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Symmetries and currents in nonholonomic mechanics

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Abstract. In this paper we derive general equations for constraint Noether--type symmetries of a first order non-holonomic mechanical system and the corresponding currents, i.e. functions constant along trajectories of the nonholonomic system. The approach is based on a consistent and effective geometrical theory of nonholonomic constrained systems on fibred manifolds and their jet prolongations, first presented and developed by Olga Rossi. As a representative example of application of the geometrical theory and the equations of symmetries and conservation laws derived within this framework we present the Chaplygin sleigh. It is a mechanical system subject to one linear nonholonomic constraint enforcing the plane motion. We describe the trajectories of the Chaplygin sleigh and show that the usual kinetic energy conservation law holds along them, the time translation generator being the corresponding constraint symmetry and simultaneously the symmetry of nonholonomic equations of motion. Moreover, the expressions for two other currents are obtained. Remarkably, the corresponding constraint symmetries are not symmetries of nonholonomic equations of motion. The physical interpretation of results is emphasized.

1 Introduction

While a wide variety of problems within the mechanics of first order systems without constraints or with holonomic constraints is solved, mechanics of nonholonomic systems is still studied relatively intensively by various authors using various approaches. Bibliography concerning nonholonomic constraints is very rich, see e.g. famous books by Neimark and Fufaev [26], Bloch and coworkers [2], Cortés Monforte [7], and Bullo [3], and others, or many papers as e.g. [9], [23], [24], [29], [34], [35], [39], [40], or recently e.g. [28] (for nonlinear constraints), to mention just a few. Most of the above cited works are concerned with linear or affine nonholonomic constraints, relevant a.e. for technical applications. A geometrical theory of

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nonholonomic systems on fibred manifolds and their jet prolongations was proposed by Olga Rossi (Krupková) in [14] and elaborated in her later works among which we can emphasize e.g. [15], [16], [19]. This theory differs from other approaches by the idea that a *nonholonomic constraint is a fibred submanifold* of the first jet prolongation of the underlying fibred manifold. The nonholonomic mechanical system is considered as a *dynamical system on this constraint submanifold* which is its true phase space. The equations of motion called the *reduced equations* are equivalent with the well known Chetaev equations [6] based on the standardly used d'Alembert's principle. In this sense the geometrical model is a generalization of the d'Alembert's principle to nonlinear as well as higher order constraints. A detailed explanation of the theory based on the nonholonomic variational principle can be found in [19].

The geometrical theory is an effective tool for solving a wide variety of problems connected with nonholonomic systems. One of them is the nonholonomic inverse problem, see e.g. [22] and [30]. The relevance and applicability of the theory was verified on examples (see [37]) and practical situations (see [8], [10], [11], [12], [13]), including the experimental verification in [12] and [13]. An interesting realistic case of a nonlinear constraint is represented by the mechanical system consisting of a mass particle in the special relativity theory. This problem is solved in [21] and [31]. Explicit results of this kind should be compared with usually applied analytic and geometric techniques which provide mostly only conclusions concerning equilibria.

Some questions concerning nonholonomic systems are still not satisfactorily understood. One of them is the problem of nonholonomic symmetries and conservation laws. On the other hand, a proper understanding of symmetries and conservation laws is a key question in mechanics including nonholonomic systems in particular. Here we emphasize a new concept of nonholonomic symmetry of a Lagrangian system and generalization of Noether theorem formulated by Olga Rossi [18] within the framework of her geometrical theory. An interesting example of the projectile motion controlled by the constant speed constraint was discussed and completely solved in [38].

In the present paper we derive general equations of constraint Noether-type symmetries for a Lagrangian first order mechanical system subjected to a quite general nonholonomic constraint and the expressions for corresponding currents, i.e. quantities conserved along trajectories. It should be emphasized that the constraint symmetries of a Lagrangian in the generalized Noether theorem need not be symmetries of the constraint equations of motion. So they play similar role as "pseudosymmetries" in nonconservative mechanics (see [4], [33], [36]). More generally, in [36] the solution of the problem of symmetries is based on the idea of generating first integrals through so called *adjoint symmetries* (a dual concept of pseudosymmetries). We focus to Noether-type symmetries defined as vector fields leaving invariant (up to a constraint form) the constraint Lepage equivalent of a Lagrangian. We illustrate the results on an example interesting from the physical point of view: the Chaplygin sleigh. It appears that the solution of the problem is technically not so simple. We present the solutions of reduced equations of the sleigh including graphical outputs, as well as conservation laws and corresponding symmetries. Moreover, we find the (non-variational) Chetaev constraint forces explicitly and emphasize the physical interpretation of the results. A brief overview (following the page restriction requirements) has been submitted for publication in the proceedings of the VIII-th International Conference Differential Geometry and Dynamical Systems (DGDS) 2014 where the results were reported, see [5].

2 Elements of the geometrical theory of nonholonomic mechanics

In this section we summarize elements of the geometrical theory of first order nonholonomic mechanical systems arising from initially Lagrangian unconstrained ones.

2.1 Underlying structures and notations

The geometrical theory of nonholonomic mechanical systems is developed on an $(m+1)$-dimensional underlying fibred manifold (Y, π, X) with the total space Y, the one-dimensional base X and the projection (surjective submersion) π. The dimension of fibres m represents the number of degrees of freedom of an unconstrained system. We use the standard notation for jet prolongations of this manifold, (J^rY, π_r, X), $r = 0, 1, 2$, $Y = J^0Y$, $\pi = \pi_0$ and for fibred manifolds $(J^rY, \pi_{r,s}, J^sY)$, $s = 0, 1$. We denote as (V, ψ) a fibred chart on Y, where $V \subset Y$ is an open set, $\psi = (t, q^\sigma)$, $1 \le \sigma \le m$. Then (U, φ), $U = \pi(V)$, $\varphi = (t)$, is the associated chart on X, and (V_r, ψ_r), $V_r = \pi_{r,0}^{-1}(V)$, $\psi_1 = (t, q^\sigma, \dot{q}^\sigma)$, $\psi_2 = (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma)$, are the associated fibred charts on J^1Y and J^2Y, respectively. Let $U \subset X$ be an open set. A section $\delta \colon U \ni t \to \delta(t) \in J^rY$, $r = 1, 2$, is called *holonomic* if there exists a section $\gamma \colon U \ni t \to \gamma(t) \in Y$ such that $\delta = J^r\gamma$.

We also use the standard concept of a vector field on Y and its prolongations connected with the fibred structure. The standard concept of differential forms is used as well. A vector field ξ on J^rY is called *π_r-projectable* if there exists a vector field ξ_0 on X such that $T\pi_r\, \xi = \xi_0 \circ \pi_r$. A vector field ξ is called *π_r-vertical* if $T\pi_r\, \xi = 0$. A vector field ξ on J^rY is called *$\pi_{r,s}$-projectable* if there exists a vector field ζ on J^sY such that $T\pi_{r,s}\, \xi = \zeta \circ \pi_{r,s}$. A vector field on J^rY is called *$\pi_{r,s}$-vertical* if $T\pi_{r,s}\, \xi = 0$. The chart expressions of the above mentioned vector fields are (for $r = 0, 1, 2$, $s = 0, 1$, $s < r$)

$$\xi = \xi^0(t)\,\frac{\partial}{\partial t} + \sum_{j=0}^{r} \xi^\sigma_{(j)}(t, q^\nu, \dots, q^\nu_r)\,\frac{\partial}{\partial q^\sigma_j}\,,$$

with $\xi^0 = 0$ for a π_r-vertical vector field, and

$$\xi = \xi^0(t, q^\nu, \dots, q^\nu_s)\frac{\partial}{\partial t} + \sum_{j=0}^{s} \xi^\sigma_j(t, q^\nu, \dots, q^\nu_s)\,\frac{\partial}{\partial q^\sigma_j} + \sum_{j=s+1}^{r} \xi^\sigma_j(t, q^\nu, \dots, q^\nu_r)\,\frac{\partial}{\partial q^\sigma_j}\,,$$

with $\xi^0 = 0$ and $\xi^\sigma_j = 0$, $j = 0, \dots, s$, for a $\pi_{r,s}$-vertical vector field. In the preceding expressions we denoted $q^\sigma = q^\sigma_0$, $\dot{q}^\sigma = q^\sigma_1$, $\ddot{q}^\sigma = q^\sigma_2$.

A differential q-form η on J^rY is called *π_r-horizontal* if $i_\xi\eta = 0$ for every π_r-vertical vector field ξ on J^rY. A q-form η on J^rY is called *$\pi_{r,s}$-horizontal* if

$i_\xi \eta = 0$ for every $\pi_{r,s}$-vertical vector field ξ on $J^r Y$. π_r-horizontal 1-forms have a chart expression

$$\eta = \eta_0(t, q^\sigma, \dots, q^\sigma_r)\,\mathrm{d}t\,.$$

Every π-projectable vector field $\xi = \xi^0(t)\frac{\partial}{\partial t} + \xi^\sigma(t,\, q^\nu)\frac{\partial}{\partial q^\sigma}$ on Y can be prolonged on $J^r Y$, $r = 1,\, 2$,

$$J^1\xi = \xi^0 \frac{\partial}{\partial t} + \xi^\sigma \frac{\partial}{\partial q^\sigma} + \tilde{\xi}^\sigma \frac{\partial}{\partial \dot{q}^\sigma}\,, \quad \text{or} \quad J^2\xi = \xi^0 \frac{\partial}{\partial t} + \xi^\sigma \frac{\partial}{\partial q^\sigma} + \tilde{\xi}^\sigma \frac{\partial}{\partial \dot{q}^\sigma} + \hat{\xi}^\sigma \frac{\partial}{\partial \ddot{q}^\sigma}\,,$$

where $\tilde{\xi}^\sigma = \frac{\mathrm{d}\xi^\sigma}{\mathrm{d}t} - \dot{q}^\sigma \frac{\mathrm{d}\xi^0}{\mathrm{d}t}$, and $\hat{\xi}^\sigma = \frac{\mathrm{d}\tilde{\xi}^\sigma}{\mathrm{d}t} - \ddot{q}^\sigma \frac{\mathrm{d}\xi^0}{\mathrm{d}t}$. A q-form η on $J^r Y$ is called *contact* if $J^r\gamma^*\eta = 0$ for every section γ of π. Contact forms on $J^r Y$ form a differential ideal $\mathcal{I}_C$ called the *contact ideal*. For expressing differential forms in coordinates we use the basis of 1-forms adapted to the contact structure, $(t,\, \omega^\sigma,\, \mathrm{d}\dot{q}^\sigma)$ and $(t,\, \omega^\sigma,\, \dot{\omega}^\sigma,\, \mathrm{d}\ddot{q}^\sigma)$ on $J^1 Y$ and $J^2 Y$, respectively, where $\omega^\sigma = \mathrm{d}q^\sigma - \dot{q}^\sigma\,\mathrm{d}t$, $\dot{\omega}^\sigma = \mathrm{d}\dot{q}^\sigma - \ddot{q}^\sigma\,\mathrm{d}t$. There exists a unique decomposition of a q-form η on $J^r Y$ into its $(q-1)$-*contact* and q-*contact component* $\pi^*_{r+1,r}\eta = p_{q-1}\eta + p_q\eta$. The chart expression of $p_{q-1}\eta$ in the basis adapted to the contact structure is a linear combination of terms with just $(q-1)$ factors of the type ω^σ or $\dot{\omega}^\sigma$ and the chart expression of $p_q\eta$ is a linear combination of terms with just q such factors. (The only contact form on Y is the trivial (zero) one.) Notice that jet prolongations of π-projectable vector fields are closely related to the contact ideal being its symmetries: $\partial_{J^r\xi}\omega \in \mathcal{I}_C$ for every $\omega \in \mathcal{I}_C$. Here $\partial_{J^r\xi}$ denotes the Lie derivative along a vector field $J^r\xi$.

