

Anton Savostianov; Sergey Zelik

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RECENT PROGRESS IN ATTRACTORS  
FOR QUINTIC WAVE EQUATIONS

ANTON SAVOSTIANOV, SERGEY ZELIK, Guildford

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*Abstract.* We report on new results concerning the global well-posedness, dissipativity and attractors for the quintic wave equations in bounded domains of  $\mathbb{R}^3$  with damping terms of the form  $(-\Delta_x)^\theta \partial_t u$ , where  $\theta = 0$  or  $\theta = 1/2$ . The main ingredient of the work is the hidden extra regularity of solutions that does not follow from energy estimates. Due to the extra regularity of solutions existence of a smooth attractor then follows from the smoothing property when  $\theta = 1/2$ . For  $\theta = 0$  existence of smooth attractors is more complicated and follows from Strichartz type estimates.

*Keywords:* damped wave equation; fractional damping; critical nonlinearity; global attractor; smoothness

*MSC 2010:* 35B40, 35B45

## 1. INTRODUCTION

We consider the nonlinear damped wave equation

$$(1.1) \quad \begin{cases} \partial_t^2 u + \gamma(-\Delta_x)^\theta \partial_t u - \Delta_x u + f(u) = g(x), \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u'_0 \end{cases}$$

in a bounded smooth domain  $\Omega \subset \mathbb{R}^3$  endowed with Dirichlet boundary conditions. Here  $\Delta_x$  is the Laplacian with respect to the variable  $x = (x^1, x^2, x^3)$ ,  $\theta \in [0, 1]$  and  $\gamma > 0$  are given exponents,  $g \in L^2(\Omega)$  is a given external force and  $f$  is a nonlinearity which satisfies some natural dissipativity and growth assumptions, say,

$$(1.2) \quad -C + \kappa|u|^q \leq f'(u) \leq C(1 + |u|^q)$$

for some positive  $C$  and  $q$ .

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Wave equations of the form (1.1) are of big interest from both the theoretical and applied points of view and have been studied by many authors, see [1], [5]–[11], [18], [23], [24] and references therein. It is remarkable that even on the linear level ( $f = g = 0$ ), these equations demonstrate rather nontrivial analytic properties in a strong dependence on the value of the exponent  $\theta$ . For the convenience of the reader, we briefly summarize them in the following table:

$\theta$	Semigroup	Smoothing	Maximal regularity
0	$C_0$	asymptotic	no
$(0, \frac{1}{2})$	$C^\infty$	instantaneous	no
$[\frac{1}{2}, 1)$	analytic	instantaneous	yes
1	analytic	instantaneous for $\partial_t u$ , asymptotic for $u$	yes

See [7]–[9], [24] for more details. As we can see from this table, there are three important borderline cases: the first ( $\theta = 0$ ) corresponds to the classical (weakly) damped wave equation, the second ( $\theta = 1$ ) gives the so-called *strongly damped* wave equations and the third ( $\theta = 1/2$ ) is often referred to as the wave equation with *structural* damping although the intermediate choices of  $\theta$  are also interesting, see e.g. [7], [18], [24] and references therein.

The situation becomes much more delicate in the presence of the nonlinearity  $f$  since the analytic properties of solutions start to depend also on the growth rate of  $f(u)$  as  $u \rightarrow \infty$  (on the exponent  $q$  in (1.2)). Recall that the solutions  $u(t)$  of problem (1.1) satisfy (at least formally) the energy identity

$$(1.3) \quad \frac{d}{dt} E(u(t), \partial_t u(t)) = -\gamma \|(-\Delta_x)^{\theta/2} \partial_t u(t)\|_{L^2}^2,$$

where  $E(u, v) = \frac{1}{2} \|\partial_t v\|_{L^2}^2 + \frac{1}{2} \|\nabla_x u\|_{L^2}^2 + (F(u), 1) - (g, u)$ . Here and below  $(u, v)$  stands for the usual inner product in  $L^2(\Omega)$  and  $F(u) := \int_0^u f(v) dv$  is a potential of the nonlinearity  $f$ . Thus, a weak *energy* solution of problem (1.1) on the interval  $t \in [0, T]$  is naturally defined as a function  $u(t)$  which has the regularity

$$(1.4) \quad u \in L^\infty(0, T; H_0^1(\Omega) \cap L^{q+2}(\Omega)), \quad \partial_t u \in L^\infty(0, T; L^2(\Omega)), \\ (-\Delta_x)^{\theta/2} \partial_t u \in L^2(0, T; L^2(\Omega))$$

