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EDGELESS GRAPHS ARE THE ONLY UNIVERSAL FIXERS

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Abstract. Given two disjoint copies of a graph G, denoted G^1 and G^2, and a permutation π of $V(G)$, the graph πG is constructed by joining $u \in V(G^1)$ to $\pi(u) \in V(G^2)$ for all $u \in V(G^1)$. G is said to be a universal fixer if the domination number of πG is equal to the domination number of G for all π of $V(G)$. In 1999 it was conjectured that the only universal fixers are the edgeless graphs. Since then, a few partial results have been shown. In this paper, we prove the conjecture completely.

Keywords: universal fixer; domination

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1. Definitions and notation

We consider only finite, simple, undirected graphs. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. The order of G, denoted by $|G|$, is the cardinality of $V(G)$. We will denote the graph consisting of n isolated vertices as $\overline{K_n}$. The *open neighborhood* of $v \in V(G)$ is $N(v) = \{u;\ uv \in E(G)\}$, and the *open neighborhood* of a subset D of vertices is $N(D) = \bigcup\limits_{v \in D} N(v)$. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$, and the closed neighborhood of a subset D of vertices is $N[D] = N(D) \cup D$. A set $S \subseteq V(G)$ is a *2-packing* of G if $N[x] \cap N[y] = \emptyset$ for every pair of distinct vertices x and y in S.

Given two sets A and B of $V(G)$, we say A *dominates* B if $B \subseteq N[A]$, and a set $D \subseteq V(G)$ dominates G if $V(G) = N[D]$. The domination number, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. A γ*-set* of G is a dominating set of G of cardinality $\gamma(G)$.

Given a graph G and any permutation π of $V(G)$, the *prism of G with respect to π* is the graph πG obtained by taking two disjoint copies of G, denoted G^1 and G^2, and joining every $u \in V(G^1)$ with $\pi(u) \in V(G^2)$. That is, the edges between G^1 and G^2 form a perfect matching in πG. For any subset $A \subseteq V(G)$, we let $\pi(A) = \bigcup\limits_{v \in A} \pi(v)$.

If π is the identity $\mathbf{1}_G$, then $\pi G \cong G \,\Box\, K_2$, the *Cartesian product* of G and K_2. The graph $G \,\Box\, K_2$ is often referred to as the *prism of* G, and the domination number of this graph has been studied by Hartnell and Rall in [6].

One can easily verify that $\gamma(G) \leqslant \gamma(\pi G) \leqslant 2\gamma(G)$ for all π of $V(G)$. If $\gamma(\pi G) = \gamma(G)$ for some permutation π of $V(G)$, then we say G is a π*-fixer*. If G is a $\mathbf{1}_G$-fixer, then G is said to be a *prism fixer*. Moreover, if $\gamma(\pi G) = \gamma(G)$ for all π, then we say G is a *universal fixer*.

In 1999, Gu [4] conjectured that a graph G of order n is a universal fixer if and only if $G = \overline{K_n}$. Clearly if $G = \overline{K_n}$, then for any π of $V(G)$ we have $\gamma(\pi G) = n = \gamma(G)$. It is the other direction, the question of whether the edgeless graphs are the only universal fixers, that is far more interesting and is the focus of this paper. Over the past decade, it has been shown that a few classes of graphs do not contain any universal fixers. In particular, given a nontrivial connected graph G, Gibson [3] showed that there exists some π such that $\gamma(G) \neq \gamma(\pi G)$ if G is bipartite. Cockayne, Gibson, and Mynhardt [2] later proved this to be true when G is claw-free. Mynhardt and Xu [7] also showed if G satisfies $\gamma(G) \leqslant 3$, then G is not a universal fixer. Other partial results can be found in [1], [5]. The purpose of this paper is to prove Gu's conjecture, which we state as the following theorem.

Theorem 1.1. *A graph G of order n is a universal fixer if and only if $G = \overline{K_n}$.*

Although the following observation is stated throughout the literature, we give a short proof here for the sake of completeness.

Observation 1.2. Let G be a disconnected graph that contains at least one edge. If G is a universal fixer, then every component of G is a universal fixer.