A *distribution* on $J^r Y$ is a mapping $\mathcal{D}\colon J^r Y \ni x \to \mathcal{D}(x) \subset T_x J^r Y$, where $\mathcal{D}(x)$ is a vector subspace of $T_x J^r Y$. A distribution is generated by local vector fields ξ_ι on $J^r Y$, $\iota \in \mathcal{I}$, where $\mathcal{I}$ is a set of indices. Equivalently, the distribution $\mathcal{D}$ can be annihilated by 1-forms η on $J^r Y$ such that $i_\xi \eta = 0$ for every vector field ξ belonging to the distribution $\mathcal{D}$.

2.2 Unconstrained systems

The geometrical theory of nonholonomic systems, as introduced in [14], is universal in the following sense: It concerns all types of nonholonomic mechanical systems given by equations of motion of the initial unconstrained system and the nonholonomic constraint, independently whether the equations of motion of the initial system are variational (Lagrangian) or not. In this paper we concentrate on the first of both situations because the concept of nonholonomic symmetries is formulated for constrained Lagrangians, not for equations.

Let λ be a first order Lagrangian, i.e. a horizontal form on $J^1 Y$, $\lambda = L(t, q^\sigma, \dot{q}^\sigma)\,\mathrm{d}t$. The pair $(\pi,\, \lambda)$ represents a *Lagrange structure*. The first order Lagrangean mechanics studies a.e. *extremals* of the Lagrange structure, i.e. sections γ of π representing critical sections γ of the variational integral (action function)

$$S_\Omega\colon\, \Gamma(\pi) \ni \gamma \to S_\Omega[\gamma] = \int_\Omega J^1\gamma^*\,\lambda$$

where $\Gamma(\pi)$ is a set of all sections of the projection π defined on open subsets of the base X, and Ω is a compact set included in the domain of γ. Critical sections of S are zero points of the variational derivative of S, i.e. integral

$$\left.\frac{\mathrm{d}S[\gamma_u]}{\mathrm{d}u}\right|_{u=0} = \int\limits_\Omega J^1\gamma^*\,\partial_{J^1\xi}\lambda\,,$$

where ξ is a π-projectable vector field called the *variation* and $\{\gamma_u\}$, $u \in (-\varepsilon,\, \varepsilon)$, is a one-parameter system of sections generated by ξ such that $\gamma_0 = \gamma$, i.e. $\gamma_u = \phi_u \circ \gamma \circ \phi_{0u}^{-1}$, where $(\phi_u,\, \phi_{0u})$ is the one-parameter group of the vector field ξ. The variational derivative of the variational integral leads to the *first variation formula*

$$\int\limits_\Omega J^1\gamma^*\partial_{J^1\xi}\lambda = \int\limits_\Omega J^1\gamma^* i_{J^1\xi}\,\mathrm{d}\theta_\lambda + \int\limits_{\partial\Omega} J^1\gamma^* i_{J^1\xi}\theta_\lambda\,, \tag{1}$$

where $\theta_\lambda = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot q^\sigma}\,\omega^\sigma$ is the Lepage equivalent of the Lagrangian (the Poincaré--Cartan form). The condition for an extremal leads to *Euler-Lagrange equations*—equations of motion of the system. The coordinate free expression of these equations reads $J^1\gamma^*\, i_{J^1\xi}\,\mathrm{d}\theta_\lambda = 0$ or $J^2\gamma^* E_\lambda = 0$, where in coordinates

$$E_\lambda = E_\sigma\,\omega^\sigma \wedge \mathrm{d}t\,, \qquad E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q^\sigma}\,, \tag{2}$$

or equivalently

$$\begin{gathered} E_\sigma \circ J^2\gamma = (A_\sigma + B_{\sigma\nu}\ddot q^\nu) \circ J^2\gamma = 0\,, \\ A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}'}{\mathrm{d}t}\frac{\partial L}{\partial \dot q}\,, \qquad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot q^\sigma \partial \dot q^\nu}\,. \end{gathered} \tag{3}$$

Here

$$\frac{\mathrm{d}'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} - \ddot q^\sigma \frac{\partial}{\partial \dot q^\sigma} = \frac{\partial}{\partial t} + \dot q \frac{\partial}{\partial q^\sigma}\,.$$

A π-projectable vector field ξ on Y is called a *symmetry* of the Lagrange structure $(\pi,\, \lambda)$ if it holds $\partial_{J^1\xi}\lambda = 0$. This condition is the *Noether equation*. For a given Lagrangian it is interpreted as a set of equations for symmetries, for a given vector field ξ it represents a functional equation for Lagrangians having the symmetry ξ. (For our purposes the first of both interpretations will be relevant.) The chart expression of the Noether equation is

$$\frac{\partial L}{\partial t}\xi^0 + \frac{\partial L}{\partial q^\sigma}\xi^\sigma + \frac{\partial L}{\partial \dot q^\sigma}\left(\frac{\mathrm{d}\xi^\sigma}{\mathrm{d}t} - \dot q^\sigma \frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right) + L\frac{\mathrm{d}\xi^0}{\mathrm{d}t} = 0\,. \tag{4}$$

Taking into account the first variation formula we can see that if ξ is a symmetry of the Lagrange structure then the quantity

$$i_{J^1\xi}\theta_\lambda = \left(L - \dot q^\sigma \frac{\partial L}{\partial \dot q^\sigma}\right)\xi^0 + \frac{\partial L}{\partial \dot q^\sigma}\xi^\sigma \tag{5}$$

(called the *current*) is constant along extremals. This result representing conservation laws is well known as the *Emmy Noether theorem*.

2.3 Nonholonomic dynamics

Suppose that an unconstrained first order Lagrangian mechanical system is subjected to a nonholonomic constraint given by k equations, $1 \leq k \leq m-1$,

$$f^a(t,\, q^\sigma,\, \dot q^\sigma) = 0\,, \quad 1 \leq a \leq k\,, \quad \text{where} \quad \operatorname{rank}\left(\frac{\partial f^a}{\partial \dot q^\sigma}\right) = k\,,$$

or in a normal form

$$\dot q^{m-k+a} = g^a(t,\, q^\sigma,\, \dot q^l)\,, \quad 1 \leq l \leq m-k\,.$$

These equations define a *constraint submanifold* $Q \subset J^1Y$ of codimension k fibred over Y (and, of course, over X as well). The corresponding projections $\bar\pi_{1,0}$ and $\bar\pi_1$ are the mappings $\pi_{1,0}$ and π_1 restricted to Q, respectively. Denote

$$\iota\colon\ Q \ni (t,\, q^\sigma,\, \dot q^l) \longrightarrow (t,\, q^\sigma,\, \dot q^l,\, g^a(t,\, q^\nu,\, \dot q^s)) \in J^1Y$$

the canonical embedding of Q into J^1Y. On the submanifold Q there arise the *induced contact ideal* $\tilde{\mathcal{I}}_C$ generated by forms $\bar\omega^\sigma = \iota^*\omega^\sigma$ and the *canonical distribution*

$$\mathcal{C} = \{\operatorname{span}\varphi^a \,|\, 1 \leq a \leq k\}\,, \qquad \varphi^a = \iota^*\omega^{m-k+a} - \frac{\partial g^a}{\partial \dot q^l}\iota^*\,\omega^l. \tag{6}$$

The $\bar\pi_1$-projectable vector fields belonging to the canonical distribution are called *Chetaev vector fields*. They represent *admissible variations* in the nonholonomic variational principle (first introduced in [19]). Let us briefly recall this principle and its consequences. Let $(\pi,\, \lambda)$ be an unconstrained Lagrangian structure and θ_λ the corresponding Poincaré-Cartan form. By the *constraint system* on Q defined by λ we mean the differential form $\iota^*\theta_\lambda$. Denote $\bar\lambda = \iota^*\lambda = (L\circ\iota)\,\mathrm{d}t$ and $\theta_{\bar\lambda} = \theta_{\iota^*\lambda}$. Calculating $\iota^*\theta_\lambda$ we obtain

$$\iota^*\theta_\lambda = \bar L\,\mathrm{d}t + \frac{\partial \bar L}{\partial \dot q^l}\bar\omega^l + \bar L_a\varphi^a = \theta_{\iota^*\lambda} + \bar L_a\varphi^a\,,$$

$$\bar L = L\circ\iota\,, \qquad \bar L_a = \frac{\partial L}{\partial \dot q^{m-k+a}}\circ\iota\,.$$

Let δ be a section of the projection $\bar\pi_1\colon\ Q \to X$ defined on an open subset $U \subset X$ containing a compact set $\Omega \subset X$. Let $Z \in \mathcal{C}$ be a $\bar\pi_1$-projectable vector field and let $(\phi_u,\, \phi_{0u})$ its one-parameter group and $\{\delta_u\} = \{\phi_u\circ\delta\circ\phi_{0u}^{-1}\}$, $\delta_0 = \delta$, the one-parameter family of sections generated by Z. The constraint variational integral and its variational derivative are

$$S_\Omega[\delta] = \int_\Omega \delta^*\iota^*\theta_\lambda, \qquad \left.\frac{\mathrm{d}S[\delta_u]}{\mathrm{d}u}\right|_{u=0} = \int_\Omega \delta^*\partial_Z\iota^*\theta_\lambda\,.$$

If we restrict to holonomic sections we obtain the variational derivative of the variational integral in the form

$$\left.\frac{\mathrm{d}S[\gamma_u]}{\mathrm{d}u}\right|_{u=0} = \int_\Omega J^1\gamma^*\partial_Z\iota^*\theta_\lambda\,.$$

Nonholonomic first variation formula reads (taking into account that $i_Z(\bar{L}^a\varphi^a) = 0$, because $Z \in \mathcal{C}$)

$$\int_\Omega J^1\gamma^*\,\partial_Z\iota^*\theta_\lambda = \int_\Omega J^1\gamma^* i_Z\,\mathrm{d}\iota^*\theta_\lambda + \int_{\partial\Omega} J^1\gamma^* i_Z\theta_{\iota^*\lambda}\,. \tag{7}$$

By a direct calculation we can justify that the integrand in the first integral on the right-hand side of (7) depends only on components of Z on Y. The requirement of vanishing of this integral (for arbitrary Ω) leads to equations of motion

$$J^1\gamma^* i_Z\,\mathrm{d}\iota^*\theta_\lambda = 0 \implies \big(\varepsilon_s(\bar{L}) - \bar{L}_a\varepsilon_s(g^a)\big)\circ J^2\gamma = 0\,, \tag{8}$$

for $1 \le s \le m-k$. In the expressions of the type

$$\varepsilon_s(f) = \frac{\partial_c f}{\partial q^s} - \frac{\mathrm{d}_c}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}^s}\,, \quad \text{where} \quad f = f(t,\, q^\sigma,\, \dot{q}^l)\,,$$

the constraint derivative operators are used

$$\frac{\partial_c}{\partial q^s} = \frac{\partial}{\partial q^s} + \frac{\partial g^a}{\partial \dot{q}^s}\frac{\partial}{\partial q^{m-k+a}}\,,$$

$$\frac{\mathrm{d}_c}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}^l\frac{\partial}{\partial q^l} + g^a\frac{\partial}{\partial q^{m-k+a}} + \ddot{q}^l\frac{\partial}{\partial \dot{q}^l} = \frac{\mathrm{d}'_c}{\mathrm{d}t} + \ddot{q}^l\frac{\partial}{\partial \dot{q}^l}\,.$$