(here  $H^s(\Omega)$  stands for the usual Sobolev space of distributions whose derivatives up to order  $s$  belong to  $L^2$ , and  $H_0^s(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  in  $H^s(\Omega)$ ) and satisfies equation (1.1) in the sense of distributions. The corresponding energy phase space is

$$(1.5) \quad \mathcal{E} := [H_0^1(\Omega) \cap L^{q+2}(\Omega)] \times L^2(\Omega), \quad \xi_u := (u, \partial_t u) \in \mathcal{E}.$$

Recall also that, due to the Sobolev embedding  $H_0^1 \subset L^6$ , we can take  $\mathcal{E} = H_0^1(\Omega) \times L^2(\Omega)$  if the growth exponent is  $q \leq 4$  (in particular, for the case of quintic nonlinearities), but the term  $L^{q+2}$  in the definition (1.5) of the energy phase space looks unavoidable if the growth exponent is  $q > 4$ . The next standard result shows that weak energy solutions exist and are dissipative for all admissible values of  $\theta$  and  $q$ .

**Theorem 1.1.** *Let  $\theta \in [0, 1]$ ,  $q \geq 0$ ,  $g \in L^2(\Omega)$ , let the nonlinearity  $f$  satisfy (1.2) and the initial data  $\xi_u(0) := (u_0, u'_0) \in \mathcal{E}$ . Then there exists at least one weak energy solution  $u(t)$  defined for all  $t \geq 0$  which satisfies the estimate:*

$$(1.6) \quad \|\xi_u(t)\|_{\mathcal{E}} + \int_t^{t+1} \|(-\Delta_x)^{\theta/2} \partial_t u(s)\|_{L^2}^2 ds \leq Q(\|\xi_u(0)\|_{\mathcal{E}}) e^{-\alpha t} + Q(\|g\|_{L^2}),$$

where the positive constant  $\alpha$  and the monotone function  $Q$  are independent of  $u$  and  $t$ .

The proof of this result is straightforward. Indeed, the dissipative estimate (1.6) formally follows by multiplication of equation (1.1) by  $\partial_t u + \beta u$  for a properly chosen positive constant  $\beta$  followed by Gronwall's inequality and the existence of a solution can be verified, say, by Galerkin approximations, see [1], [10].

In contrast to this, the uniqueness and further regularity of energy solutions is more difficult and requires essential restrictions on the growth exponent  $q$ . To the best of our knowledge this problem has been solved before only for the values of  $\theta$  and  $q$  collected in the following table:

$\theta$	0	$(0, \frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2}, \frac{3}{4})$	$[\frac{3}{4}, 1]$
$q$	$[0, 2]$	?	$[0, 4)$	$0 \leq q(\theta) < 8\theta/(3 - 4\theta)$	$(0, \infty)$

See [1], [7], [18], [24] for more details. E.g., for the case of structural damping  $\theta = 1/2$ , the open problem was to treat the “critical” case of quintic nonlinearities  $q = 4$ , and for the classical case  $\theta = 0$  in bounded domains with Dirichlet boundary conditions the theory has been developed only for cubic and sub-cubic nonlinearities ( $q \leq 2$ ) although  $q = 4$  has been conjectured as the critical exponent here. For domains without boundary:  $\Omega = \mathbb{R}^3$  or  $\Omega = \mathbb{T}^3$  (periodic boundary conditions), a reasonable theory exists for  $q < 4$  in [13], [19]. In the quintic case ( $\theta = 0$ ,  $q = 4$ ) and  $\Omega = \mathbb{R}^3$ , the global existence of regular solutions has not been known for a long time (see [16], [20], [26]), but the attractor theory has been not developed for this case.

