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ON GENERALIZED PARTIAL TWISTED SMASH PRODUCTS

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Abstract. We first introduce the notion of a right generalized partial smash product and explore some properties of such partial smash product, and consider some examples. Furthermore, we introduce the notion of a generalized partial twisted smash product and discuss a necessary condition under which such partial smash product forms a Hopf algebra. Based on these notions and properties, we construct a Morita context for partial coactions of a co-Frobenius Hopf algebra.

Keywords: partial bicomodule algebra; partial twisted smash product; partial bicoinvariant; Morita context

MSC 2010: 16T05

Introduction

Partial group actions were first defined by Exel in the context of operator algebras and they turned out to be a powerful tool in the study of C^*-algebras generated by partial isometries on a Hilbert space in [8]. The developments originated by the definition of partial group actions, and soon became an independent topic of interest in ring theory in [6]. Now, the results are formulated in a purely algebraic way, independently of the C^* algebraic techniques which originated them.

Partial Hopf actions were motivated by an attempt to generalize the notion of partial Galois extensions of commutative rings in [7] to a broader context. The definitions of partial Hopf actions and coactions were introduced by Caenepeel and Janssen in [5], using the notions of partial entwining structures. In particular, partial actions of a group G determine partial actions of the group algebra kG in a natural way. In the same article, the authors also introduced the concept of partial smash

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product, which in the case of the group algebra kG turns out to be the crossed product by a partial action $A \rtimes_\alpha G$. Further developments in the theory of partial Hopf actions were done by Lomp in [9], Alves and Batista extended several results from the theory of partial group actions to the Hopf algebra setting in [1]. They also constructed a Morita context relating the fixed point subalgebra for partial actions of finite dimensional Hopf algebras, and constructed the partial smash product in [3].

Motivated by the above ideas, this paper is organized as follows. In Section 2, we study the generalized partial smash product $\underline{A\#_l^H B^{\mathrm{op}}}$ where A is a left H-module algebra and B^{op} is a left partial H-comodule algebra and explore some properties of the generalized partial smash products $\underline{A\#_l^H B^{\mathrm{op}}}$ and $\underline{A\#_l^L B^{\mathrm{op}}}$ (see Proposition 2.5). In Section 3, we first study the generalized partial smash product and discuss a necessary condition for $\underline{A \star H^*}$ to be a Hopf algebra (see Theorem 3.5). In Section 4, we show a Morita context relating the generalized partial smash product $\underline{A \star H^{*\mathrm{rat}}}$ and the partial bicoinvariants $A^{\underline{\mathrm{bico}H}}$ for co-Frobenius Hopf algebra H, where A is a partial H-bicomodule algebra (see Theorem 4.4).

1. Preliminaries

Throughout the paper, we let k be a fixed field and we work over k. Let M be a vector space over k and let id_M denote the usual identity map. Let $\otimes$ be over k. For the comultiplication Δ in a coalgebra C with a counit ε_C, we use the Sweedler-Heyneman's notation (see Sweedler [10]): $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$.

We recall some basic results and propositions that we will need later from Alves and Batista [3] and Beattie et al. [4].

1.1. Right partial comodule algebra. Let H be a Hopf algebra and A an algebra. A is said to be a right partial H-comodule algebra if there exists a k-linear map $\varrho\colon A \to A \otimes H$ which is a partial comodule structure, such that the following conditions are satisfied:

$$(\mathrm{id}_A \otimes \varepsilon)\varrho^r = \mathrm{id}_A;$$
$$(\varrho^r \otimes \mathrm{id}_H)\varrho^r(a) = (\varrho(1_A) \otimes \mathrm{id}_H)(\mathrm{id}_A \otimes \Delta)\varrho^r(a);$$
$$\varrho^r(ab) = \varrho^r(a)\varrho^r(b)$$

for all $a, b \in A$; we use the standard notation $\varrho^r(a) = a_{[0]} \otimes a_{[1]}$ for $a \in A$.