Note that these operators have an important geometrical meaning: Vector fields

$$\frac{\partial_c}{\partial t} = \frac{\mathrm{d}'_c}{\mathrm{d}t} - \dot{q}^l\frac{\partial_c}{\partial q^l}\,, \quad \frac{\partial_c}{\partial q^l}, \quad \frac{\partial}{\partial \dot{q}^l}, \quad 1 \le l \le m-k\,,$$

generate the canonical distribution $\mathcal{C}$. The equations (8) can be written as follows

$$\bar{A}_s + \bar{B}_{sr}\ddot{q}^r = 0\,, \quad 1 \le s \le m-k\,, \tag{9}$$

$$\bar{A}_s = \frac{\partial_c\bar{L}}{\partial q^s} - \frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial\bar{L}}{\partial\dot{q}^s} - \bar{L}_a\left(\frac{\partial_c g^a}{\partial q^s} - \frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial g^a}{\partial \dot{q}^s}\right), \quad \bar{B}_{sr} = -\frac{\partial^2\bar{L}}{\partial\dot{q}^s\partial\dot{q}^r} + \bar{L}_a\frac{\partial^2 g^a}{\partial\dot{q}^s\partial\dot{q}^r}\,,$$

or, via functions A_σ and $B_{\sigma\nu}$ (3),

$$\bar{A}_s = \Bigg[A_s + \sum_{a=1}^{k} A_{m-k+a}\frac{\partial g^a}{\partial \dot{q}^s} + \sum_{a=1}^{k}\left(B_{s,m-k+a} + \sum_{b=1}^{k} B_{m-k+b,m-k+a}\frac{\partial g^b}{\partial \dot{q}^s}\right)\left(\frac{\partial g^a}{\partial t} + \frac{\partial g^a}{\partial q^\sigma}\dot{q}^\sigma\right)\Bigg]\circ\iota$$

$$\bar{B}_{sr} = \Bigg[B_{sr} + \sum_{a=1}^{k}\left(B_{s,m-k+a}\frac{\partial g^a}{\partial \dot{q}^r} + B_{m-k+a,r}\frac{\partial g^a}{\partial \dot{q}^s}\right) + \sum_{a,b=1}^{k} B_{m-k+b,m-k+a}\frac{\partial g^b}{\partial \dot{q}^s}\frac{\partial g^a}{\partial \dot{q}^r}\Bigg]\circ\iota\,.$$

The last relations are universal in the following sense: They hold for both types of equations of motion of an initial unconstrained mechanical system, i.e. variational as well as non-variational ones.

We obtained $m-k$ *reduced equations* of a nonholonomic system. These equations together with k equations of the constraint form a complete set of equations of motion of the system for its trajectories $\gamma\colon t \to \gamma(t) = (t,\, q^\sigma\gamma(t)) \in Y$, $1 \le \sigma \le m$.

2.4 Chetaev equations

In the framework of the geometrical theory of nonholonomic systems the well known Chetaev equations of motion can be derived. We present them for completeness. These equations are obtained by introducing the *Chetaev constraint force* into equations of motion. Suppose that $A_\sigma + B_{\sigma\nu}\ddot{q}^\nu = 0$, $1 \le \sigma, \nu \le m$, are equations of motion of an unconstrained system. The Chetaev force is defined as the form $\phi = \mu^a \frac{\partial f^a}{\partial \dot{q}^\sigma}\, \omega^\sigma \wedge \mathrm{d}t$. The coefficients μ^a, $1 \le a \le k$, on J^1Y are *Lagrange multipliers*. The Chetaev equations read

$$\left(A_\sigma + B_{\sigma\nu}\ddot{q}^\nu - \mu^a \frac{\partial f^a}{\partial \dot{q}^\sigma} \right) \circ J^2\gamma = 0\,. \tag{10}$$

Together with the equations of the constraint $f^a = 0$, $1 \le a \le k$, we obtain $m+k$ equations for trajectories and Lagrange multipliers. Knowing the Lagrange multipliers we can determine the constraint force ϕ which is important for interpretation of results from the point of view of physics.

3 Nonholonomic constraint symmetries

In this section we present the definition of a (nonholonomic) constraint symmetry and derive general equations for symmetries of a constrained mechanical system arising from an initially unconstrained first order Lagrangian structure.

3.1 The concept of constraint symmetries

Let Z be a Chetaev vector field, i.e. $Z \in \mathcal{C}$. The chart expression of Z is

$$Z = Z^0 \frac{\partial}{\partial t} + Z^l \frac{\partial}{\partial q^l} + Z^{m-k+a} \frac{\partial}{\partial q^{m-k+a}} + \tilde{Z}^l \frac{\partial}{\partial \dot{q}^l}\,,$$

$$Z^{m-k+a} = Z^0 g^a + (Z^s - \dot{q}^s Z^0)\frac{\partial g^a}{\partial \dot{q}^s}\,. \tag{11}$$

The condition for components Z^{m-k+a} follows from the assumption that Z belongs to the canonical distribution, i.e. $i_Z\varphi^a = 0$ for $1 \le a \le k$. We say that Z is a *constraint symmetry* of the nonholonomic mechanical system arising from a primarily unconstrained Lagrangean structure $(\pi,\, \lambda)$ subjected to nonholonomic constraints $\dot{q}^{m-k+a} = g^a(t,\, q^\sigma,\, \dot{q}^l)$ if the constrained system $\iota^*\theta_\lambda$ on Q defined by λ remains invariant under transformations given by the one-parameter group of the vector field Z up to a constraint form. This means that

$$\partial_Z \iota^*\theta_\lambda = i_Z\, \mathrm{d}\iota^*\theta_\lambda + \mathrm{d}i_Z\iota^*\theta_\lambda = F_a\varphi^a, \tag{12}$$

where F_a are some functions on Q. Relation (12) represents the *constraint Noether equation*. From the nonholonomic variation formula (7) we can see that if Z is a constraint symmetry of a nonholonomic mechanical system and γ is a solution of the corresponding reduced equations together with constraints, then $\mathrm{d}J^1\gamma^* i_Z \iota^* \theta_\lambda = 0$, i.e. $(i_Z \iota^* \theta_\lambda) \circ J^1\gamma = \text{const.}$ This means that the quantities $\Phi = i_Z \iota^* \theta_\lambda$ are constant along solutions. We obtain

$$\Phi = \left(\bar{L} - \dot{q}^l \frac{\partial \bar{L}}{\partial \dot{q}^l} \right) Z^0 + \frac{\partial \bar{L}}{\partial \dot{q}^l} Z^l. \tag{13}$$

The quantities Φ are called *Noether-type currents* and the conditions $\Phi = \text{const.}$ are the corresponding *conservation laws*.

3.2 Equations for constraint symmetries

Using the definition of constraint symmetries and relations (9) we obtain after some tedious calculations the following set of partial differential equations for $(2(m-k)+1)$ components of these symmetries:

$$Z^0 \left[\frac{\mathrm{d}'_c \bar{L}}{\mathrm{d}t} - \left(\frac{\partial_c \bar{L}}{\partial q^l} - \bar{L}_a \varepsilon'_l(g^a) \right) \dot{q}^l \right] + Z^l \left(\frac{\partial_c \bar{L}}{\partial q^l} - \bar{L}_a \varepsilon'_l(g^a) \right) + \frac{\mathrm{d}'_c Z^0}{\mathrm{d}t} \left(\bar{L} - \frac{\partial \bar{L}}{\partial \dot{q}^l} \dot{q}^l \right) + \frac{\mathrm{d}'_c Z^l}{\mathrm{d}t} \frac{\partial \bar{L}}{\partial \dot{q}^l} = 0 \,, \tag{14}$$

$$\begin{aligned} Z^0 \Bigg[& \frac{\mathrm{d}'_c}{\mathrm{d}t} \left(\frac{\partial \bar{L}}{\partial \dot{q}^l} \right) - \frac{\partial_c}{\partial q^s} \left(\frac{\partial \bar{L}}{\partial \dot{q}^l} \right) \dot{q}^s + \bar{L}_a \varepsilon'_l(g^a) \\ & - \bar{L}_a \dot{q}^s \left(\frac{\partial_c}{\partial q^l} \right) \left(\frac{\partial g^a}{\partial \dot{q}^s} - \frac{\partial_c}{\partial q^s} \left(\frac{\partial g^a}{\partial \dot{q}^l} \right) \right) \Bigg] \\ & + Z^s \left[\frac{\partial_c}{\partial q^s} \left(\frac{\partial \bar{L}}{\partial \dot{q}^l} \right) + \bar{L}_a \left(\frac{\partial_c}{\partial q^l} \left(\frac{\partial g^a}{\partial \dot{q}^s} \right) - \frac{\partial_c}{\partial q^s} \left(\frac{\partial g^a}{\partial \dot{q}^l} \right) \right) \right] \\ & + \tilde{Z}^s \left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} - \bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} \right) + \frac{\partial_c Z^0}{\partial q^l} \left(\bar{L} - \frac{\partial \bar{L}}{\partial \dot{q}^s} \dot{q}^s \right) + \frac{\partial_c Z^s}{\partial q^l} \frac{\partial \bar{L}}{\partial \dot{q}^s} = 0 \,, \end{aligned} \tag{15}$$

$$\bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} (Z^s - \dot{q}^s Z^0) + \left(\bar{L} - \frac{\partial \bar{L}}{\partial \dot{q}^s} \dot{q}^s \right) \frac{\partial Z^0}{\partial \dot{q}^l} + \frac{\partial \bar{L}}{\partial \dot{q}^s} \frac{\partial Z^s}{\partial \dot{q}^l} = 0 \,, \tag{16}$$

for $1 \le l \le m-k$. The following expression represents the coefficients F_a of the constraint form $F_a \varphi^a$ (we present them for completeness):

$$\begin{aligned} F_a = i_Z \, \mathrm{d}\bar{L}_a + \bar{L}_b & \left(\frac{\partial g^b}{\partial q^{m-k+a}} Z^0 + \frac{\partial^2 g^b}{\partial q^{m-k+a} \partial \dot{q}^s} (Z^s - \dot{q}^s Z^0) \right) \\ & + \left(\bar{L} - \frac{\partial \bar{L}}{\partial \dot{q}^s} \dot{q}^s \right) \frac{\partial Z^0}{\partial q^{m-k+a}} + \frac{\partial \bar{L}}{\partial \dot{q}^s} \frac{\partial Z^s}{\partial q^{m-k+a}} \,. \end{aligned}$$

For a special but in practical situations frequent case of a semiholonomic constraint (linear constraint with $\varepsilon'_l(g^a) = 0$, $1 \le l \le m-k$, $1 \le a \le k$) the equations for symmetries take a simplified form

$$\begin{gathered}\frac{\mathrm{d}'_c\bar{L}}{\mathrm{d}t}Z^0+\frac{\partial_c\bar{L}}{\partial q^l}(Z^l-\dot{q}^lZ^0)+\left(\bar{L}-\dot{q}^l\frac{\partial\bar{L}}{\partial\dot{q}^l}\right)\frac{\mathrm{d}'_cZ^0}{\mathrm{d}t}+\frac{\partial\bar{L}}{\partial\dot{q}^l}\frac{\mathrm{d}'_cZ^l}{\mathrm{d}t}=0\,,\\
\frac{\mathrm{d}'}{\mathrm{d}t}\left(\frac{\partial\bar{L}}{\partial\dot{q}^l}\right)Z^0+\frac{\partial_c}{\partial q^s}\left(\frac{\partial\bar{L}}{\partial\dot{q}^l}\right)(Z^s-\dot{q}^sZ^0)\\
+\tilde{Z}^s\frac{\partial^2\bar{L}}{\partial\dot{q}^l\partial\dot{q}^s}+\frac{\partial\bar{L}}{\partial\dot{q}^s}\frac{\partial_cZ^s}{\partial q^l}+\frac{\partial_cZ^0}{\partial q^l}\left(\bar{L}-\dot{q}^s\frac{\partial\bar{L}}{\partial\dot{q}^s}\right)=0\,,\\
\left(\bar{L}-\dot{q}^s\frac{\partial\bar{L}}{\partial\dot{q}^s}\right)\frac{\partial Z^0}{\partial\dot{q}^l}+\frac{\partial\bar{L}}{\partial\dot{q}^s}\frac{\partial Z^s}{\partial\dot{q}^l}=0\,,\end{gathered}$$

$1\le l,\, s\le m-k$. These relations are fully consistent with equations for symmetries of Poincaré-Cartan form of unconstrained systems, $\partial_{J^1\xi}\theta_\lambda=0$, taking into account that for unconstrained systems ξ is a π-projectable vector field on Y, i.e. $\xi^0=\xi^0(t)$, $\xi^\sigma=\xi^\sigma(t,\,q^\nu)$, $1\le\sigma,\,\nu\le m$, and components $\tilde{\xi}^\sigma$ are uniquely given by ξ^0 and ξ^ν (see relations in Section 2.1). It is obvious that for a nonholonomic case the constraint differential operators are used instead of the usual ones.