The aim of these notes is to present our recent results concerning global well-posedness, dissipativity and existence of smooth global attractors for quintic ( $q = 4$ ) wave equations in bounded domains, see [17] and [25] for a detailed exposition. We

restrict ourselves to discussing only two borderline cases  $\theta = 1/2$  and  $\theta = 0$  although our conjecture is that analogous results hold for all  $\theta \in (0, 1/2)$ . Note that in both cases, the regularity of solutions provided by the energy estimate is not sufficient and the results are obtained by verifying some extra space-time regularity of the solutions although how this regularity is obtained is very different: in the case of structural damping, we derive an extra Lyapunov type estimate based on the multiplication of equation (1.1) by  $(-\Delta_x)^{1/2}u$  while in the weakly damped case it is achieved by utilizing the so-called Strichartz type estimates which have been recently extended to the case of bounded domains, see [3], [4]. We discuss each of these two cases in more details below.

## 2. QUINTIC WAVE EQUATION: THE CASE OF STRUCTURAL DAMPING $\theta = 1/2$

The key novelty here is the following theorem proved in [25].

**Theorem 2.1.** *Let  $\theta = 1/2$ ,  $g \in L^2(\Omega)$  and let the nonlinearity  $f(u)$  be odd and satisfy (1.2) with  $q = 4$ . Then any weak energy solution  $u(t)$  of problem (1.1) belongs to the space  $L^2(0, T; H^{3/2}(\Omega))$  and satisfies the estimate*

$$(2.1) \quad \|u\|_{L^2(t, t+1; H^{3/2}(\Omega))} \leq Q(\|\xi_u\|_{L^\infty(t, t+1; \mathcal{E})}) + Q(\|g\|_{L^2}),$$

where the monotone function  $Q$  is independent of  $t$  and  $u$ .

*Proof.* We sketch the proof of this theorem for the case of periodic boundary conditions  $\Omega = \mathbb{T}^3$ . The case of Dirichlet boundary conditions can be reduced to that one by using the odd extension of the solution  $u$  through the boundary (to this end, we need the assumption that the nonlinearity  $f$  is odd) together with a proper cut-off procedure, see [25] for the details. To this end, we use the following identity obtained by multiplying (1.1) by  $(-\Delta_x)^{1/2}u$ :

$$\begin{aligned} \frac{d}{dt} \left( (\partial_t u, (-\Delta_x)^{1/2}u) + \frac{\gamma}{2} \|(-\Delta_x)^{1/2}u\|_{L^2}^2 \right) &+ \|(-\Delta_x)^{3/4}u\|_{L^2}^2 + (f(u), (-\Delta_x)^{1/2}u) \\ &= (g, (-\Delta_x)^{1/2}u) + \|(-\Delta_x)^{1/4}\partial_t u\|_{L^2}^2. \end{aligned}$$

The integration of this identity over the interval  $(t, t + 1)$  gives the desired estimate (2.1) in a straightforward way if we are able to estimate the term containing the nonlinearity  $f$ . To this end, we utilize the following lemma which can be verified using Fourier series and the Parseval equality.

**Lemma 2.1.** *Let  $s \in (0, 1)$  and  $u, v \in H^s(\mathbb{T}^3)$ . Then*

$$(2.2) \quad (v, (-\Delta_x)^s u) = c \int_{\mathbb{R}^3} \int_{\mathbb{T}^3} \frac{(v(x+h) - v(x))(u(x+h) - u(x))}{|h|^{3+2s}} dx dh,$$

where the constant  $c$  depends only on  $s$ .

This lemma, together with the assumption  $f'(u) \geq -K$  and the mean value theorem gives the desired estimate

$$(f(u), (-\Delta_x)^{1/2} u) \geq -K \|(-\Delta_x)^{1/4} u\|_{L^2}^2,$$

which completes the proof of the theorem in the case of periodic boundary conditions.  $\square$

With the extra regularity of energy solutions proved, their uniqueness and further regularity can be obtained in a standard way, so we omit the technicalities and only recall below (see [1], [12], [21]) the definition of the global attractor followed by the statement of the main result, proved in [25].

