_m)_{m\in\mathbb{Z}^*}$ is a biorthogonal sequence to $(\mathrm{e}^{\mu_n t})_{n\in\mathbb{Z}^*}$. Finally, from Plancherel's Theorem we deduce that (3.1) holds and the proof of the theorem is complete. □

To prove the uniform controllability of the beam equation with vanishing viscosity, we have to construct a new biorthogonal sequence to the family $(\mathrm{e}^{\mu_n t})_{n\in\mathbb{Z}^*}$ with better norm properties than those of the biorthogonal $(\theta_m)_{m\in\mathbb{Z}^*}$ from Theorem 3.1. The following theorem gives us such a biorthogonal sequence.

Theorem 3.2. *Let T_0 be any number larger that $\widetilde{T}$. There exists a biorthogonal sequence $(\zeta_m)_{m\in\mathbb{Z}^*}$ to the family $(\mathrm{e}^{t\mu_m})_{m\in\mathbb{Z}^*}$ in $L^2(-T_0/2,T_0/2)$ such that for any finite sequence $(\alpha_m)_{m\in\mathbb{Z}^*}$, we have*

$$\int_{-T_0/2}^{T_0/2}\Big|\sum_{m\in\mathbb{Z}^*}\alpha_m\zeta_m(t)\Big|^2\,\mathrm{d}t\leqslant C(T_0)\sum_{m\in\mathbb{Z}^*}|\alpha_m|^2\exp(2\alpha|\Re(\mu_m)|), \tag{3.10}$$

where α is the same as in Theorem 3.1 *and $C(T_0)$ is a constant depending only on T_0.*

Proof. Let $a=(T_0-\widetilde{T})/2>0$ and $k_a=\sqrt{2\pi}a^{-2}(\chi_a*\chi_a)$, where χ_a represents the characteristic function $\chi_{[-a/2,a/2]}$. Evidently $\operatorname{supp}(k_a)\subset[-a,a]$. Also, we have

$$\widehat{k}_a(\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}k_a(t)\mathrm{e}^{-\mathrm{i}t\xi}\,\mathrm{d}t=\frac{2\pi}{a^2}\widehat{\chi}_a(\xi)\widehat{\chi}_a(\xi)=\frac{4}{a^2}\frac{\sin^2(\xi a/2)}{\xi^2}.$$

We define $\varrho_m(x)=\mathrm{e}^{\mathrm{i}x\Im(\mu_m)}k_a(x)$, so $\operatorname{supp}(\varrho_m)\subset[-a,a]$.

Let $(\theta_m)_{m\in\mathbb{Z}^*}$ be the biorthogonal sequence from Theorem 3.1. We define

$$\zeta_m=\frac{1}{\sqrt{2\pi}\widehat{\varrho}_m(\mathrm{i}\,\overline{\mu}_m)}(\theta_m*\varrho_m),\qquad m\in\mathbb{Z}^*. \tag{3.11}$$

Let us show that the sequence $(\zeta_m)_{m\in\mathbb{Z}^*}$ fulfils the requirements from our theorem. First of all, let us remark that $\zeta_m\in L^2(-T_0/2,T_0/2)$ and, for any $m,n\in\mathbb{Z}^*$, the following relation takes place:

$$\int_{-\widetilde{T}/2-a}^{\widetilde{T}/2+a}\zeta_m(t)\mathrm{e}^{\overline{\mu}_n t}\,\mathrm{d}t=\sqrt{2\pi}\widehat{\zeta}_m(\mathrm{i}\,\overline{\mu}_n)=\frac{\widehat{\theta}_m(\mathrm{i}\,\overline{\mu}_n)\widehat{\varrho}_m(\mathrm{i}\,\overline{\mu}_n)}{\widehat{\varrho}_m(\mathrm{i}\,\overline{\mu}_m)}=\delta_{mn}.$$

It follows that $(\zeta_m)_{m\in\mathbb{Z}^*}$ is a biorthogonal sequence to the family $(\mathrm{e}^{t\mu_n})_{n\in\mathbb{Z}^*}$ in $L^2(-T_0/2,T_0/2)$.

In order to prove (3.10) remark that

$$\widehat{\varrho}_m(\mathrm{i}\overline{\mu}_m)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\varrho_m(t)\mathrm{e}^{t\overline{\mu}_m}\,\mathrm{d}t=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}t\Im(\mu_m)}k_a(t)\mathrm{e}^{t\overline{\mu}_m}\,\mathrm{d}t=\widehat{k}_a(\mathrm{i}\Re(\mu_m))\geqslant 1.$$