1.2. Integral. Let H be a Hopf algebra. A left (right) integral for H is a k-linear form $\lambda \in H^*$ such that, for all $f \in H^*$ ($g \in H^*$),

$$f\lambda = f(1)\lambda \quad (\lambda g = g(1)\lambda).$$

Recall that $H^{*\mathrm{rat}}$ is the unique maximal left (right) rational submodule of the left (right) H^*-module H^*. Since $H^{*\mathrm{rat}}$ is an ideal of H^* equal to the sum of all finite dimensional left (right) ideals of H^*, cf. [10], $H^{*\mathrm{rat}}$ is an H^*-H^*-bimodule.

1.3. Co-Frobenius Hopf algebra. A Hopf algebra H is called co-Frobenius if H has a nonzero space of left (right) integral $\int_l \left(\int_r \right)$.

Let H be a co-Frobenius Hopf algebra. We have:

(1) There exists a group like element x of H such that $\lambda h^* = \langle h^*, x\rangle \lambda$, for all $h^* \in H^*$; $\lambda(S(h)) = \lambda(hx)$ and $\lambda(S^{-1}(h)) = \lambda(xh)$, for all $h \in H$.
(2) H^* is a free left (right) H-module for action defined for any $f \in H^*$ and $h, l \in H$, by $(h \rightharpoonup f)(l) = f(lh)$ $((f \leftharpoonup h)(l) = f(hl))$. The subalgebra $H^{*\mathrm{rat}}$ of H^* is a H-H-bimodule under these actions.

2. Generalized partial smash product

Now, we give the definition of a left partial H-comodule algebra.

Definition 2.1. Let H be a Hopf algebra and A an algebra. A is called a left partial H-comodule algebra if there exists a k-linear map $\varrho^l\colon A \to H \otimes A$ such that the following conditions are satisfied:

$$(\varepsilon \otimes \mathrm{id}_A)\varrho^l = \mathrm{id}_A;$$
$$(\mathrm{id}_H \otimes \varrho^l)\varrho^l(a) = (\Delta \otimes \mathrm{id}_A)\varrho^l(a)(\mathrm{id}_H \otimes \varrho^l(1_A));$$
$$\varrho^l(ab) = \varrho^l(a)\varrho^l(b)$$

for all $a, b \in A$. We use the standard notation $\varrho^l(a) = a_{[-1]} \otimes a_{[0]}$ for $a \in A$.

Let A be a left H-module algebra and B^{op} a left partial H-comodule algebra. We first define a multiplication on the vector space $A \otimes B^{\mathrm{op}}$ by

$$(a \#_l^H b)(c \#_l^H d) = a(b_{[-1]} \rightharpoonup c) \#_l^H b_{[0]} d$$

for all $a, c \in A$, $b, d \in A$, which is automatically associative. In order to make a unital algebra, we project onto

$$\underline{A \#_l^H B^{\mathrm{op}}} = (1_A \otimes 1_{B^{\mathrm{op}}})(A \otimes B^{\mathrm{op}}),$$

then we can deduce directly the form and the properties of typical elements of this algebra

$$\underline{a \#_l^H b} = 1_{[-1]} \rightharpoonup a \otimes 1_{[0]} b,$$

and finally verify that the product of typical elements satisfies

$$(2.1)\qquad (\underline{a\#_l^H b})(\underline{c\#_l^H d}) = \underline{a(b_{[-1]} \rightharpoonup c)\#_l^H b_{[0]}d}$$

for all $a, c \in A$, $b, d \in B^{\mathrm{op}}$.

Proposition 2.2. *$\underline{A\#_l^H B^{\mathrm{op}}}$ is an associative algebra with the multiplication given by Equation* (2.1) *and with the unit $1_A\#_l^H 1_{B^{\mathrm{op}}}$.*

Proof. It is straightforward to check the associativity of the multiplication. We only check the unitary properties as follows:

$$(1_A\#_l^H 1_{B^{\mathrm{op}}})(\underline{a\#_l^H b}) = \underline{(1_{[-1]} \rightharpoonup a)\#_l^H 1_{[0]}b} = \underline{a\#_l^H b},$$

and

$$(\underline{a\#_l^H b})(1_A\#1_B) = \underline{a(b_{[-1]} \rightharpoonup 1_A)\#_l^H b_{[0]}1_{B^{\mathrm{op}}}} = \underline{a\#_l^H b}.$$