**Definition 2.1.** *A compact subset  $\mathcal{A}$  of a Banach space  $\mathcal{E}$  is called a global attractor for a semigroup  $S(t): \mathcal{E} \rightarrow \mathcal{E}$ , if*

- (1)  $\mathcal{A}$  is strictly invariant, i.e.,  $S(t)\mathcal{A} = \mathcal{A}$  for all  $t \geq 0$ ;
- (2) for any bounded set  $B \subset \mathcal{E}$  and any neighbourhood  $\mathcal{O}(\mathcal{A})$  of the set  $\mathcal{A}$ , there exists a time  $T = T(B, \mathcal{O})$  such that  $S(t)B \subset \mathcal{O}(\mathcal{A})$  for all  $t \geq T$ .

**Theorem 2.2.** *Let the assumptions of Theorem 2.1 hold. Then the energy solution of problem (1.1) is unique and the solution semigroup  $S(t)$  associated with problem (1.1) in the phase space  $\mathcal{E}$  possesses a global attractor  $\mathcal{A}$  which is bounded in the more regular space  $\mathcal{E}_1 := [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ .*

**Remark 2.1.** The existence of an exponential attractor bounded in  $\mathcal{E}_1$  that, in particular, implies finite fractal dimension of the global attractor  $\mathcal{A}$  is also verified in [25].

### 3. QUINTIC WAVE EQUATION: THE CASE OF WEAK DAMPING $\theta = 0$

The extra space-time regularity of energy solutions is based here on the following nontrivial result concerning Strichartz type estimates for the linear wave equation in a bounded domain, see [3], [4].

**Lemma 3.1.** *Let  $\xi_0 \in \mathcal{E}$ ,  $G(t) \in L^1([0, T]; L^2(\Omega))$  and let  $v$  solve the linear equation*

$$(3.1) \quad \partial_t^2 v - \Delta v = G(t), \quad v|_{\partial\Omega} = 0, \quad \xi_v|_{t=0} = \xi_0.$$

*Then  $v \in L^4([0, T]; L^{12}(\Omega))$  and the following estimate holds:*

$$(3.2) \quad \|v\|_{L^4([0, T]; L^{12}(\Omega))} \leq C_T (\|\xi_0\|_{\mathcal{E}} + \|G\|_{L^1([0, T]; L^2(\Omega))}),$$

*where  $C$  may depend on  $T$  but it is independent of  $\xi_0$  and  $G$ .*

However, in contrast to the previous case, we do not know whether or not *all* energy solutions possess this extra regularity. For this reason, we restrict ourselves to considering only such energy solutions  $u(t)$  of (1.1) which belong to the space  $L^4(0, T; L^{12}(\Omega))$ . We will refer in the sequel to such solutions as *Shatah-Struwe* solutions of problem (1.1).

The next theorem which gives the global well-posedness of problem (1.1) in the class of Shatah-Struwe solutions can be proved with help of the so-called Pohozaev-Morawetz identity and the above mentioned Strichartz estimate, see [4].

**Theorem 3.1.** *Let  $\theta = 0$ ,  $g \in L^2(\Omega)$  and let the nonlinearity  $f$  satisfy (1.2) with  $q = 4$  as well as the following extra assumptions*

1.  $|f''(u)| \leq C(1 + |u|^3)$ ,
2.  $f(u)u - 4F(u) \geq -C$ .

*Then, for any  $(u_0, u'_0) \in \mathcal{E}$ , there exists a unique Shatah-Struwe solution  $u(t)$  of problem (1.1) defined for all  $t \geq 0$ .*

Thus, the (Shatah-Struwe) solution semigroup  $S(t): \mathcal{E} \rightarrow \mathcal{E}$  associated with problem (1.1) is well-defined. Moreover, these Shatah-Struwe solutions have a number of good properties: they satisfy the energy identity (1.3) and the dissipative estimate (1.6); they are more regular, say,  $\xi_u(t) \in \mathcal{E}_1$  for all  $t$  if  $\xi_u(0) \in \mathcal{E}_1$ , etc.