Proof. Let G be a disconnected graph containing at least one edge, and let $C_1, \ldots, C_k$ represent the components of G where $k \geqslant 2$. Suppose, for some $j \in \{1, \ldots, k\}$, that C_j is not a universal fixer. There exists a permutation $\pi_j \colon V(C_j) \to V(C_j)$ such that $\gamma(\pi_j C_j) > \gamma(C_j)$. Now define $\pi \colon V(G) \to V(G)$ by

$$\pi(x) = \begin{cases} x & \text{if } x \in V(G) \setminus V(C_j), \\ \pi_j(x) & \text{if } x \in V(C_j). \end{cases}$$

Note that πG is a disconnected graph which can be written as the disjoint union

$$\Big(\bigcup_{i \neq j} C_i \,\Box\, K_2\Big) \cup\ \pi_j C_j.$$

Thus,

$$\begin{aligned}\gamma(\pi G) &= \gamma\left(\bigcup_{i\neq j} C_i \,\Box\, K_2\right) + \gamma(\pi_j C_j)\\ &> \sum_{i\neq j} \gamma(C_i \,\Box\, K_2) + \gamma(C_j)\\ &\geqslant \gamma(G).\end{aligned}$$

Therefore, if there exists a permutation π of a component C_j of G such that C_j is not a π-fixer, then G is not a universal fixer. The result follows. □

Observation 1.2 allows us to consider only nontrivial connected graphs. Therefore, we focus on proving the following theorem.

Theorem 1.3. *If a connected graph G is a universal fixer, then $G = K_1$.*

The remainder of the paper is organized as follows. Section 2 is dedicated to previous results that will be useful in the proof of Theorem 1.3. The proof of Theorem 1.3 is given in Section 3.

2. Known results

In order to study π-fixers, we will make use of the following results.

Lemma 2.1 ([7])**.** *Let G be a connected graph of order $n \geqslant 2$ and π a permutation of $V(G)$. Then $\gamma(\pi G) = \gamma(G)$ if and only if G has a γ-set D such that*

(a) *D admits a partition $D = D_1 \cup D_2$ where D_1 dominates $V(G) \setminus D_2$;*
(b) *$\pi(D)$ is a γ-set of G and $\pi(D_2)$ dominates $V(G) \setminus \pi(D_1)$.*

Note that if a graph G is a universal fixer, then G is also a prism fixer. So applying Lemma 2.1 to the permutation $\mathbf{1}_G$, we get the following type of γ-set.

Definition 2.2. A γ-set D of G is said to be *symmetric* if D admits a partition $D = D_1 \cup D_2$ where

1. D_1 dominates $V(G) \setminus D_2$, and
2. D_2 dominates $V(G) \setminus D_1$.

We write $D = [D_1, D_2]$ to emphasize properties 1 and 2 of this partition of D.

The following two results were shown by Hartnell and Rall [6], where some statements are in a slightly different form.

Lemma 2.3 ([6]). *If $D = [D_1, D_2]$ is a symmetric γ-set of G, then:*

(a) *D is independent.*

(b) *G has minimum degree at least 2.*

(c) *D_1 and D_2 are maximal 2-packings of G.*

(d) *For $i \in \{1, 2\}$, $\sum\limits_{x \in D_i} \deg x = |V(G)| - \gamma(G)$.*

Theorem 2.4 ([6]). *The conditions below are equivalent for any nontrivial, connected graph G.*

(a) *G is a prism fixer.*

(b) *G has a symmetric γ-set.*

(c) *G has an independent γ-set D that admits a partition $D = [D_1, D_2]$ such that each vertex in $V(G) \setminus D$ is adjacent to exactly one vertex in D_i for $i \in \{1, 2\}$, and each vertex in D is adjacent to at least two vertices in $V(G) \setminus D$.*

We shall add to this terminology that if a symmetric γ-set $D = [D_1, D_2]$ exists such that $|D_1| = |D_2|$, then D is an *even* symmetric γ-set.

3. Proof of Theorem 1.3

The proof of Theorem 1.3 is broken into three cases depending on the type of symmetric γ-sets a graph possesses. The following property will be useful in each of these cases.

Property 3.1. Let $A = [A_1, A_2]$ and $B = [B_1, B_2]$ be symmetric γ-sets of G such that $|A_1| \leqslant |A_2|$ and $|B_1| \leqslant |B_2|$.

(a) If $|A_1| < |B_1|$, then $A_2 \cap B_1 \neq \emptyset$.

(b) If $|B_1| = |A_1| < |A_2|$, then $A_2 \cap B_2 \neq \emptyset$.