Using the expressions for currents and for coefficients of reduced equations $\bar{A}_l$ and $\bar{B}_{ls}$ given by (9) we obtain a more suitable form of equations (14)–(16):

$$\frac{\mathrm{d}'_c\Phi}{\mathrm{d}t}+\bar{A}_l(Z^l-\dot{q}^lZ^0)=0\,,\tag{17}$$

$$\frac{\partial_c\Phi}{\partial q^l}-\bar{A}_lZ^0+\left\{\frac{\partial\bar{A}_s}{\partial\dot{q}^l}+\frac{\partial\bar{L}_a}{\partial\dot{q}^l}\varepsilon'_s(g^a)\right\}_{\mathrm{alt}(l,s)}(Z^s-\dot{q}^sZ^0)-\bar{B}_{ls}\tilde{Z}^s=0\,,\tag{18}$$

$$\frac{\partial\Phi}{\partial\dot{q}^l}+\bar{B}_{ls}(Z^s-\dot{q}^sZ^0)=0\,,\tag{19}$$

where $1\le l,\, s\le m-k$. (The equations are expressed via currents, for clarity. Nevertheless, the constraint derivatives of the current Φ depend on symmetry components and their derivatives. There arises, of course, the problem of solution of these equations for concrete situations.)

Equations (17)–(19) enable us to obtain symmetries of the mechanical system via currents: For a regular matrix B denote $\mathcal{B}=\bar{B}^{-1}$. Multiplying the system of equations (19) by the matrix $\mathcal{B}$ we get

$$Z^l-\dot{q}^lZ^0=-\mathcal{B}^{ls}\frac{\partial\Phi}{\partial\dot{q}^s}\,.$$

Putting the obtained expressions for $Z^l-\dot{q}^lZ^0$ into (13) we can express the component Z^0 explicitly. Putting the result into (17)–(19) we finally obtain explicit expressions for the components of symmetries:

$$
\begin{aligned}
Z^0 &= \frac{1}{\bar{L}}\left(\Phi + \mathcal{B}^{ls}\frac{\partial \bar{L}}{\partial \dot{q}^l}\frac{\partial \Phi}{\partial \dot{q}^s}\right), \\
Z^l &= \dot{q}^l Z^0 - \mathcal{B}^{ls}\frac{\partial \Phi}{\partial \dot{q}^s}, \\
\tilde{Z}^l &= \mathcal{B}^{ls}\left(\frac{\partial_c \Phi}{\partial q^s} - \bar{A}_s Z^0 + \left\{\frac{\partial \bar{A}_r}{\partial \dot{q}^s} + \frac{\partial \bar{L}_a}{\partial \dot{q}^s}\varepsilon'_r(g^a)\right\}_{\mathrm{alt}(r,s)}(Z^r - \dot{q}^r Z^0)\right).
\end{aligned}
\tag{20}
$$

The problem of computing symmetries simplifies if we know the currents (constants of motion). This might happen during the process of solving the equations of motion. It is obvious that for vector fields obtained by such a way the verification of conditions (14)–(16) should be made. In particular we take advantage of this simplification in the example presented in Section 4.

3.3 Classification of constraint symmetries

There is a possibility to classify the constraint Noether-type symmetries in the context of constraint equations of motion. For a regular matrix $\bar{B}$ the equations of motion (9) can be written in the explicit form $\ddot{q}^l = -\mathcal{B}^{ls}\bar{A}_s$ (recall that $\mathcal{B} = \bar{B}^{-1}$). The holonomic paths of these equations in Q are integral sections of local vector field belonging to the canonical distribution $\mathcal{C}$ called *constraint semispray* (see [14])

$$
\Gamma = \frac{\partial}{\partial t} + \dot{q}^l\frac{\partial}{\partial q^l} + g^a\frac{\partial}{\partial q^{m-k+a}} + \tilde{\Gamma}^l\frac{\partial}{\partial \dot{q}^l}, \qquad \tilde{\Gamma}^l = -\mathcal{B}^{ls}\bar{A}_s. \tag{21}
$$

The constraint semispray Γ spans a distribution $\mathcal{D}_\Gamma$ of rank one called a *constraint connection*. Let Z be a vector field on Q. It is a *symmetry of equations of motion* of the corresponding nonholonomic mechanical system if $[\Gamma, Z] = f\Gamma$, where $[\Gamma, Z]$ is the Lie bracket of vector fields Γ and Z and $f = f(t, q^\sigma, \dot{q}^l)$ is a function on Q. Let Φ be a current, i.e. quantity conserved along trajectories of the nonholonomic system (not necessarily a Noether-type current). Then $\Gamma(\Phi) = \partial_\Gamma\Phi = 0$. If Z is a symmetry of equations of motion then $[\Gamma, Z](\Phi) = f\Gamma(\Phi) = 0$. On the other hand, $[\Gamma, Z](\Phi) = \partial_\Gamma\partial_Z\Phi - \partial_Z\partial_\Gamma\Phi = \partial_\Gamma(\partial_Z\Phi)$. This means that $\partial_Z\Phi$ is the current as well.

Let Z be a constraint symmetry of a nonholonomic system. Let us discuss possible relationship between distributions spanned by vector fields Γ, Z and $[\Gamma, Z]$. First of all let us answer the question whether and under what conditions a vector field belonging to the distribution $\mathcal{D}_\Gamma$ can be a constraint symmetry. Putting components of the vector field $f\Gamma$, $f = f(t, q^\sigma, \dot{q}^l)$ being a function on Q, into conditions (14)–(16) we obtain

$$
\frac{\mathrm{d}'_c\bar{L}}{\mathrm{d}t} = 0, \quad \frac{\partial_c\bar{L}}{\partial q^l} = 0, \quad \frac{\partial \bar{L}}{\partial \dot{q}^l} = 0, \quad 1 \le l \le m-k.
$$

Because of the relation

$$
\mathrm{d}F = \frac{\mathrm{d}'_c F}{\mathrm{d}t}\,\mathrm{d}t + \frac{\partial_c F}{\partial q^l}\,\omega^l + \frac{\partial F}{\partial \dot{q}^l}\,\mathrm{d}\dot{q}^l + \frac{\partial F}{\partial q^{m-k+a}}\,\varphi^a
$$

for every function $F = F(t, q^\sigma, \dot q^l)$ on Q this means that $\mathrm{d}\bar L \in \operatorname{annih}\mathcal{C}$ and $\bar L$ is constant along the distribution $\mathcal{D}_\Gamma$. In the following considerations we exclude this trivial situation.

Another question is whether and under what conditions the Lie bracket $[\Gamma, Z]$ belongs to the canonical distribution. For general vector fields $\xi, \zeta \in \mathcal{C}$ it holds $i_{[\xi,\zeta]}\varphi^a = -\mathrm{d}\varphi^a(\xi,\zeta)$. As $\mathrm{d}\varphi^a$ need not belong to the constraint ideal $\mathcal{I}_\mathcal{C}$, it is evident that $[\xi,\,\zeta]$ need not belong to $\mathcal{C}$. For $\mathrm{d}\varphi^a$ we obtain from (6)

$$\begin{aligned}\mathrm{d}\varphi^a = &- \varepsilon'_s(g^a)\,\bar\omega^s\wedge \mathrm{d}t + \frac{\partial_c}{\partial q^r}\left(\frac{\partial g^a}{\partial \dot q^s}\right)\bar\omega^s\wedge\bar\omega^r + \frac{\partial^2 g^a}{\partial \dot q^r\partial \dot q^s}\,\bar\omega^s\wedge \mathrm{d}\dot q^r\\ &- \frac{\partial g^a}{\partial q^{m-k+b}}\,\varphi^b\wedge \mathrm{d}t - \frac{\partial}{\partial q^{m-k+b}}\left(\frac{\partial g^a}{\partial \dot q^s}\right)\varphi^b\wedge\bar\omega^s.\end{aligned}$$

Calculating the Lie bracket $[\Gamma,\,Z]$ using relations (20) and (21) we obtain after some technical calculations

$$\begin{aligned}i_{[\Gamma,Z]}\,\varphi^a &= -\mathrm{d}\varphi^a(\Gamma,Z)\\ &= \mathcal{B}^{ls}\Bigg[\varepsilon'_l(g^a)\frac{\partial\Phi}{\partial \dot q^s}\\ &\qquad + \dot q^p\frac{\partial^2 g^a}{\partial \dot q^p\partial \dot q^l}\left(\frac{\partial_c\Phi}{\partial q^s} - \frac{\partial\Phi}{\partial \dot q^v}\mathcal{B}^{rv}\left\{\frac{\partial \bar A_r}{\partial \dot q^s} + \varepsilon'_r(g^a)\frac{\partial \bar L_a}{\partial \dot q^s}\right\}_{\mathrm{alt}(r,s)}\right)\Bigg],\end{aligned}\tag{22}$$

where Φ is the Noether-type current corresponding to the constraint symmetry Z. We can see that for a semiholonomic constraint this condition is fulfilled and thus $[\Gamma,\,Z] \in \mathcal{C}$. For a general linear constraint this conditions reduces to

$$i_{[\Gamma,Z]}\,\varphi^a = \mathcal{B}^{ls}\varepsilon'_l(g^a)\frac{\partial\Phi}{\partial \dot q^s}\,. \tag{23}$$

There can be, of course, special cases with a general constraint for which the condition is fulfilled too. We shall see various situations in the example presented in Section 4.

Now let us discuss the relation of the Lie bracket $[\Gamma,\,Z]$ with respect to distributions spanned by vector fields Γ and Z. Let Φ be again the Noether-type current corresponding to the constraint symmetry Z (not belonging to $\mathcal{D}_\Gamma$). Then $\partial_Z\Phi = 0$ and thus $\partial_{[\Gamma,Z]}\Phi = \partial_\Gamma\partial_Z\Phi - \partial_Z\partial_\Gamma\Phi = 0$. This means that the quantity Φ is conserved along the vector field $[\Gamma,\,Z]$. On the other hand, let ζ be a vector field belonging to the distribution $\mathcal{D}_{(\Gamma,\,Z)}$ spanned by vector fields Γ and Z, i.e. $\zeta = a\Gamma + bZ$, where $a = a(t, q^\sigma,\,\dot q^l)$ and $b = b(t, q^\sigma, \dot q^l)$ are functions on Q. Then $\zeta(\Phi) = \partial_\zeta\Phi = a\,\partial_\Gamma\Phi + b\,\partial_Z\Phi = 0$ and Φ is conserved along the distribution $\mathcal{D}_{(\Gamma,Z)}$. Moreover, because of the relation $[\Gamma, Z](\Phi) = 0$ it is conserved along the distribution $\mathcal{D}$ spanned by vector fields Γ, Z and $[\Gamma, Z]$. There are three possibilities for the relation of a symmetry Z to the vector field Γ:

1) Z is a symmetry of equations of motion, i.e. $[\Gamma,\,Z] = a\Gamma$, $a = a(t, q^\sigma, \dot q^l)$.