However, in contrast to the previous case, we do not know whether or not the analogue of estimate (2.1) holds for the Strichartz norm  $\|u\|_{L^4(t, t+1; L^{12}(\Omega))}$ , and the previous theorem actually does not give any control of this norm as  $t \rightarrow \infty$ . Thus, the control of the Strichartz norm may be a priori lost when passing to the limit  $t \rightarrow \infty$  and even if we initially consider the Shatah-Struwe solutions only, the other types of energy solutions (for which we have neither the energy equality nor the uniqueness theorem) may a priori appear on the attractor.

For this reason, despite the fact that the considered Shatah-Struwe solutions are unique, as an intermediate step, we need to exploit the existence of a weak attractor in the class of energy solutions where the uniqueness theorem is not known. Namely,

as proved in [27], if we restrict ourselves to considering the energy solutions which can be obtained as limits of Galerkin approximations only, then the trajectory of the dynamical system associated with these solutions possesses a trajectory attractor in the weak-star topology of  $L_{\text{loc}}^\infty(\mathbb{R}_+, \mathcal{E})$  and (which is almost the same), the *multi-valued* semigroup  $\overline{S}(t)$  associated with these solutions (the uniqueness is not known for that type of solutions) possesses a global attractor  $\mathcal{A}_w$  in a weak topology of  $\mathcal{E}$ , see also [10] for more details. Moreover, as shown in [27] (see also [14], [15]), the solutions on the attractor  $\mathcal{A}_w$  possess the following backward regularity.

**Theorem 3.2.** *Under the above assumptions, the global attractor  $\mathcal{A}_w$  is generated by complete energy solutions  $u(t)$ ,  $t \in \mathbb{R}$  which are bounded in  $\mathcal{E}$  for all  $t \in \mathbb{R}$ . Moreover, for any such solution  $u$  there exists  $T = T(u)$  such that  $u(t) \in \mathcal{E}_1$  for  $t \leq -T$  and*

$$\|u\|_{L^\infty(-\infty, -T; \mathcal{E}_1)} \leq C,$$

where the constant  $C$  is independent of  $u$ .

Combining the result of Theorem 3.1 with the above backward regularity is enough to verify the analogue of Theorem 2.2 for that case as well.

**Theorem 3.3.** *Let the assumptions of Theorem 3.1 hold. Then the (Shatah-Struwe) solution semigroup  $S(t)$  associated with problem (1.1) possesses a global attractor  $\mathcal{A}$  in  $\mathcal{E}$  which is a bounded set in  $\mathcal{E}_1$ .*

Indeed, combining the aforementioned backward regularity, the fact that for any initial data  $\xi_0 \in \mathcal{E}_1$ , the corresponding Shatah-Struwe solution remains in  $\mathcal{E}_1$ , and the proper version of weak-strong uniqueness, we establish that the weak attractor satisfies  $\mathcal{A}_w \subset \mathcal{E}_1$ , see [17] for more details. Thus, the energy equality holds for any solution belonging to this attractor. The asymptotic compactness of the solution semigroup in  $\mathcal{E}$  can be then established using the so-called energy method, see [2], [22]. Finally, the additional regularity of the global attractor  $\mathcal{A}$  is based on this asymptotic compactness by using more or less standard bootstrapping arguments, see [17].

**Remark 3.1.** In the *subcritical* case  $q < 4$ , the following analogue of the dissipative estimate for the Strichartz norm holds:

$$(3.3) \quad \|u\|_{L^4([t, t+1]; L^{12}(\Omega))} \leq Q(\|\xi_0\|_{\mathcal{E}})e^{-\alpha t} + Q(\|g\|),$$

where the monotone function  $Q$  is independent of  $t$  and  $u$  and the attractor theory in this case is essentially simpler, see [17].



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*Authors' address: Anton Savostianov, Sergey Zelik, University of Surrey, Department of Mathematics, Guildford, Surrey GU2 7XH, United Kingdom, e-mail: a.savostianov@surrey.ac.uk, s.zelik@surrey.ac.uk.*