Proof. (a) By assumption, $|B_1 \setminus A_1| > 0$ and A_1 dominates $V(G) \setminus A_2$. If $A_2 \cap B_1 = \emptyset$, then by the pigeonhole principle there exists $v \in A_1$ such that v dominates at least two vertices in B_1. This contradicts the fact that B_1 is a 2-packing. Therefore, $A_2 \cap B_1 \neq \emptyset$.

(b) Since $|B_2| = |A_2| > |A_1|$, replacing B_1 with B_2 in the above argument gives the desired result. □

We call the reader's attention to the fact that any universal fixer is inherently a prism fixer. Therefore, in each of the following proofs, we show that for every nontrivial connected prism fixer G there exists a permutation α such that $\gamma(\alpha G) > \gamma(G)$. Furthermore, the results of Mynhardt and Xu [7] allow us to consider only connected graphs with domination number at least 4.

To prove the next three theorems, we introduce the following notation. Let G be a graph and let π be a permutation of $V(G)$. For each vertex $v \in V(G)$, we let v^1 represent the copy of v in G^1 and v^2 represent the copy of v in G^2; conversely, for $i = 1, 2$, if $v^i \in V(G^i)$, let v be the corresponding vertex of G. If $A \subseteq V(G)$, we define $A^i = \{v^i \colon v \in A\}$ for $i = 1, 2$. Conversely, if $A^i \in V(G^i)$, then $A = \{v \in V(G) \colon v^i \in A^i\}$, $i = 1, 2$. If B is a set of vertices in the graph πG, we write $B = X^1 \cup Y^2$, for some symbols X and Y, where $X^1 = B \cap V(G^1)$ and $Y^2 = B \cap V(G^2)$. Thus we navigate between G and πG: the absence of superscripts indicates vertices or sets of vertices in G, and the superscript $i \in \{1, 2\}$ indicates the corresponding vertices or sets of vertices in the subgraph G^i of πG.

Theorem 3.2. *Let G be a nontrivial connected prism fixer with $\gamma(G) \geqslant 4$. If G has a symmetric γ-set that intersects every even symmetric γ-set of G nontrivially, then G is not a universal fixer.*

Proof. Let $D = [D_1, D_2]$ be a symmetric γ-set of G that intersects every even symmetric γ-set of G nontrivially. By Lemma 2.3(c), D_1 and D_2 are 2-packings. Assume without loss of generality that $|D_1| \geqslant |D_2|$ and let $D_1 = \{x_1, \ldots, x_k\}$. Since D_1 is nonempty and a 2-packing, there exists a vertex $u \in N(x_1)$ such that $u \notin \bigcup\limits_{i=2}^{k} N(x_i)$. Define the permutation α of $V(G)$ by $\alpha(x_i) = x_{i+1}$, $i = 1, \ldots, k-1$, $\alpha(x_k) = u$, $\alpha(u) = x_1$, and $\alpha(v) = v$ for $v \in V(G) \setminus (D_1 \cup \{u\})$. Figure 1 illustrates αG with this particular permutation.

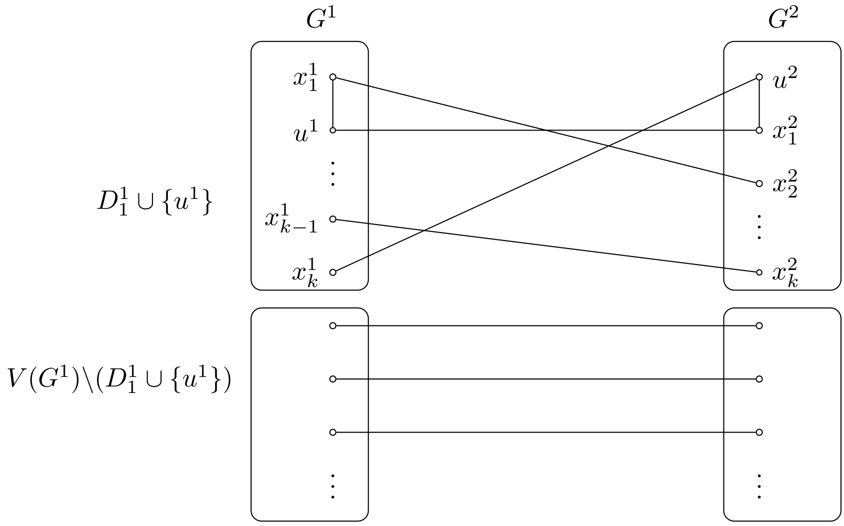

Figure 1. αG where D is a symmetric γ-set that nontrivially intersects every even symmetric γ-set of G.