2) The Lie bracket of vector fields Γ and Z belongs to the distribution spanned by these vector fields, i.e. $[\Gamma, Z] = a\Gamma + bZ$, where $a = a(t, q^\sigma, \dot q^l)$ and $b = b(t, q^\sigma, \dot q^l)$ are functions on Q.

3) There is no specific relation of the symmetry Z to the vector field Γ.

In the cases 1) and 2) the the distribution spanned by vector fields Γ, Z and $[\Gamma, Z]$ has the rank two, in the case 3) its rank is three. (Recall that this distribution need not be a subdistribution of the canonical distribution $\mathcal{C}$, because $[\Gamma, Z]$ need not belong to $\mathcal{C}$.) We shall derive the conditions under which situations 1) take place. After some tedious technical calculations we obtain components of the vector field $\Xi = [\Gamma, Z]$ for a vector field Z belonging to the canonical distribution (i.e. relations (11) are considered). It holds

$$\begin{aligned}
\Xi &= \Xi^0 \frac{\partial}{\partial t} + \Xi^l \frac{\partial}{\partial q^l} + \Xi^{m-k+a} \frac{\partial}{\partial q^{m-k+a}} + \widetilde{\Xi}^l \frac{\partial}{\partial \dot{q}^l}\,, \qquad (24)\\
\Xi^0 &= -\frac{\mathrm{d}'_c Z^0}{\mathrm{d}t} + \mathcal{B}^{sr} \bar{A}_r \frac{\partial Z^0}{\partial \dot{q}^s}\,,\\
\Xi^l &= -\frac{\mathrm{d}'_c Z^l}{\mathrm{d}t} + \mathcal{B}^{sr} \bar{A}_r \frac{\partial Z^l}{\partial \dot{q}^s} + \widetilde{Z}^l,\\
\Xi^{m-k+a} &= \Xi^0 g^a + (\Xi^l - \dot{q}^l \Xi^0) \frac{\partial g^a}{\partial \dot{q}^l} + \left[(Z^l - \dot{q}^l Z^0) \varepsilon'_l(g^a) - Z^0 \mathcal{B}^{sr} \bar{A}_r \dot{q}^l \frac{\partial g^a}{\partial \dot{q}^l \partial \dot{q}^s} \right],\\
\widetilde{\Xi}^l &= Z^0 \frac{\mathrm{d}'_c}{\mathrm{d}t} (-\mathcal{B}^{lr} \bar{A}_r) + (Z^s - \dot{q}^s Z^0) \frac{\partial_c}{\partial q^s} (-\mathcal{B}^{lr} \bar{A}_r)\\
&\quad + \widetilde{Z}^s \frac{\partial}{\partial \dot{q}^s} (-\mathcal{B}^{lr} \bar{A}_r) - \frac{\mathrm{d}'_c \widetilde{Z}^l}{\mathrm{d}t} + \frac{\partial \widetilde{Z}^l}{\partial \dot{q}^s} \mathcal{B}^{sr} \bar{A}_r\,.
\end{aligned}$$

The requirement $[\Gamma,\, Z] = a\Gamma$ (in such a case the constraint symmetry Z is a symmetry of equations of motion as well) means that there exists a function $a = a(t,\, q^\sigma,\, \dot{q}^l)$ on the constraint submanifold Q such that $\Xi^0 = a$, $\Xi^l = a\dot{q}^l$, $\Xi^{m-k+a} = ag^a$, $\widetilde{\Xi}^l = -\mathcal{B}^{ls} \bar{A}_s$, $1 \le l,\, s \le m-k$, $1 \le a \le k$. This leads to conditions

$$\left(\frac{\mathrm{d}'_c}{\mathrm{d}t} - \mathcal{B}^{sr} \bar{A}_r \frac{\partial}{\partial \dot{q}^s} \right) (Z^l - \dot{q}^l Z^0) - \mathcal{B}^{lr} \bar{A}_r Z^0 - \tilde{Z}^l = 0\,, \qquad (25)$$

$$(Z^l - \dot{q}^l Z^0) \varepsilon'_l(g^a) - \dot{q}^l Z^0 \mathcal{B}^{sr} \bar{A}_r \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} = 0\,, \qquad (26)$$

$$\begin{aligned}
-\frac{\mathrm{d}'_c}{\mathrm{d}t} (\mathcal{B}^{lr} \bar{A}_r Z^0) - (Z^s - \dot{q}^s Z^0) \frac{\partial_c}{\partial q^s} (\mathcal{B}^{lr} \bar{A}_r) + \tilde{Z}^s \frac{\partial}{\partial \dot{q}^s} (\mathcal{B}^{lr} \bar{A}_r)&\\
-\frac{\mathrm{d}'_c \tilde{Z}^l}{\mathrm{d}t} + \frac{\partial \tilde{Z}^l}{\partial \dot{q}^s} (\mathcal{B}^{sr} \bar{A}_r) + \mathcal{B}^{lr} \mathcal{B}^{sp} \bar{A}_r \bar{A}_p \frac{\partial Z^0}{\partial \dot{q}^s} &= 0\,. \qquad (27)
\end{aligned}$$

It is evident that the condition (26) is automatically satisfied if the constraint is semiholonomic. The constraint symmetries (vector fields $Z \in \mathcal{C}$ which are solutions of equations (14)–(16)) are simultaneously symmetries of constraint equations of motion iff they obey the above derived conditions (25)–(27).

4 Example: Chaplygin sleigh

In this section we use the geometrical theory for solving the motion of so called Chaplygin sleigh. This example is exposed in [26], where the motion of Chaplygin sleigh is described in another way without considering the problem of symmetries and conservations laws. We study this problem using our results obtained in Section 3.

4.1 Chaplygin sleigh and its motion

The sleigh consists of a rigid body sliding on the horizontal plane without friction (see the figure 1). The constraint is imposed by a sharp blade placed at a point A

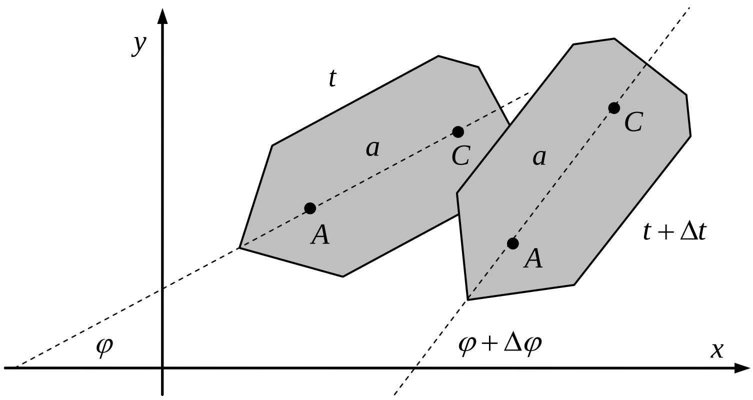

Figure 1: Chaplygin sleigh.

such that the distance between this point and the center of mass of the body C is $AC = a$. The blade prevents the sleigh to move in the direction perpendicular to the straight line AC. The constraint defining the constraint submanifold Q in the fibred chart with coordinates $(t,\, \varphi,\, x,\, y,\, \dot{\varphi},\, \dot{x},\, \dot{y})$, i.e. $m = 3$, reads

$$\dot{y}\cos\varphi - \dot{x}\sin\varphi = 0 \implies \dot{y} = \dot{x}\tan\varphi\,. \tag{28}$$

The canonical embedding $\iota :\, Q \to J^1Y$ has the form

$$\iota :\, Q \ni (t,\, \varphi,\, x,\, y,\, \dot{\varphi},\, \dot{x}) \to (t,\, \varphi,\, x,\, y,\, \dot{\varphi},\, \dot{x},\, \dot{x}\tan\varphi) \in J^1Y\,.$$

The canonical distribution is annihilated by the form φ^1 obtained by putting the constraint equation into the general expression (6). We obtain

$$\varphi^1 = \mathrm{d}y - \tan\varphi\,\mathrm{d}x\,.$$

The unconstrained Lagrangian is $\lambda = L\,\mathrm{d}t$, with

$$L = \frac{1}{2}m\left[(\dot{x} - a\dot{\varphi}\sin\varphi)^2 + (\dot{y} + a\dot{\varphi}\cos\varphi)^2\right] + \frac{1}{2}J\dot{\varphi}^2,$$

where m and J are the mass and inertia (with respect to the axis perpendicular to the coordinate plane xy and going through C) of the sleigh, respectively. Constraint

Lagrangian functions are

$$\bar{L} = L \circ \iota = \frac{m}{2}\left(\frac{\dot{x}^2}{\cos^2\varphi} + a^2k^2\dot{\varphi}^2\right), \tag{29}$$

$$\bar{L}_1 = \frac{\partial L}{\partial \dot{y}} \circ \iota = m(\dot{x}\tan\varphi + a\dot{\varphi}\cos\varphi), \quad k^2 = 1 + \frac{J}{ma^2}.$$

Putting this into (9) we obtain the matrices $\bar{A}$, $\bar{B}$ and $\mathcal{B} = \bar{B}^{-1}$,

$$\bar{A} = \begin{pmatrix} -\frac{ma\dot{\varphi}\dot{x}}{\cos\varphi} & , & \frac{ma\dot{\varphi}^2}{\cos\varphi} - \frac{m\dot{\varphi}\dot{x}\sin\varphi}{\cos^3\varphi} \end{pmatrix}, \tag{30}$$

$$\bar{B} = \begin{pmatrix} -ma^2k^2 & 0 \\ 0 & -\frac{m}{\cos^2\varphi} \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} -\frac{1}{ma^2k^2} & 0 \\ 0 & -\frac{\cos^2\varphi}{m} \end{pmatrix},$$

and the equations of motion

$$0 = -ma^2k^2\ddot{\varphi} - \frac{ma}{\cos\varphi}\dot{\varphi}\dot{x} \qquad \Longrightarrow \qquad \ddot{\varphi} + \frac{\dot{\varphi}\dot{x}}{ak^2\cos\varphi} = 0, \tag{31}$$

$$0 = -\frac{m}{\cos^2\varphi}\ddot{\varphi} - \frac{m\sin\varphi}{\cos^3\varphi}\dot{\varphi}\dot{x} \qquad \Longrightarrow \qquad \ddot{x} - a\dot{\varphi}^2\cos\varphi + \dot{\varphi}\dot{x}\tan\varphi = 0. \tag{32}$$

Solutions of these equations take the following form:

$$\varphi(t) = k \arcsin\tanh\left(\frac{C_1}{k^2}(t - C_2)\right) + C_3, \quad \varphi = k\psi + C_3,$$

$$x(t) = ak^2 \int \cos(k\psi + C_3)\tan\psi\,\mathrm{d}\psi,$$

$$y(t) = ak^2 \int \sin(k\psi + C_3)\tan\psi\,\mathrm{d}\psi,$$

where C_1, C_2 and C_3 are integration constants. Using the initial conditions $\varphi(0) = 0$, $\dot{\varphi}(0) = \omega_0 > 0$, $\dot{x}(0) = 0$ we obtain constants $C_1 = k\omega_0$, $C_2 = 0$, $C_3 = 0$ and the corresponding particular solution

$$\varphi(t) = k \arcsin\tanh\frac{\omega_0 t}{k}, \quad \tan\psi = \sinh\frac{\omega_0 t}{k},$$

$$x(t) = ak^2 \int \cos k\psi \tan\psi\,\mathrm{d}\psi, \tag{33}$$

$$y(t) = ak^2 \int \sin k\psi \tan\psi\,\mathrm{d}\psi.$$

The graphical outputs for some special situations ($a = 1$, $\frac{\omega_0}{k} = 1$, $m = 2$, values $k = 1, 2, 3, 4$) are presented in figures 2–5 for illustration.