This completes the proof. □

Corollary 2.3. *If $A = H$, then $\underline{H\#_l^H B^{\mathrm{op}}}$ is an associative algebra with the unit $1_H\#_l^H 1_{B^{\mathrm{op}}}$.*

Similarly, L is a Hopf algebra. Suppose that B^{op} is a right L-module algebra and A is a right partial L-comodule algebra. We can form a generalized right partial smash product denoted by $\underline{A\#_r^L B^{\mathrm{op}}}$, with the multiplication $(\underline{a\#_r^L b})(\underline{c\#_r^L d}) = \underline{ac_{[0]}\#_l^L b \leftharpoonup c_{[1]}d}$ for all $a, c \in A$, $b, d \in B^{\mathrm{op}}$.

Example 2.4. Let H be a finite dimensional Hopf algebra; the algebra $H^{*\mathrm{rat}}$ is a right H-module algebra via $(f \leftharpoonup h)(g) = f(hg)$, $g, h \in H$, $f \in H^{*\mathrm{rat}}$. Thus if A is a right partial H-comodule algebra, we may form the right partial smash product $\underline{A\#H^{*\mathrm{rat}}}$.

Proposition 2.5. *Suppose that A is a left H-module algebra and B^{op} is a left partial H-comodule algebra, and furthermore that A is also a right partial L-comodule algebra and B^{op} is a right L-module algebra such that for all $a \in A$, $b \in B^{\mathrm{op}}$,*

$$a_{[0]} \otimes b \leftharpoonup a_{[1]} = b_{[-1]} \rightharpoonup a \otimes b_{[0]}.$$

Then there is a natural algebra isomorphism from $\underline{A\#_l^H B^{\mathrm{op}}}$ to $\underline{A\#_r^L B^{\mathrm{op}}}$ defined by the mapping $\underline{a\#_l^H b}$ to $\underline{a\#_r^L b}$.

Proof. Defining $\xi\colon \underline{A\#_l^H B^{\mathrm{op}}} \to \underline{A\#_r^L B^{\mathrm{op}}}$ by $\varphi(\underline{a\#_l^H b}) = \underline{a\#_r^L b}$ for $a \in A$ and $b \in B^{\mathrm{op}}$, we have

$$\begin{aligned}\xi((\underline{a\#_l^H b})(\underline{c\#_l^H d})) &= \xi(\underline{a(b_{[-1]} \rightharpoonup c)\#_l^H b_{[0]}d}) = \underline{a(b_{[-1]} \rightharpoonup c)\#_l^L b_{[0]}d} \\ &= \underline{ac_{[0]}\#_l^L b \leftharpoonup c_{[1]}d} = (\underline{a\#_r^L b})(\underline{c\#_r^L d}) \\ &= \xi(\underline{a\#_l^H b})\xi(\underline{c\#_l^H d}).\end{aligned}$$

□

This example of partial coaction comes from [2]. Let G be a finite group. If N is a normal group of G with $\mathrm{char}(k) \nmid |N|$, then $e_N = |N|^{-1} \sum\limits_{n\in N} n$ is a central idempotent in kG. Let $B = e_N kG$ be the ideal generated by e_N. Consider the partial kG-coaction induced on A by $\Delta\colon kG \to kG \otimes kG$, i.e.,

$$\varrho(e_N g) = \Delta(e_N g)(1 \otimes e_N) = e_N g \otimes e_N g = \frac{1}{|N|^2} \sum_{m,n\in N} mg \otimes ng.$$

Then B is a left partial kG-comodule algebra.

Example 2.6. Suppose that $A = e_M kG'$ is a left kG-module algebra and $B = e_N kG$ is a right kG'-module algebra, where M is a normal group of G' with $\mathrm{char}(k) \nmid |M|$. Then $e_m = |M|^{-1} \sum\limits_{m\in M} m$ is a central idempotent in kG', then $B = e_N kG$ is a left partial kG-comodule algebra and $A = e_M kG'$ is also a right partial kG'-comodule algebra such that for any $g \in G$, $h \in G'$,

$$e_M h \otimes e_N g \leftharpoonup e_M h = e_N g \rightharpoonup e_M h \otimes e_N g.$$

Then there is a natural algebra isomorphism from $\underline{A\#_l^{kG} B}$ to $\underline{A\#_r^{kG'} B}$ defined by the mapping $\underline{a\#_l^{kG} b}$ to $\underline{a\#_r^{kG'} b}$.