Suppose $\gamma(\alpha G) = \gamma(G)$ and let $Q^1 \cup R^2$ be a γ-set of αG. Let S^1 consist of the vertices of G^1 that are not dominated by Q^1. Then S^1 is dominated by R^2, that is, for each $s^1 \in S^1$, $\alpha(s) \in R$ and thus $\alpha(S) \subseteq R$. Suppose $r^2 \in R^2$ is adjacent to $a^1 \in V(G^1) - S^1$. Then $\alpha^{-1}(r) = a$ and Q^1 dominates a^1; hence each vertex of G^1 is dominated by a vertex in Q^1 or a vertex in $R^2 \setminus \{r^2\}$, implying $(Q \cup \alpha^{-1}(R)) \setminus \{a\}$ is a dominating set of G of cardinality less than $\gamma(\alpha G) = \gamma(G)$, which is impossible. Hence the neighbor in G^1 of each vertex in R^2, as determined by α, belongs to S^1, that is $\alpha^{-1}(R) \subseteq S$. It follows that $\alpha(S) = R$. Similarly, if T^2 consists of the vertices of G^2 that are not dominated by R^2, then $\alpha(Q) = T$. Furthermore, S and T are 2-packings, otherwise G would also have a dominating set of cardinality less than $\gamma(G)$. We consider four cases.

Case 1. Assume that $S \cap (D_1 \cup \{u\}) = \emptyset$ and $Q \cap (D_1 \cup \{u\}) = \emptyset$. By definition of α, $\alpha(v) = v$ for each $v \in S \cup Q$. Since $\alpha(S) = R$, $R = S$. Similarly, $Q = T$. Since Q^1 dominates $V(G^1) \setminus S^1$ and R^2 dominates $V(G^2) \setminus T^2$, it follows that T dominates $V(G) \setminus S$ and S dominates $V(G) \setminus T$. Hence $[S, T]$ is a symmetric γ-set of G, where we may assume without loss of generality that $|S| \leqslant |T|$.

If $|S| = \gamma(G)/2$, then $[S, T]$ is an even symmetric γ-set of G. By the choice of D, $D \cap (S \cup T) \neq \emptyset$. Since $\alpha(v) = v$ for each $v \in S \cup Q = S \cup T$, we know that $D \cap (S \cup T) \subseteq D$. Hence, by the assumptions of Case 1, $D \cap (S \cup T) \subseteq D_2$. Without loss of generality, assume there exists $y \in D_2 \cap T$. By Lemma 2.3(a), y does not dominate any vertex in D_1. Now each vertex in D_1 is either dominated by a vertex in T or is contained in S. But $S \cap D_1 = \emptyset$ and y does not dominate any vertex in D_1. Hence $T \setminus \{y\}$ dominates D_1. But $|T| = \gamma(G)/2$, so by the choice of D_1, $|T \setminus \{y\}| < \gamma(G)/2 \leqslant |D_1|$. Therefore D_1 is not a 2-packing, contradicting Lemma 2.3(c).

Hence assume $|S| < \gamma(G)/2$. Letting S represent A_1 and D_1 represent B_1 in Property 3.1(a), and recalling that $|D_1| \geqslant \gamma(G)/2$, we see that $T \cap D_1 \neq \emptyset$. But then $Q \cap D_1 \neq \emptyset$, contrary to the assumption of Case 1. Hence Case 1 cannot occur.

Case 2. Assume that $u \in Q \cup S$. First suppose that $u \in Q$. Then $\alpha(u) = x_1 \in T$. Since $ux_1 \in E(G)$ and T is a 2-packing, $u \notin T$. Hence $u \in N(R)$ by definition of T. Let v be a vertex in R adjacent to u. Since x_1 is the only vertex of D_1 adjacent to u, $\alpha(v) = v$. Since $\alpha(S) = R$, it follows that $v \in S$. But now uv joins $u \in Q$ to $v \in S$, contrary to the definition of S.