Notice that in [26] equivalent equations of motion are obtained for variables u and v representing components of the sleigh velocity with respect to non-inertial reference frame connected with the sleigh, and the variable ω representing the angular velocity $\dot{\varphi}$. The equations of motion are obtained by formulating the second Newton's law in the above mentioned non-inertial reference frame. Thus

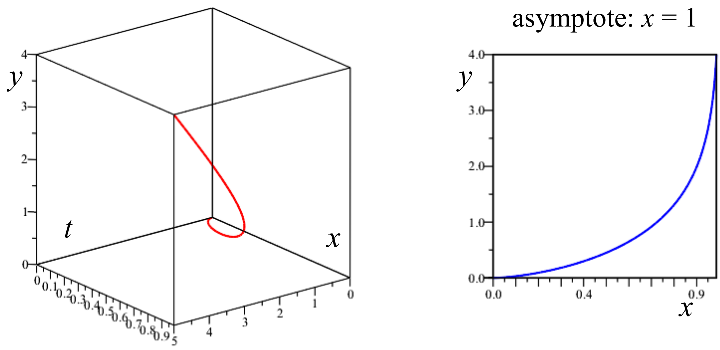

Figure 2: Chaplygin sleigh motion: $k = 1$.

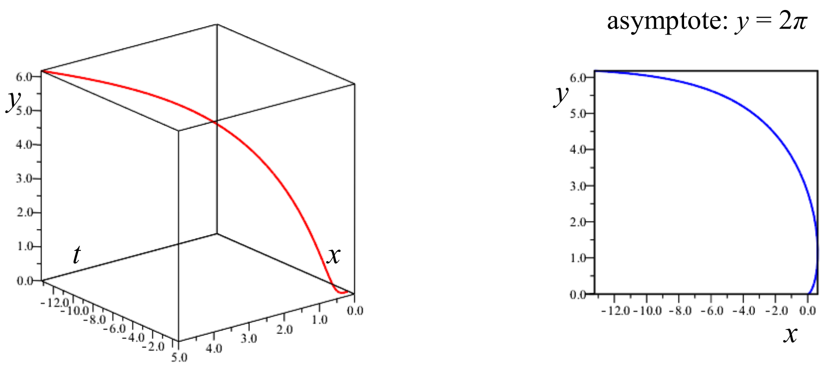

Figure 3: Chaplygin sleigh motion: $k = 2$.

they contain the "fictive" forces $\vec{F}^*$. Moreover, the "reaction" force $\vec{R}$ normal to the straight line AC and representing the constraint is included. Its magnitude is considered as an unknown quantity and it is obtained by solving the equations of motion as well. The solution of these equations of motion is then transformed into the inertial reference frame. Our solution is the same as the last cited one. Recall that in [26] the conservation laws are not discussed.

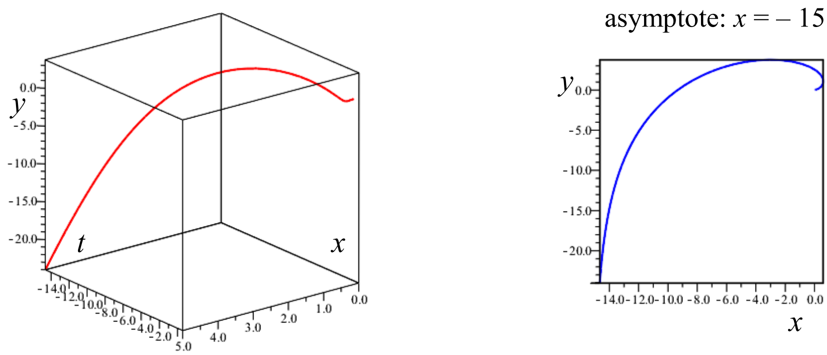

Figure 4: Chaplygin sleigh motion: $k = 3$.

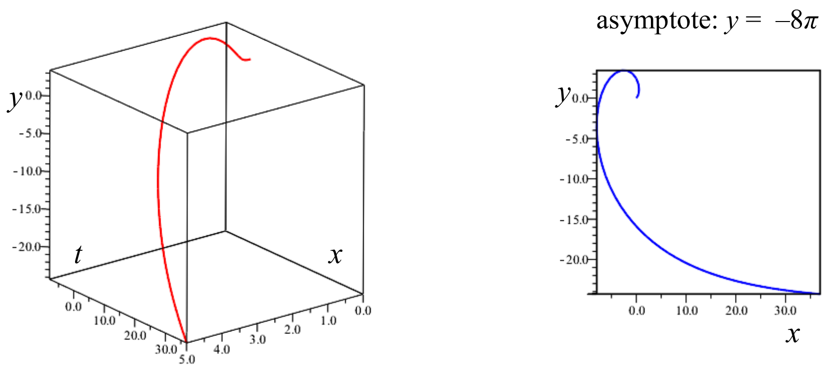

Figure 5: Chaplygin sleigh motion: $k = 4$.

4.2 Constraint symmetries and currents

Putting the expressions (29) for constraint Lagrange functions into equations (17), (18), (19) we obtain

$$\begin{aligned}\frac{\mathrm{d}'_c\Phi}{\mathrm{d}t}-\frac{ma\dot{\varphi}\dot{x}}{\cos\varphi}(Z^\varphi-\dot{\varphi}Z^0)+m\left(\frac{a\dot{\varphi}^2}{\cos\varphi}-\frac{\dot{\varphi}\dot{x}\tan\varphi}{\cos^2\varphi}\right)(Z^x-\dot{x}Z^0)&=0\,,\\ \frac{\partial_c\Phi}{\partial\varphi}+\frac{m\dot{x}^2\tan\varphi}{\cos^2\varphi}Z^0-m\left(\frac{\dot{x}\tan\varphi}{\cos^2\varphi}-\frac{a\dot{\varphi}}{\cos\varphi}\right)Z^x+ma^2k^2\tilde{Z}^\varphi&=0\,,\\ \frac{\partial_c\Phi}{\partial x}+m\left(\frac{\dot{x}\tan\varphi}{\cos^2\varphi}-\frac{a\dot{\varphi}}{\cos\varphi}\right)Z^\varphi+\frac{m}{\cos^2\varphi}\tilde{Z}^x&=0\,,\\ \frac{\partial\Phi}{\partial\dot{\varphi}}-ma^2k^2(Z^\varphi-\dot{\varphi}Z^0)&=0\,,\\ \frac{\partial\Phi}{\partial\dot{x}}-\frac{m}{\cos^2\varphi}(Z^x-\dot{x}Z^0)&=0\,.\end{aligned}\tag{34}$$

Expressing the components $(Z^\varphi-\dot{\varphi}Z^0)$ and $(Z^x-\dot{x}Z^0)$ from the last two of these equations, putting them into the first equation and substituting $v=\frac{\dot{x}}{\cos\varphi}$ we obtain

$$\left(\frac{\partial}{\partial t}+\dot{\varphi}\frac{\partial}{\partial\varphi}+v\cos\varphi\frac{\partial}{\partial x}+v\sin\varphi\frac{\partial}{\partial y}-\frac{\dot{\varphi}v}{ak^2}\frac{\partial}{\partial\dot{\varphi}}+a\dot{\varphi}^2\frac{\partial}{\partial v}\right)\Phi=0\,.\tag{35}$$

So, we have the characteristics ODE's

$$\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}\varphi}{\dot{\varphi}}=\frac{\mathrm{d}x}{v\cos\varphi}=\frac{\mathrm{d}y}{v\sin\varphi}=-ak^2\frac{\mathrm{d}\dot{\varphi}}{\dot{\varphi}v}=\frac{\mathrm{d}v}{a\dot{\varphi}^2}.$$

Integrating the last equation we obtain

$$\frac{1}{2}v^2+\frac{1}{2}a^2k^2\dot{\varphi}^2=\text{const.},\quad\text{i.e.}\quad\frac{1}{2}\frac{\dot{x}^2}{\cos^2\varphi}+\frac{1}{2}a^2k^2\dot{\varphi}^2=\text{const.}$$

This quantity multiplied by the sleigh mass m represents the total mechanical energy E_0 of the sleigh which is the sum of the translational energy $E_T=\frac{1}{2}\frac{m\dot{x}^2}{\cos^2\varphi}$ and the rotational energy $E_R=\frac{1}{2}(J+ma^2)\dot{\varphi}^2$ with respect to the vertical axis going through the point A. Recall that due to the Steiner theorem $J+ma^2$ is the inertia of the sleigh with respect to this axis. More precisely, the total mechanical energy of the sleigh expressed via the components of the velocity of the center od mass $(x_C,\,y_C)$ is

$$E=\frac{m}{2}\left(\dot{x}_C^2+\dot{y}_C^2\right)+\frac{1}{2}J\dot{\varphi}^2.$$

Taking into account that $x_C=x+a\cos\varphi$, $y_C=y+a\sin\varphi$ and considering the constraint we can immediately see that $E=E_0$. For the particular solution of equations of motion presented in the previous section we have

$$E_0=\frac{1}{2}ma^2k^2\omega_0^2,\quad C_1=k\omega_0=\sqrt{\frac{2E_0}{ma^2}}\,.$$

The corresponding conserved current can be obtained using the equations of motion and the fact that the constrained Lagrange function $\bar{L}$ does not depend on time explicitly,

$$\Phi_1 = -\frac{m}{2}\left(\frac{\dot{x}^2}{\cos^2\varphi} + a^2k^2\dot{\varphi}^2\right) . \tag{36}$$

Putting this expression into equations (20) we can verify that the corresponding symmetry is $Z = \frac{\partial}{\partial t}$. Taking into account the solution of equations of motion (section 4.2) we obtain the following expressions for the translational and rotational energy and the angle φ as functions of time (see also figure 6):

$$E_T = E_0 \tanh^2 \frac{\omega_0 t}{k}, \quad E_R = E_0 \cosh^{-2} \frac{\omega_0 t}{k}, \quad \sin \frac{\varphi}{k} = \tanh \frac{\omega_0 t}{k} . \tag{37}$$

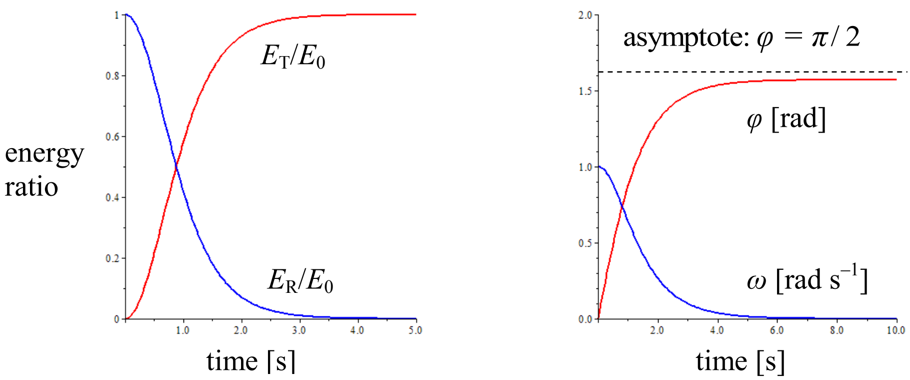

Figure 6: Conservation of energy, damping of rotation.

The graphs show the asymptotic behavior of the sleigh motion: the translational motion accelerates at the expense of the rotational motion which is asymptotically damped.
The decomposition of the energy into the term corresponding to translational motion of the point A and the energy corresponding to the rotation of the sleigh around the axis going through this point is "induced" by the formulation of the problem itself (the constraint concerns the motion of the point A). On the other hand, more correct from the point of view of physics is the energy decomposition into the translational energy of the center of mass C, $E_{T,C} = \frac{1}{2}m(\dot{x}_C^2 + \dot{y}_C^2)$, and the rotational energy of the sleigh with respect to the center of mass, $E_{R,C} = \frac{1}{2}J\dot{\varphi}^2$.