Now, we evaluate the left hand side of (3.10) as follows:

$$
\begin{aligned}
\int_{-\widetilde{T}/2-a}^{\widetilde{T}/2+a}\Big|\sum_{m\in\mathbb{Z}^*}\alpha_m\zeta_m(t)\Big|^2\,\mathrm{d}t &= \int_{\mathbb{R}}\Big|\sum_{m\in\mathbb{Z}^*}\alpha_m\widehat{\zeta}_m(x)\Big|^2\,\mathrm{d}x\\
&= \int_{\mathbb{R}}\Big|\sum_{m\in\mathbb{Z}^*}\alpha_m\frac{\widehat{\theta}_m(x)\widehat{\varrho}_m(x)}{\sqrt{2\pi}\widehat{\varrho}_m(\mathrm{i}\,\overline{\mu}_m)}\Big|^2\,\mathrm{d}x\\
&\leqslant \frac{1}{2\pi}\int_{\mathbb{R}}\Big|\sum_{m\in\mathbb{Z}^*}\frac{|\alpha_m|}{|\widehat{\varrho}_m(\mathrm{i}\overline{\lambda}_m)|}\|\widehat{\theta}_m\|_{L^\infty(\mathbb{R})}|\widehat{\varrho}_m(x)|\Big|^2\,\mathrm{d}x\\
&\leqslant \frac{\widetilde{T}}{2\pi}\int_{\mathbb{R}}\Big|\sum_{m\in\mathbb{Z}^*}|\alpha_m|\|\theta_m\|_{L^2(-\widetilde{T}/2,\widetilde{T}/2)}|\widehat{\varrho}_m(x)|\Big|^2\,\mathrm{d}x\\
&= \frac{\widetilde{T}}{2\pi}\int_{\mathbb{R}}\Big|\sum_{m\in\mathbb{Z}^*}|\alpha_m|\|\theta_m\|_{L^2(-\widetilde{T}/2,\widetilde{T}/2)}|\widehat{k}_a(x-\Im(\mu_m))|\Big|^2\,\mathrm{d}x\\
&\leqslant \frac{\widetilde{T}}{2\pi}\int_{-a}^{a}k_a^2(t)\Big|\sum_{m\in\mathbb{Z}^*}|\alpha_m|\|\theta_m\|_{L^2(-\widetilde{T}/2,\widetilde{T}/2)}\mathrm{e}^{\mathrm{i}\Im(\mu_m)t}\Big|^2\,\mathrm{d}t\\
&\leqslant \frac{\widetilde{T}}{a^2}\int_{-a}^{a}\Big|\sum_{m\in\mathbb{Z}^*}|\alpha_m|\|\theta_m\|_{L^2(-\widetilde{T}/2,\widetilde{T}/2)}\mathrm{e}^{\mathrm{i}\Im(\mu_m)t}\Big|^2\,\mathrm{d}t.
\end{aligned}
$$

Since $|\Im(\mu_{m+1})-\Im(\mu_m)|\geqslant\sqrt{4-\varepsilon^2}m$, Ingham's inequality (see [2], [10]) yields that for any $a>0$ there exists a positive constant C such that

$$
\int_{-a}^{a}\Big|\sum_{m\in\mathbb{Z}^*}|\alpha_m|\|\theta_m\|_{L^2(-\widetilde{T}/2,\widetilde{T}/2)}\mathrm{e}^{\mathrm{i}\Im(\mu_m)t}\Big|^2\,\mathrm{d}t\leqslant C\sum_{m\in\mathbb{Z}^*}|\alpha_m|^2\|\theta_m\|^2_{L^2(-\widetilde{T}/2,\widetilde{T}/2)}.
$$

Now, taking into account Theorem 3.1, we deduce immediately (3.10) and the proof is complete. □

4. Controllability results

Now we are able to prove the main result of this paper.

P r o o f of Theorem 1.1. Let $T>\max\{2\alpha,\widetilde{T}\}$ and $(\zeta_m)_{m\in\mathbb{Z}^*}$ be as in Theorem 3.2 with $T_0=T$. We construct a control $v_\varepsilon\in L^2(0,T)$ of (1.7) corresponding to the initial data $(u^0,u^1)\in\mathcal{H}$, as follows:

$$
v_\varepsilon(t)=\sum_{m\in\mathbb{Z}^*}\frac{(-1)^{m+1}}{m^3-\varepsilon m\mu_m}(-\widehat{u}^1_m+(\mu_m-\varepsilon m^2)\widehat{u}^0_m)\mathrm{e}^{-T/2\mu_m}\widetilde{\zeta}_m\Big(t-\frac{T}{2}\Big),\quad t\in(0,T), \tag{4.1}
$$

where $\widetilde{\zeta}_m$ is the extension by zero of ζ_m to the interval $(-T/2, T/2)$. From the properties of the biorthogonal sequence $(\zeta_m)_{m\in\mathbb{Z}^*}$ it is easy to see that v_ε verifies (1.9). Now, to conclude that v_ε is a control for (1.7), we only have to prove that the series from (4.1) converges in $L^2(0,T)$. This follows immediately from Theorem 3.2 and the fact that $(u^0, u^1) \in \mathcal{H}$. Indeed, we have

$$\begin{aligned}
&\int_0^T |v_\varepsilon(t)|^2 \,\mathrm{d}t \\
&= \int_0^T \left| \sum_{m\in\mathbb{Z}^*} \frac{(-1)^{m+1}}{m^3 - \varepsilon m \mu_m} (-\widehat{u}_m^1 + (\mu_m - \varepsilon m^2)\widehat{u}_m^0) \mathrm{e}^{-T/2\mu_m} \widetilde{\zeta}_m\Big(t - \frac{T}{2}\Big) \right|^2 \mathrm{d}t \\
&= \int_{-\widetilde{T}/2-a}^{\widetilde{T}/2+a} \left| \sum_{m\in\mathbb{Z}^*} \frac{(-1)^{m+1}}{m^3 - \varepsilon m \mu_m} (-\widehat{u}_m^1 + (\mu_m - \varepsilon m^2)\widehat{u}_m^0) \mathrm{e}^{-T/2\mu_m} \zeta_m(t) \right|^2 \mathrm{d}t \\
&\leqslant C(\widetilde{T}, a) \|(u^0, u^1)\|_{\mathcal{H}}.
\end{aligned}$$