Hence we may assume that $u \in S$. Then $\alpha(u) = x_1 \in R$. Since $ux_1 \in E(G)$ and S is a 2-packing, $x_1 \notin S$. Hence $x_1 \in N(Q)$ by definition of S. Let v be a vertex in Q adjacent to x_1. As above, $\alpha(v) = v$, and since $\alpha(Q) = T$, $v \in T$. Therefore there exists an edge between R and T, contrary to the definition of T.

Case 3. Assume for some $j \in \{2, \ldots, k-1\}$ that $x_j \in Q \cup S$. Suppose we can show that $x_1, u \in Q \cap R$. Since $\alpha(Q) = T$ and $u \in Q$, it will follow that $\alpha(u) = x_1 \in T$, contrary to the fact that $R \cap T = \emptyset$. Hence this is what we do next.

Since $x_j \in Q \cup S$, $\alpha(x_j) = x_{j+1} \in R \cup T$. Suppose there exists a vertex $v \in R$ such that $vx_j \in E(G)$. By the choice of u, $v \neq u$. Since D_1 is independent, $v \notin D_1$. Therefore $v \in V(G) \setminus (D_1 \cup \{x\})$ and so $\alpha(v) = v$, which implies that $v \in S$. Since S is a 2-packing, $x_j \notin S$, and since no vertex in Q dominates a vertex in S, $x_j \notin Q$, contrary to the assumption of Case 3. Hence no such vertex v exists and thus, by definition of R and T, $x_j \in R \cup T$. Therefore $\alpha^{-1}(x_j) = x_{j-1} \in Q \cup S$. A similar argument shows that x_{j+1} is not adjacent to any vertex in Q and so $x_{j+1} \in Q \cup S$. We can now apply the same argument inductively to $x_{j+1} \in Q \cup S$ and $x_{j-1} \in Q \cup S$ until we arrive at the conclusion that $\{x_1, x_2, \ldots, x_k\} \subseteq (Q \cup S) \cap (R \cup T)$. Then $\alpha^{-1}(x_1) = u \in (Q \cup S)$ and $\alpha(x_k) = u \in (R \cup T)$. Since x_1 and u are adjacent, the definitions of Q and S imply that x_1 and u are both in Q or both in S; but since S is a 2-packing, $x_1, u \in Q$. Similarly, $x_1, u \in R$ and thus $x_1, u \in Q \cap R$, as required.

Case 4. Assume that either x_1 or x_k is in $Q \cup S$. Applying arguments similar to those in Case 3 yields the same contradiction. Therefore, this case cannot occur either.

Thus, no such dominating set $Q^1 \cup R^2$ exists for αG and the result follows. □

If a nontrivial connected prism fixer G with $\gamma(G) \geqslant 4$ has at most one even symmetric γ-set, then the premise of Theorem 3.2 is true and we immediately obtain the following corollary.

Corollary 3.3. *Let G be a nontrivial connected prism fixer with $\gamma(G) \geqslant 4$. If G contains at most one even symmetric γ-set, then G is not a universal fixer.*

Theorem 3.2 also implies that if a nontrivial connected universal fixer G with $\gamma(G) \geqslant 4$ exists, then for each even symmetric γ-set D of G there exists another even symmetric γ-set E of G such that $D \cap E = \emptyset$. We now consider graphs that contain at least two pairwise disjoint even symmetric γ-sets. Note that in this case $\gamma(G)$ is an even integer.

Theorem 3.4. *Let G be a nontrivial connected prism fixer with $\gamma(G) = 2k$ where $k \geqslant 2$. If G contains at least two disjoint even symmetric γ-sets, then G is not a universal fixer.*

P r o o f. Let $D_1, \ldots, D_m$ be a maximal set of pairwise disjoint even symmetric γ-sets. Since D_i is symmetric, for each $1 \leqslant i \leqslant m$ we can write $D_i = [X_i, Y_i]$ such that X_i dominates $V(G) \setminus Y_i$ and Y_i dominates $V(G) \setminus X_i$. We let $X = \bigcup\limits_i X_i$.

We know that each X_i is a 2-packing of size k. Thus, we can index the vertices of X_i as $x_{i,1}, x_{i,2}, \ldots, x_{i,k}$ such that $x_{i+1,j}$ is adjacent to $x_{i,j}$ for $1 \leqslant i \leqslant m-1$ and $1 \leqslant j \leqslant k$.