Definition 2.7. We call an algebra A a left (right) L-H-dimodule algebra if A is a left (right) L-module algebra and a left (right) partial H-comodule algebra such that the H-comodule structure map is an L-module map, i.e.,

$$(m \rightharpoonup a)_{[-1]} \otimes (m \rightharpoonup a)_{[0]} = a_{[-1]} \otimes m \rightharpoonup a_{[0]}$$

and

$$((a \leftharpoonup m)_{[0]} \otimes (a \leftharpoonup m)_{[1]} = a_{[0]} \leftharpoonup m \otimes a_{[1]})$$

for all $m \in L$, $a \in A$.

Remark 2.8. Definition 2.7 which involves partial actions of two different groups is considered as follows. Let $e \in kG$ be an idempotent such that $e \otimes e = \Delta(e)(e \otimes 1)$ and $\varepsilon(e) = 1$. Obviously $A = k$ is a left (right) kG'-module algebra, and a left (right) partial kG-comodule algebra, then the algebra A is called a left (right) kG'-kG-dimodule algebra.

Lemma 2.9. *Let H and L be two Hopf algebras. Then we have the following statements:*

(1) *Suppose A is a left H-module algebra and B is a left L-H-dimodule algebra. Then $\underline{A\#_l^H B}$ is a left L-module algebra under the left L-action induced by that on B, i.e., $l \rightharpoonup (\underline{a\#_l^H b}) = \underline{a\#_l^H (l \rightharpoonup b)}$ for all $l \in L$.*
(2) *Suppose A is a left L-H-dimodule algebra and B is a left partial L-comodule algebra. Then $\underline{A\#_l^L B}$ is a left partial H-comodule algebra under the left partial H-coaction induced by A, i.e., $(\underline{a\#_l^L b})_{[-1]} \otimes (\underline{a\#_l^L b})_{[0]} = a_{[-1]} \otimes \underline{a_{[0]}\#_l^L b}$.*

Proof. Straightforward. $\square$

Example 2.10. Let G and G' be two groups. Then we have the following statements:

(1) Suppose A is a left kG-module algebra and $B = k$ is a left kG'-kG-dimodule algebra. Then $\underline{A\#_l^{kG} B}$ is a left kG'-module algebra under the left kG'-action induced by that on B, i.e., $h \rightharpoonup (\underline{a\#_l^{kG} b}) = \underline{a\#_l^{kG} b}$ for all $h \in G'$, $b \in B$.
(2) Let $e \in kG$ be an idempotent such that $e \otimes e = \Delta(e)(e \otimes 1)$ and $\varepsilon(e) = 1$. One can easily check that $A = k$ is a left kG'-kG-dimodule algebra and $B = e_M kG'$ is a left partial kG'-comodule algebra. Then $\underline{A\#_l^{kG'} B}$ is a left partial kG-comodule algebra under the left partial H-coaction induced by A, i.e., $(\underline{x\#_l^{kG'} b})_{[-1]} \otimes (\underline{a\#_l^{kG'} b})_{[0]} = e \otimes \underline{x\#_l^{kG'} b}$ for any $x \in A$.

Theorem 2.11. *Suppose A is a left H-module algebra, B a left L-H-dimodule algebra, and C a left partial L-comodule algebra. Then the map taking $\underline{(\underline{a\#_l^H b})\#_l^L c}$ to $\underline{a\#_l^H (\underline{b\#_l^L c})}$ is a natural isomorphism from $\underline{(\underline{A\#_l^H B})\#_l^L C}$ to $\underline{A\#_l^H (\underline{B\#_l^L C})}$ where the partial smash products $(\underline{A\#_l^H B})$ and $(\underline{B\#_l^L C})$ have the left L-module and left partial H-comodule structures defined in Lemma 2.9 (1) and (2), respectively.*