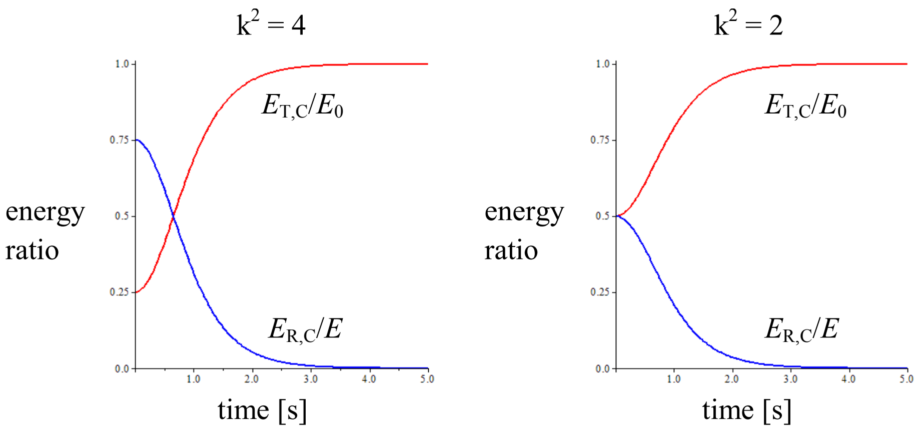

Figure 7: Energy decomposition with respect to the center of mass.

Considering the solution of equations of motion we obtain

$$E_{T,C} = E_0 \left(\tanh^2 \frac{\omega_0 t}{k} + \frac{1}{k^2 \cosh^2 \frac{\omega_0 t}{k}} \right), \; E_{R,C} = \frac{E_0}{\cosh^2 \frac{\omega_0 t}{k}} \left(1 - \frac{1}{k^2} \right). \tag{38}$$

Figure 7 shows the behavior of both types of kinetic energy during the time for two different values k. Relations (37) represent the limit case of (38) for $J \gg ma^2$, i.e. $k \to \infty$, as expected. Notice that for $k = 1$ (zero inertia with respect to the center of mass, or, more exactly, $J \ll ma^2$) we have $E_{T,C} = E_0$ and $E_{R,C} = 0$. This result is not in contradiction with the initial conditions. $E_{R,C}$ vanishes because of zero inertia, even though $\omega_0 \neq 0$. (Figures 6 and 7 are drawn for $\omega_0/k = 1$ for simplicity.)

Expressing the quantities C_2 and C_3 (the fact that they are zeros for the chosen initial conditions does not affect their general meaning of integration constants) we obtain the following currents

$$\Phi_2 = \frac{m\dot{x}}{\cos\varphi} \sin\left(\frac{\varphi}{k}\right) + mak\dot{\varphi}\cos\left(\frac{\varphi}{k}\right)$$

$$\Phi_3 = \frac{1}{2}ma^2k^2 \ln \left(\frac{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2} + \frac{\dot{x}}{a\cos\varphi}}{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2} - \frac{\dot{x}}{a\cos\varphi}} \right) - ma^2t\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2},$$

and in shortened notation with help of energies

$$\Phi_3 = \frac{1}{2}ma^2k^2 \ln \frac{\sqrt{E_0} + \sqrt{E_T}}{\sqrt{E_0} - \sqrt{E_T}} - at\sqrt{2mE_0}$$

For the special case of zero inertia J, i.e. $k = 1$, the current Φ_2 represents the y-component of the impulse of the sleigh, $p_{C,y} = m\dot{y}_C = m(\dot{x}\tan\varphi + a\dot{\varphi}\cos\varphi)$.

(We shall see later that in such a case the component $p_{C,x}$ must be conserved as well.)

The corresponding symmetries are (denoting $\psi = \frac{\varphi}{k}$ as above)

$$\begin{aligned} Z(\Phi_2) &= \frac{1}{ak}\cos\psi\frac{\partial}{\partial\varphi} + \cos\varphi\sin\psi\frac{\partial}{\partial x} + \sin\varphi\sin\psi\frac{\partial}{\partial y} \\ &\quad - \frac{\dot{x}\cos\psi}{a^2k^3\cos\varphi}\frac{\partial}{\partial\dot{\varphi}} + \frac{1}{ak}\left(a\dot{\varphi}\cos\varphi - \dot{x}\tan\varphi\right)\cos\psi\frac{\partial}{\partial\dot{x}}\,, \\ Z(\Phi_3) &= \frac{k^2}{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2}\ln\left(\frac{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2} + \frac{\dot{x}}{a\cos\varphi}}{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2} - \frac{\dot{x}}{a\cos\varphi}}\right)\Gamma \\ &\quad - \frac{\dot{\varphi}t + \frac{\dot{x}}{a\dot{\varphi}\cos\varphi}}{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2}}\left(\frac{\partial}{\partial\varphi} - \left(a\dot{\varphi}\cos\varphi - \dot{x}\tan\varphi\right)\frac{\partial}{\partial\dot{x}}\right) \\ &\quad + \frac{ak^2\cos\varphi - \dot{x}t}{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2}}\left(\frac{\partial}{\partial x} + \tan\varphi\frac{\partial}{\partial y} + \frac{\dot{\varphi}}{ak^2\cos\varphi}\frac{\partial}{\partial\dot{\varphi}}\right)\,, \end{aligned}$$

or, in shortened notation via energies

$$\begin{aligned} Z(\Phi_3) &= \frac{ma^2k^2}{2E_0}\ln\left(\frac{\sqrt{E_0} + \sqrt{E_T}}{\sqrt{E_0} - \sqrt{E_T}}\right)\Gamma \\ &\quad - \sqrt{\frac{ma^2}{2E_0}}\left(\dot{\varphi}t + \frac{\dot{x}}{a\dot{\varphi}\cos\varphi}\right)\left(\frac{\partial}{\partial\varphi} - \left(a\dot{\varphi}\cos\varphi - \dot{x}\tan\varphi\right)\frac{\partial}{\partial\dot{x}}\right) \\ &\quad + \sqrt{\frac{ma^2}{2E_0}}\left(ak^2\cos\varphi - \dot{x}t\right)\left(\frac{\partial}{\partial x} + \tan\varphi\frac{\partial}{\partial y} + \frac{\dot{\varphi}}{ak^2\cos\varphi}\frac{\partial}{\partial\dot{\varphi}}\right) \end{aligned}$$

where the vector field Γ reads

$$\Gamma = \frac{\partial}{\partial t} + \dot{\varphi}\frac{\partial}{\partial\varphi} + \dot{x}\frac{\partial}{\partial x} + \dot{x}\tan\varphi\frac{\partial}{\partial y} - \frac{\dot{\varphi}\dot{x}}{ak^2\cos\varphi}\frac{\partial}{\partial\dot{\varphi}} + \left(a\dot{\varphi}^2\cos\varphi - \dot{\varphi}\dot{x}\tan\varphi\right)\frac{\partial}{\partial\dot{x}}\,,$$

which is the vector field representing the equations of motion on the submanifold Q. Keep in mind that the above presented shortened notation via energies is given only for better clarity. For eventual further calculations the full expression in coordinates $(t,\,\varphi,\,x,\,y,\,\dot{\varphi},\,\dot{x})$ on Q must be used, i.e. it is necessary to put

$$E_0 = \frac{ma^2}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2\right), \quad E_T = \frac{m\dot{x}^2}{2\cos^2\varphi}, \quad \psi = \frac{\varphi}{k}$$

into corresponding expressions.

The equations (14)–(16) take the form

$$\begin{aligned} &-\frac{\dot{x}\dot{\varphi}}{a\cos\varphi}Z^{\varphi} + \left(\frac{\dot{x}\dot{\varphi}\sin\varphi}{a^2\cos^3\varphi} + \frac{\dot{\varphi}^2}{a\cos\varphi}\right)Z^x \\ &\quad - \frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2\right)\frac{\mathrm{d}'_c Z^0}{\mathrm{d}t} + k^2\dot{\varphi}\,\frac{\mathrm{d}'_c Z^{\varphi}}{\mathrm{d}t} + \frac{\dot{x}}{a^2\cos^2\varphi}\,\frac{\mathrm{d}'_c Z^x}{\mathrm{d}t} = 0\,, \qquad (39) \end{aligned}$$

$$\left(\frac{\dot{x}\sin\varphi}{a^2\cos^3\varphi}+\frac{\dot{\varphi}}{a\cos\varphi}\right)Z^x+k^2\tilde{Z}^\varphi$$
$$-\frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos\varphi}+k^2\dot{\varphi}^2\right)\frac{\partial_c Z^0}{\partial\varphi}+k^2\dot{\varphi}\frac{\partial_c Z^\varphi}{\partial\varphi}+\frac{\dot{x}}{a^2\cos^2\varphi}\frac{\partial_c Z^x}{\partial\varphi}=0\,, \qquad (40)$$
$$\left(\frac{\dot{x}\sin\varphi}{a^2\cos^3\varphi}-\frac{\dot{\varphi}}{a\cos\varphi}\right)Z^\varphi+\frac{1}{a^2\cos^2\varphi}\tilde{Z}^x$$
$$-\frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi}+k^2\dot{\varphi}^2\right)\frac{\partial_c Z^0}{\partial x}+k^2\dot{\varphi}\frac{\partial_c Z^\varphi}{\partial x}+\frac{\dot{x}}{a^2\cos^2\varphi}\frac{\partial_c Z^x}{\partial x}=0\,, \qquad (41)$$
$$-\frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi}+k^2\dot{\varphi}^2\right)\frac{\partial Z^0}{\partial\dot{\varphi}}+k^2\dot{\varphi}\frac{\partial Z^\varphi}{\partial\dot{\varphi}}+\frac{\dot{x}}{a^2\cos^2\varphi}\frac{\partial Z^x}{\partial\dot{\varphi}}=0\,, \qquad (42)$$
$$-\frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi}+k^2\dot{\varphi}^2\right)\frac{\partial Z^0}{\partial\dot{x}}+k^2\dot{\varphi}\frac{\partial Z^\varphi}{\partial\dot{x}}+\frac{\dot{x}}{a^2\cos^2\varphi}\frac{\partial Z^x}{\partial\dot{x}}=0\,. \qquad (43)$$

Putting the components of vector fields $Z(\Phi_1)$, $Z(\Phi_2)$ and $Z(\Phi_3)$ into equations (39)–(43) we can verify that they are constraint symmetries. Thus Φ_1, Φ_2 and Φ_3 are Noether-type currents. Nevertheless, the physical interpretation of symmetries $Z(\Phi_2)$ and $Z(\Phi_3)$ and their currents is not completely clear in a general situation.

The relation for the current Φ_2 is linear in variables velocity and angular velocity. This enables us to conclude that for a general description of the sleigh motion it is satisfactory to consider special initial conditions $\dot{x}(0)=0$ and $\dot{\varphi}(0)=\omega(0)\neq 0$. If $v(0)\neq 0$ and $\dot{\varphi}=\omega_0$, then $v(\tau)=0$ and $\dot{\varphi}(\tau)=\Omega_0\neq\omega_0$ at some other time τ.