In order to define our permutation of $V(G)$, we first assign an additional index to X_m, since we will map X_m to X_1. Note that we have already indexed X_m so that $x_{m,j} \in N(x_{m-1,j})$ for $j = 1, \ldots, k$, and this index will be used to map X_{m-1} to X_m. Now for $1 \leqslant j \leqslant k$, define a_j such that $x_{m,a_j} \in N(x_{1,j})$, and this index will be used to map X_m to X_1. We may define the following permutation of $V(G)$:

$$\alpha(v) = \begin{cases} x_{i+1,j} & \text{if } v = x_{i,j} \text{ for } 1 \leqslant j \leqslant k \text{ and } 1 \leqslant i \leqslant m-1, \\ x_{1,j+1} & \text{if } v = x_{m,a_j} \text{ for } 1 \leqslant j \leqslant k-1, \\ x_{1,1} & \text{if } v = x_{m,a_k}, \\ v & \text{otherwise.} \end{cases}$$

Notice in Figure 2 that when we consider the indices of X_m as $x_{m,a_j} \in N(x_{1,j})$, we can write the vertices of X_1 and X_m as a cyclic permutation

$$\beta = (x_{m,a_1}, x_{1,2}, x_{m,a_2}, x_{1,3}, \ldots, x_{m,a_k}, x_{1,1}),$$

where for each $1 \leqslant j \leqslant k$:

(1) $\beta(x_{1,j}) = x_{m,a_j}$; i.e., x_{m,a_j} is adjacent to the vertex immediately preceding it in β, and
(2) $\beta(x_{m,a_j}) = \alpha(x_{m,a_j}) = x_{1,j+1}$; i.e., α maps x_{m,a_j} to the vertex immediately following it in β.

Furthermore, by the definitions of α and a_j, $1 \leqslant j \leqslant k$, β cannot be written as a product of subcycles that exhibit the same properties.

Suppose $\gamma(\alpha G) = 2k$ and let $Q^1 \cup R^2$ be a γ-set of αG. Define S^1 and T^2 as in Theorem 3.2 with all the associated properties.

We first claim that $Q \cap X \neq \emptyset$. To see this, suppose neither S nor Q contains a vertex of X. By definition of α, $T = \alpha(Q) = Q$ and $R = \alpha(S) = S$. Thus Q and R are disjoint 2-packings and $[Q, R]$ is a symmetric γ-set of G.

By the symmetry of αG we need only to consider two cases. If $|Q| = k = |R|$, then $[Q, R]$ is an even symmetric γ-set. By the choice of $D_1, \ldots, D_m$, $D_i \cap (Q \cup R) \neq \emptyset$ for some $1 \leqslant i \leqslant m$. Because $\alpha(Q \cup R) = Q \cup R$, the definition of α implies that $D_i \cap (Q \cup R) \subseteq Y_i$. Assume without loss of generality that $y_{i,j} \in Q$ for some $1 \leqslant j \leqslant k$. Then each vertex of X_i is dominated by a vertex of $Q \setminus \{y_{i,j}\}$ or is contained in S. But by the assumption, $S \cap X = \emptyset$, hence $Q \setminus \{y_{i,j}\}$ dominates X_i. Since $|Q \setminus \{y_{i,j}\}| = k - 1 < |X_i|$, this contradicts X_i being a 2-packing. Therefore

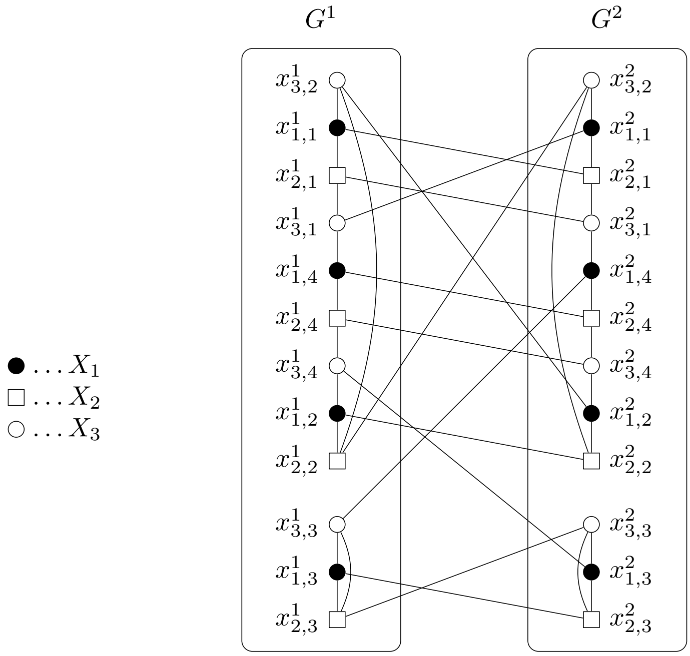

Figure 2. Specific case when $m = 3$ and $k = 4$. Note that $\alpha(v) = v$ for all other vertices of G not depicted.