Example 2.12. Let $e \in kG$ be an idempotent such that $e \otimes e = \Delta(e)(e \otimes 1)$ and $\varepsilon(e) = 1$. One can easily check that $B = k$ is a left kG'-kG-dimodule algebra and $C = e_M kG'$ a left partial kG'-comodule algebra. Suppose A is a left kG-module algebra. Then the map taking $\underline{(\underline{a\#_l^{kG} b})\#_l^{kG'} c}$ to $\underline{a\#_l^{kG} (\underline{b\#_l^{kG'} c})}$ is a natural isomorphism from $\underline{(\underline{A\#_l^{kG} B})\#_l^{kG'} C}$ to $\underline{A\#_l^{kG} (\underline{B\#_l^{kG'} C})}$ where the partial smash products $(\underline{A\#_l^{kG} B})$

and $(\underline{B\#_l^{kG'}C})$ have the left kG'-module and left partial kG-comodule structures defined in Example 2.10 (1) and (2), respectively.

Remark 2.13. We can get a right version of Theorem 2.11 for another generalized right partial smash product. We omit it.

3. Generalized partial twisted smash product

In this section, we introduce the notion of partial coactions of a Hopf algebra containing partial left and right coaction, and define a partial bicomodule algebra. On the base of these notions, we introduce a new partial twisted smash product $\underline{A \star H^*}$. Furthermore, we find a necessary condition for $\underline{A \star H^*}$ to be a Hopf algebra.

Definition 3.1. Let H be a Hopf algebra with antipode S and A an algebra. A is called a partial H-bicomodule algebra if A is not only a left partial H-comodule algebra with the left partial comodule coaction ϱ^l but also a partial right H-comodule algebra with the right partial comodule coaction ϱ^r, and satisfies the compatibility condition, i.e., $(\varrho^l \otimes \mathrm{id}_H)\varrho^r = (\mathrm{id}_H \otimes \varrho^r)\varrho^l$.

We denote

$$a_{[-1]} \otimes a_{[0]} \otimes a_{[1]} = a_{[0][-1]} \otimes a_{[0][0]} \otimes a_{[1]} = a_{[-1]} \otimes a_{[0][0]} \otimes a_{[0][1]}.$$

Let H be a finite dimensional Hopf algebra and A a partial H-bicomodule algebra. Then A is a partial H^*-bimodule algebra via $f \rightharpoonup a = \sum\langle f, a_{[1]}\rangle a_{[0]}$ and $a \leftharpoonup g = \langle g, a_{[-1]}\rangle a_{[0]}$ for $a \in A,\ f, g \in H^*$.

We first propose a multiplication on the vector space $A \otimes H^*$,

$$(a \star f)(b \star g) = ab_{[0]} \star (S(b_{[-1]}) \rightarrow f \leftarrow b_{[1]})g$$

for all $a, c \in A$, $b, d \in A$, which is automatically associative. In order to make a unital algebra, we project onto

$$\underline{A \star H^*} = (A \otimes H^*)(1_A \otimes 1_{H^*});$$

then we can deduce directly the form and the properties of typical elements of this algebra

$$\underline{a\#_l^H b} = 1_{[-1]} \rightharpoonup a \otimes 1_{[0]}b,$$

and finally verify that the product typical elements satisfies

$$(\underline{a \star f})(\underline{b \star g}) = \underline{ab_{[0]} \star (S(b_{[-1]}) \rightarrow f \leftarrow b_{[1]})g} \tag{3.1}$$

for all $a, b \in A,\ f, g \in H^*$.