The last inequality results from (3.10) and the constant C does not depend on ε and m. Thus, the sequence of controls $(v_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(0,T)$. Let v be a weak limit of this sequence. In order to prove that v is a control for (1.1) we only have to pass to the limit as ε goes to zero in (1.9). □

Remark 4.1. The space of uniformly controllable initial data $\mathcal{H}$ from Theorem 1.1 coincides with that obtained in [13], Theorem 4.4, page 301, for the limit case $\varepsilon = 0$ when two controls are considered. Also, we note that, since $H_0^3(0,\pi) \subset V$, the dual space V' is included in $H^{-3}(0,\pi)$. Moreover, since $(\sin(nx))_{n\geqslant 1}$ is an orthogonal basis of V, it is also a basis of V' and the following characterization holds:

$$V' = \Big\{ u = \sum_{n\geqslant 1} a_n \sin(nx);\ \sum_{n\geqslant 1} \frac{|a_n|^2}{n^6} < \infty \Big\}.$$

References

[1] *S. A. Avdonin, S. A. Ivanov*: Families of Exponentials. The method of moments in controllability problems for distributed parameter systems. Cambridge University Press, Cambridge, 1995.

[2] *J. M. Ball, M. Slemrod*: Nonharmonic Fourier series and the stabilization of distributed semilinear control systems. Commun. Pure Appl. Math. *32* (1979), 555–587.

[3] *R. J. DiPerna*: Convergence of approximate solutions to conservation laws. Arch. Ration. Mech. Anal. *82* (1983), 27–70.

[4] *J. Edward*: Ingham-type inequalities for complex frequencies and applications to control theory. J. Math. Anal. Appl. *324* (2006), 941–954.

[5] *J. Edward, L. Tebou*: Internal null-controllability for a structurally damped beam equation. Asymptotic Anal. *47* (2006), 55–83.

[6] *H. O. Fattorini, D. L. Russell*: Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations. Q. Appl. Math. *32* (1974), 45–69.
[7] *H. O. Fattorini, D. L. Russell*: Exact controllability theorems for linear parabolic equations in one space dimension. Arch. Ration. Mech. Anal. *43* (1971), 272–292.
[8] *L. I. Ignat, E. Zuazua*: Dispersive properties of numerical schemes for nonlinear Schrödinger equations. Foundations of Computational Mathematics (L. M. Pardo et al., eds.). Santander, Spain, 2005, London Math. Soc. Lecture Note Ser. 331, Cambridge University Press, Cambridge, 2006, pp. 181–207.
[9] *L. I. Ignat, E. Zuazua*: Dispersive properties of a viscous numerical scheme for the Schrödinger equation. C. R., Math., Acad. Sci. Paris *340* (2005), 529–534.
[10] *A. E. Ingham*: Some trigonometrical inequalities with applications to the theory of series. Math. Z. *41* (1936), 367–379.
[11] *V. Komornik, P. Loreti*: Fourier Series in Control Theory. Springer Monographs in Mathematics, Springer, New York, 2005.
[12] *G. Lebeau, E. Zuazua*: Null-controllability of a system of linear thermoelasticity. Arch. Ration. Mech. Anal. *141* (1998), 297–329.
[13] *J.-L. Lions*: Exact controllability, perturbations and stabilization of distributed systems. Volume 1: Exact controllability. Research in Applied Mathematics 8, Masson, Paris, 1988. (In French.)
[14] *S. Micu, I. Rovenţa*: Uniform controllability of the linear one dimensional Schrödinger equation with vanishing viscosity. ESAIM, Control Optim. Calc. Var. *18* (2012), 277–293.
[15] *T. I. Seidman, S. A. Avdonin, S. A. Ivanov*: The 'window problem' for series of complex exponentials. J. Fourier Anal. Appl. *6* (2000), 233–254.
[16] *R. M. Young*: An Introduction to Nonharmonic Fourier Series. Pure and Applied Mathematics 93, Academic Press, New York, 1980.
[17] *J. Zabczyk*: Mathematical Control Theory: An Introduction. Systems & Control: Foundations & Applications, Birkhäuser, Boston, 1992.

Author's address: Ioan Florin Bugariu, University of Craiova, Faculty of Exact Sciences, Department of Computer Science, Strada Alexandru Ioan Cuza 13, 200 585 Craiova, Romania, e-mail: florin_bugariu_86@yahoo.com.