either $Q \cap X \neq \emptyset$, and we are done, or $S \cap X \neq \emptyset$. In the latter case, we interchange the labels G^1 and G^2 and obtain $Q \cap X \neq \emptyset$.

On the other hand, if $|Q| < k$, then $S \cap X_i \neq \emptyset$ for each $1 \leqslant i \leqslant m$, since each X_i is a 2-packing and every vertex of G is either dominated by Q or is contained in S. This implies for each $1 \leqslant i \leqslant m$ that $R \cap X_i \neq \emptyset$ by definition of α. As before, simply relabel G^1 and G^2 so that $|Q| \geqslant k$ and obtain $Q \cap X \neq \emptyset$.

We next claim that $T \cap X_1 \neq \emptyset$. From the above, we may assume $|Q| \geqslant k$. If $|Q| > k$, then $|R| < k$. This implies that $T \cap X_1 \neq \emptyset$, since X_1 is a 2-packing and every vertex of G is either dominated by R or is contained in T. So assume that $|Q| = k$, and let $x_{i,a} \in Q$ for some $1 \leqslant i \leqslant m$ and $1 \leqslant a \leqslant k$. If $i = m$, then by definition of α we have $T \cap X_1 \neq \emptyset$. So assume $i \neq m$. Since Y_i is a 2-packing and no vertex of Y_i is adjacent to a vertex of X_i, there exist at least $|Q \cap D_i|$ vertices in $S \cap Y_i$. Moreover, since each vertex of Y_i is mapped to itself under α, we know there exist at least $|Q \cap D_i|$ vertices in $R \cap Y_i$ as well. This, together with the fact that $|Q| = k = |R|$, gives

$$\begin{aligned} |R \setminus Y_i| &\leqslant k - |Q \cap D_i| \\ &\leqslant k - 1. \end{aligned}$$

Therefore, since X_i is a 2-packing and each vertex of G is either dominated by R or is contained in T, $T \cap X_i \neq \emptyset$. So assume $x_{i,b} \in T$ for some $1 \leqslant b \leqslant k$. If $i = 1$ or if $m = 2$, then we are done with the proof of this claim. So assume $m > 2$ and $i \notin \{1, m\}$. By definition of α, $x_{i-1,b} \in Q$. Applying the above argument inductively, eventually we have $T \cap X_1 \neq \emptyset$. Let $r = |T \cap X_1| > 0$.

We next claim that $r < k$. To see this, suppose that $r = k$. Then $X_1 \subseteq T$. Because X_1 dominates $V(G) \setminus Y_1$, $R \subseteq Y_1$. If $R \subset Y_1$, then T contains X_1 and some vertex $y_{1,j} \in Y_1$. Since Y_1 dominates $V(G) \setminus X_1$, some $x_{1,i}$ and $y_{1,j}$ have a common neighbor in $V(G) \setminus D_1$, contrary to T being a 2-packing. Therefore $R = Y_1$ and so $T = X_1$. Then $Q = \alpha^{-1}(T) = \alpha^{-1}(X_1) = X_m$ and $S = \alpha^{-1}(R) = Y_1$. By the choice of the D_i, $D_1 \cap D_m = \emptyset$. Hence $X_m = Q$ dominates $Y_1 = S$, contradicting the fact that, by definition, $S = V(G) \setminus N[Q]$. Thus, we may conclude that $r < k$.