Calculating $[\Gamma,\, Z]$ for all three obtained symmetries $Z(\Phi_1)$, $Z(\Phi_2)$ and $Z(\Phi_3)$ we can see that only the symmetry $Z(\Phi_1)=\frac{\partial}{\partial t}$ is simultaneously the symmetry of constrained (reduced) equations of motion. Concretely, it is evident that $[\Gamma,\, \frac{\partial}{\partial t}]=0$. Moreover, using the condition (23) we can check that it holds

$$i_{[\Gamma,Z_1]}\varphi^1=0\,, \qquad i_{[\Gamma,Z_2]}\varphi^1=-\frac{\dot{x}}{ak\cos^2\varphi}\cos\frac{\varphi}{k}+\frac{\dot{\varphi}}{\cos\varphi}\sin\frac{\varphi}{k}\,,$$

$$i_{[\Gamma,Z_3]}\varphi^1=-\frac{a}{\dot{\varphi}\cos\varphi}\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi}+k^2\dot{\varphi}^2}.$$

This means that for the symmetry Z_1 the vector field $[\Gamma, Z_1]$ belongs to the canonical distribution unlike the vector fields $[\Gamma, Z_2]$ and $[\Gamma, Z_3]$.

4.3 Chetaev equations and constraint forces

Finally, let us express Chetaev equations of motion and the constraint forces as exposed in section 2.4 (equations (10)). Rewriting the constraint as

$$f(t,\,\varphi,\,x,\,y,\,\dot{\varphi},\,\dot{x},\,\dot{y})\equiv\dot{y}-\dot{x}\tan\varphi=0$$

we obtain the following equations:

$$\begin{aligned} -mak^2\ddot{\varphi} + m\ddot{x}\sin\varphi - m\ddot{y}\cos\varphi &= \frac{\mu}{a}\frac{\partial f}{\partial\dot{\varphi}}\,, & \frac{\partial f}{\partial\dot{\varphi}} &= 0\,, \\ ma\ddot{\varphi}\sin\varphi - m\ddot{x} + ma\dot{\varphi}^2\cos\varphi &= \mu\frac{\partial f}{\partial\dot{x}}\,, & \frac{\partial f}{\partial\dot{x}} &= -\tan\varphi\,, \\ -ma\ddot{\varphi}\cos\varphi - m\ddot{y} + ma\dot{\varphi}^2\sin\varphi &= \mu\frac{\partial f}{\partial\dot{y}}\,, & \frac{\partial f}{\partial\dot{y}} &= 1\,, \end{aligned} \tag{44}$$

μ being a Lagrange multiplier. The constraint force is

$$\phi = \mu\left(\frac{1}{a}\frac{\partial f}{\partial\dot{\varphi}}, \frac{\partial f}{\partial\dot{x}}, \frac{\partial f}{\partial\dot{y}}\right) = \mu(0, -\tan\varphi, 1)\,. \tag{45}$$

It has a clear physical meaning in the reference frame connected with the point A and rotating with the sleight: Denote $\vec{r}\,' = (0,\, a\cos\varphi,\, a\sin\varphi)$, $\vec{\omega} = (\dot{\varphi},\, 0,\, 0)$, $\vec{\varepsilon} = (\ddot{\varphi},\, 0,\, 0)$, $\vec{A}(0,\, \ddot{x},\, \ddot{y})$. (Note that $\vec{r}\,'$ determines the position of the center of mass C of the sleigh with respect to the point A.) Denoting ϕ as $\vec{F}^*$ as it is usual in physics, we obtain

$$\vec{F}^* = \left(ma\ddot{\varphi}\sin\varphi - m\ddot{x} + ma\dot{\varphi}^2\cos\varphi,\, -ma\ddot{\varphi}\cos\varphi - m\ddot{y} + ma\dot{\varphi}^2\sin\varphi,\, 0\right)\,,$$

$$\vec{F}^* = -m\vec{\varepsilon}\times\vec{r}\,' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}\,') - m\vec{A}\,. \tag{46}$$

This force is the sum of three terms: the Euler force, the centrifugal force and the translational force. The Coriolis force is missing because the velocity of the center of mass with respect to the reference system connected with the point A is zero.

Using the constraint to write $\ddot{y} = \ddot{x}\tan\varphi + \frac{\dot{\varphi}\dot{x}}{\cos^2\varphi}$ and substituting into (44) we obtain after some calculations the Lagrange multiplier μ and the constraint force ϕ:

$$\mu = -\frac{mJ}{J+ma^2}\dot{\varphi}\dot{x}, \quad \phi = \frac{mJ}{J+ma^2}\,(0,\, \dot{\varphi}\dot{x}\tan\varphi,\, -\dot{\varphi}\dot{x})\,. \tag{47}$$

Notice that these forces are not variational in the sense of e.g. [17], [25], [27], [32]. Thus the Chaplygin sleigh cannot be alternatively described as an unconstrained variational system with an appropriately modified Lagrangian. For $k = 1$ the constraint force vanishes. This is consistent with the (non-realistic, of course) limit case $J \to 0$ in relations (38): The motion of the center of mass is uniform and straightforward (both components of the impulse of the center of mass are conserved), while the sleigh rotates around it with the initial angular velocity ω_0 but with zero energy due to $J = 0$.

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References

[1] L.Y. Bahar: A unified approach to non-holonomic dynamics. *Int. J. Non-Linear Mech.* 35 (2000) 613–625.

[2] A.M. Bloch, (with the collaboration of J. Baillieul, P.E. Crouch and J.E. Marsden): *Nonholonomic Mechanics and Control, Interdisciplinary Applied Mathematics 24.* Springer Science + Business Media, LLC (2003).

[3] F. Bullo, A.D. Lewis: *Geometric Control of Mechanical Systems, Texts in Applied Mathematics 49.* Springer Science + Business Media, Inc., New York (2005).

[4] F. Cantrijn: Vector fields generating invariants for classical dissipative systems. *J. Math. Phys.* 23 (1982) 1589–1595.

[5] M. Čech, J. Musilová: Symmetries and conservation laws for Chaplygin sleigh. Balkan J. Geom. Appl. Submitted.

[6] N.G. Chetaev: On the Gauss principle. *Izv. Kazan Fiz.-Mat. Obsc.* 6 (1932–1933) 323–326. (in Russian).

[7] J. Cortés Monforte: Geometric, Control and Numerical Aspects of Nonholonomic Systems, Lecture Notes in Mathematics 1793. (2002).

[8] L. Czudková, J. Musilová: A practical application of the geometrical theory on fibred manifolds to a planimeter motion. *Int. J. Non-Linear Mech.* 50 (2012) 19–24.

[9] M. de León, J.C. Marrero, D. Martín de Diego: Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic mechanics. *J. Geom. Mech.* 2 (2010) 159–198. (See also arXiv: 0801.4358v3 [mat-ph] 13 Nov 2009.)

[10] J. Janová, J. Musilová: Non-holonomic mechanics: A geometrical treatment of general coupled rolling motion. *Int. J. Non-Linear Mech.* 44 (2009) 98–105.

[11] J. Janová, J. Musilová: The streetboard rider: an appealing problem in non-holonomic mechanics. *Eur. J. Phys.* 31 (2010) 333–345.

[12] J. Janová, J. Musilová: Coupled rolling motion: considering rolling friction in non-holonomic mechanics. *Eur. J. Phys.* 32 (2011) 257–269.

[13] J. Janová, J. Musilová, J. Bartoš: Coupled rolling motion: a student project in non-holonomic mechanics. *Eur. J. Phys.* 30 (2010) 1257–1269.

[14] O. Krupková: Mechanical systems with nonholonomic constraints. *J. Math. Phys.* 38 (1997) 5098–5126.

[15] O. Krupková: Higher order mechanical systems with nonholonomic constraints. *J. Math. Phys.* 41 (2000) 5304–5324.

[16] O. Krupková: Recent results in the geometry of constrained systems. *Rep. Math. Phys.* 49 (2002) 269–278.

[17] O. Krupková: Variational metric structures. *Publ. Math. Debrecen* 62 (3–4) (2003) 461–495.

[18] O. Krupková: Noether Theorem, 90 years on. In: F. Etayo et al.: *XVII. International Fall Workshop.* American Institute of Physics (2009) 159–170.

[19] O. Krupková: The nonholonomic variational principle. *J. Phys. A: Math. Theor.* 42 (2009). 185201 (40pp)

[20] O. Krupková: The geometric mechanics on nonholonomic submanifolds. *Comm. Math.* 18 (2010) 51–77.

[21] O. Krupková, J. Musilová: The relativistic particle as a mechanical system with non-holonomic constraints. *J. Phys. A: Math. Gen.* 34 (2001) 3859–3875.

[22] O. Krupková, J. Musilová: Nonholonomic variational systems. *Rep. Math. Phys.* 55 (2) (2005) 211–220.

[23] E. Massa, E. Pagani: Classical mechanics of non-holonomic systems: a geometric approach. *Ann. Inst. Henri Poincaré* 55 (1991) 511–544.

[24] E. Massa, E. Pagani: A new look at classical mechanics of constrained systems. *Ann. Inst. Henri Poincaré* 66 (1997) 1–36.

[25] M. Mráz, J. Musilová: Variational compatibility of force laws in mechanics. In: I. Kolář el al.: *Differential Geometry and its Applications.* Masaryk Univ., Brno (1999) 553–560.

[26] Ju.I. Neimark, N.A. Fufaev: *Dynamics of Nonholonomic Systems, Translations of Mathematical Monographs, vol. 33.* American Mathematical Society, Rhode Island (1972).

[27] J. Novotný: On the inverse variational problem in the classical mechanics. In: O. Kowalski: *Proc. Conf. on Diff. Geom. and Its Appl. 1980.* Universita Karlova, Prague (1981) 189–195.

[28] P. Popescu, Ch. Ida: Nonlinear constraints in nonholonomic mechanics. arXiv: submit/1026356 [marh-ph] 20 Jul 2014.

[29] C.M. Roithmayr, D.H. Hodges: Forces associated with non-linear non-holonomic constraint equations. *Int. J. Non-Linear Mech.* 45 (2010) 357–369.

[30] O. Rossi, J. Musilová: On the inverse variational problem in nonholonomic mechanics. *Comm. Math.* 20 (1) (2012) 41–62.

[31] O. Rossi, J. Musilová: The relativistic mechanics in a nonholonomic setting: A unified approach to particles with non-zero mass and massless particles. *J. Phys A: Math. Theor.* 45 (2012). 255202

[32] O. Rossi, R. Paláček: On the Zermelo problem in Riemannian manifolds. *Balkan Journal of Geometry and Its Applications* 17 (2) (2012) 77–81.

[33] W. Sarlet, F. Cantrijn: Special symmetries for Lagrangian systems snd their analogues in nonconservative mechanics. In: D. Krupka: *Difrerential Geometry and its Applications. Proc. Conf. Nové Město na Moravě, Czechoslovakia, September 1983.* J.E. Purkyně University, Brno (1984) 247–260.

[34] W. Sarlet, F. Cantrijn, D.J. Saunders: A geometrical framework for the study of non-holonomic Lagrangian systems. *J. Phys. A: Math. Gen.* 28 (1995) 3253–3268.

[35] W. Sarlet, D.J. Saunders, F. Cantrijn: A geometrical framework for the study of non-holonomic Lagrangian systems II. *J. Phys. A: Math. Gen.* 29 (1996) 4265–4274.

[36] W. Sarlet, D.J. Saunders, F. Cantrijn: Adjoint symmetries and the generation of first integrals in non-holonomic mechanics. *Journal of Geometry and Physics* 55 (2005) 207–225.

[37] M. Swaczyna: Several examples of nonholonomic mechanical systems. *Comm. Math.* 19 (2011) 27–56.

[38] M. Swaczyna, P. Volný: Uniform projectile motion: Dynamics, symmetries and conservation laws. *Rep. Math. Phys.* 73 (2) (2014) 177–200.

[39] F.E. Udwadia: Equations of motion for mechanical systems: A unified approach. *Int. J. Non-Linear Mech.* 31 (1996) 951–958.

[40] F.E. Udwadia, R.E. Kalaba: On the foundations of analytical dynamics. *Int. J. Non-Linear Mech.* 37 (2002) 1079–1090.

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