Let $x_{1,b_1}, x_{1,b_2}, \ldots, x_{1,b_r}$ be the vertices of $T \cap X_1$. There exist exactly r vertices in $Q \cap X_m$; call them $x_{m,c_1}, x_{m,c_2}, \ldots, x_{m,c_r}$. We claim for some $x_{1,b_j} \in T \cap X_1$ that $x_{1,b_j} \notin N(Q \cap X_m)$. So assume not; that is, assume $\{x_{1,b_1}, x_{1,b_2}, \ldots, x_{1,b_r}\} \subset N(Q \cap X_m)$. This implies there exists a relabeling of the b_j's and c_j's such that $x_{m,c_j} \in N(x_{1,b_j})$ and $\alpha(x_{m,c_j}) = x_{1,b_j+1}$ for $b_j \in \{1, \ldots, k-1\}$ and $\alpha(x_{m,c_j}) = x_{11}$ if $c_j = a_k$ where a_k is the index first given to x_m to define α. Consequently, there exists a subcycle of β consisting of the vertices $x_{1,b_1}, x_{1,b_2}, \ldots, x_{1,b_r}, x_{m,c_1}, x_{m,c_2}, \ldots, x_{m,c_r}$ such that for each $1 \leqslant j \leqslant r$:

(1) x_{m,c_j} is adjacent to the vertex immediately preceding it within its subcycle; and
(2) x_{m,c_j} is mapped under α to the vertex immediately following it within its subcycle.

However, this contradicts the construction of α unless $r = k$, which we know to be false. Thus, for some $x_{1,b_j} \in T \cap X_1$, $x_{1,b_j} \in S$ or $x_{1,b_j} \in N[Q \setminus X_m]$.

If $x_{1,b_j} \in S$, then by definition of α, $x_{2,b_j} \in R$. Since $x_{1,b_j} \in N(x_{2,b_j})$, this implies there exists an edge between R and T. This contradiction shows $x_{1,b_j} \in N[Q \setminus X_m]$. So assume $v \in Q$ where $x_{1,b_j} \in N[v]$. If $\alpha(v) = v$, then v and x_{1,b_j} are both in T, which contradicts T being a 2-packing. On the other hand, if $\alpha(v) \neq v$, then $v = x_{i,d}$ for some $i \neq m$ and $1 \leqslant d \leqslant k$.

Case 1. Assume that $i = 1$. Since X_i is a 2-packing, it follows that $v = x_{1,b_j} \in Q$. Thus, $x_{2,b_j} \in T$ by definition of α. But x_{1,b_j} was assumed to be in T, so this violates T being a 2-packing. Therefore, this case cannot occur.

Case 2. Assume that $i \notin \{1, m\}$. Immediately this implies that $m > 2$. Furthermore, $\alpha(x_{i,d}) = x_{i+1,d}$, and we have $x_{i,d} \in N(x_{1,b_j}) \cap N(x_{i+1,d})$, which contradicts T being a 2-packing, as shown in Figure 3. Thus, this case cannot occur either.

Having considered all cases, we have shown such a dominating set $Q^1 \cup R^2$ of αG does not exist of order $2k$. Hence, the result follows. $\square$

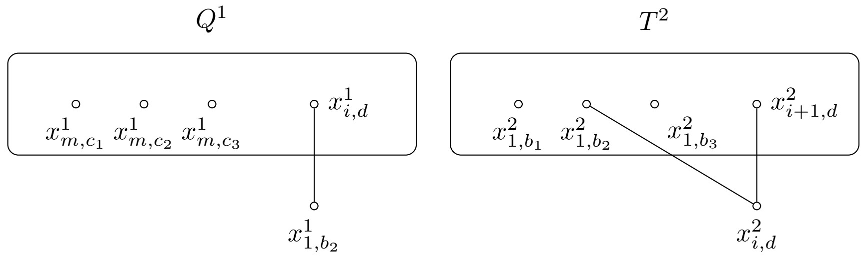

Figure 3. Specific case when $|T \cap X_1| = 3$.

We now use the results of this section to prove Theorem 1.3.

P r o o f of Theorem 1.3. Assume that G is a connected universal fixer of order $n \geqslant 2$. By Mynhardt and Xu [7], we may assume that $\gamma(G) \geqslant 4$. Since G is a universal fixer, G is a prism fixer. Theorem 3.2 implies that for every even symmetric γ-set D of G, there exists an even symmetric γ-set D' of G such that $D \cap D' = \emptyset$. However, this contradicts Theorem 3.4, which states that G cannot contain a pair of disjoint even symmetric γ-sets. Therefore, no such connected universal fixer of order at least 2 exists. That is, if G is a connected universal fixer, then $G = K_1$. $\square$

In conclusion, we know that any component of a universal fixer must be an isolated vertex. It follows that edgeless graphs are the only universal fixers.

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