From the above definition and using the compatibility condition, we have

Proposition 3.2. *Let H be a finite dimensional Hopf algebra and A a partial H-bicomodule algebra. Then the tensor space $\underline{A \star H^*}$ is an associative algebra with the multiplication in* (3.1) *and the unit $\underline{1_A \star 1_{H^*}}$.*

Proof. We only prove the unit and omit the associativity.

$$\begin{aligned}(\underline{a \star f})(\underline{1_A \star 1_{H^*}}) &= \sum \underline{a1_{[0]} \star S(1_{[-1]}) \to f \leftarrow 1_{[1]}} \\ &= \sum a1_{[0]}\hat{1}_{[0]} \otimes S(1_{[-1]}\hat{1}_{[-1]}) \to f \leftarrow 1_{[1]}\hat{1}_{[1]} \\ &= \underline{a \star f} = (\underline{1_A \star 1_{H^*}})(\underline{a \star f}).\end{aligned}$$

□

Proposition 3.4. *Let $\underline{a \star 1_{H^*}}, \underline{1_A \star f} \in \underline{A \star H^*}$. Then*

(i) $(\underline{a \star 1_{H^*}})(\underline{1_A \star f}) = \underline{a \star f}$,
(ii) $(\underline{1_A \star f})(\underline{a \star 1_{H^*}}) = \underline{a_{[0]} \star (S(a_{[-1]}) \to f \leftarrow a_{[1]})}$,
(iii) $(\underline{a \star 1_{H^*}})(\underline{b \star 1_{H^*}}) = \underline{ab \star 1_{H^*}}$.

Proof. Straightforward. □

Theorem 3.5. *Let H be a finite dimensional Hopf algebra with antipode S, let A be a bialgebra and a partial H-bicomodule algebra.*

(1) *The partial twisted smash product algebra $\underline{A \star H^*}$ equipped with the tensor product coalgebra structure makes $\underline{A \star H^*}$ into a bialgebra, if the following conditions hold:*
(a) $\sum \varepsilon_A(f_1 \rightharpoonup a \leftharpoonup S^*(f_2)) = \varepsilon_A(a)\varepsilon_{H^*}(f)$,
(b) $\Delta_A\big(\sum f_1 \rightharpoonup a \leftharpoonup S^*(f_2)\big) = \sum (f_1 \rightharpoonup a_1 \leftharpoonup S^*(f_2)) \otimes (f_3 \rightharpoonup a_2 \leftharpoonup S^*(f_4))$,
(c) $\sum (f_1 \rightharpoonup a) \otimes f_2 = \sum (f_2 \rightharpoonup a) \otimes f_1$,
(d) $\sum (a \leftharpoonup S^*(f_1)) \otimes f_2 = \sum (a \leftharpoonup S^*(f_2)) \otimes f_1$.
(2) *Furthermore, if A is a Hopf algebra, and we assume that the formula*

$$\sum f_1 \to 1_A \leftarrow S^*(f_2) = \varepsilon_{H^*}(f)1_A$$

holds, then $\underline{A \star H^*}$ is a Hopf algebra with antipode $S_{\underline{A \star H^*}}$ defined by:

$$S_{\underline{A \star H^*}}(\underline{a \star f}) = (\underline{1 \star S^*(f)})(\underline{S_A(a) \star 1}).$$

P r o o f. (1) First we verify that $\Delta_{\underline{A\star H^*}}$ is an algebra morphism with the multiplication on $\underline{A\star H^*}$ and the tensor product coalgebra structure on $\underline{A\star H^*}$:

$$
\begin{aligned}
\Delta_{\underline{A\star H^*}}((\underline{a\star f})(\underline{b\star g})) &= \sum \Delta_{\underline{A\star H^*}}(\underline{ab_{[0]}\star (S(b_{[-1]})\rightarrow f\leftarrow b_{[1]})g})\\
&= \sum \Delta_{\underline{A\star H^*}}(\underline{a_1(f_1\rightharpoonup b\leftharpoonup S^*(f_3))\star f_2 g})\\
&= \sum \underline{(a_1(f_1\rightharpoonup b\leftharpoonup S^*(f_3)))_1\star (f_2g)_1}\otimes \underline{(a_1(f_1\rightharpoonup b\leftharpoonup S^*(f_3)))_2\star (f_2g)_2}\\
&= \sum \underline{a_1(f_1\rightharpoonup b\leftharpoonup S^*(f_4))_1\star f_2g_1}\otimes \underline{a_2(f_1\rightharpoonup b\leftharpoonup S^*(f_4))_2\star f_3g_2}\\
&\overset{\text{(d)}}{=} \sum \underline{a_1(f_1\rightharpoonup b\leftharpoonup S^*(f_3))_1\star f_2g_1}\otimes \underline{a_2(f_1\rightharpoonup b\leftharpoonup S^*(f_3))_2\star f_4g_2}\\
&\overset{\text{(d)}}{=} \sum \underline{a_1(f_1\rightharpoonup b\leftharpoonup S^*(f_2))_1\star f_3g_1}\otimes \underline{a_2(f_1\rightharpoonup b\leftharpoonup S^*(f_2))_2\star f_4g_2}\\
&\overset{\text{(b)}}{=} \sum \underline{a_1(f_1\rightharpoonup b_1\leftharpoonup S^*(f_2))\star f_5g_1}\otimes \underline{a_2(f_3\rightharpoonup b_2\rightharpoonup S^*(f_4))\star f_6g_2}\\
&\overset{\text{(d)}}{=} \sum \underline{a_1(f_1\rightharpoonup b_1\leftharpoonup S^*(f_2))\star f_4g_1}\otimes \underline{a_2(f_3\rightharpoonup b_2\leftharpoonup S^*(f_5))\star f_6g_2}\\
&\overset{\text{(d)}}{=} \sum \underline{a_1(h_1\rightharpoonup b_1\leftharpoonup S^*(f_2))\star f_4g_1}\otimes \underline{a_2(f_3\rightharpoonup b_2\leftharpoonup S^*(f_6))\star f_5g_2}\\
&\overset{\text{(c)}}{=} \sum \underline{a_1(f_1\rightharpoonup b_1\leftharpoonup S^*(f_2))\star f_3g_1}\otimes \underline{a_2(f_4\rightharpoonup b_2\leftharpoonup S^*(f_6))\star f_5g_2}\\
&\overset{\text{(d)}}{=} \sum \underline{a_1(f_1\rightharpoonup b_1\leftharpoonup S^*(f_3))\star f_2g_1}\otimes \underline{a_2(f_4\rightharpoonup b_2\leftharpoonup S^*(f_6))\star f_5g_2}\\
&= \Delta(\underline{a\star f})\Delta(\underline{b\star g}).
\end{aligned}
$$

Next, we verify that $\varepsilon_{\underline{A\star H^*}}$ is also an algebra morphism. It is easy to verify that

$$\varepsilon_{\underline{A\star H^*}}(\underline{a\star f}) = \varepsilon_A(a)\varepsilon_{H^*}(f).$$

In fact,

$$
\begin{aligned}
\varepsilon_{\underline{A\star H^*}}(\underline{a\star f}) &= \sum \varepsilon_{\underline{A\star H^*}}(a(f_1\rightharpoonup 1_A\leftharpoonup S^*(f_3))\otimes f_2)\\
&= \sum \varepsilon_A(a(f_1\rightharpoonup 1_A\leftharpoonup S^*(f_3)))\varepsilon_{H^*}(f_2)\\
&= \varepsilon_A(a)\varepsilon_{H^*}(f),\\
\varepsilon_{\underline{A\star H^*}}((\underline{a\star f})(\underline{b\star g})) &= \sum \varepsilon_{\underline{A\star H^*}}(\underline{ab_{[0]}\star (S(b_{[-1]})\rightarrow f\leftarrow b_{[1]})g})\\
&= \sum \varepsilon_{\underline{A\star H^*}}(\underline{a(f_1\rightharpoonup b\leftharpoonup S^*(f_3))\star f_2g})\\
&= \sum \varepsilon_A(a(f_1\rightharpoonup b\leftharpoonup S^*(f_3)))\varepsilon_{H^*}(f_2g)\\
&\overset{\text{(a)}}{=} \varepsilon_A(a)\varepsilon_{H^*}(f)\varepsilon_A(b)\varepsilon_{H^*}(g)\\
&= \varepsilon_{\underline{A\star H^*}}(\underline{a\star f})\varepsilon_{\underline{A\star H^*}}(\underline{b\star g}).
\end{aligned}
$$

Hence, $\underline{A\star H^*}$ is a bialgebra.

(2) It is easy to check that $(S_{\underline{A\star H^*}} * \mathrm{id})(\underline{a\star f}) = \varepsilon_A(a)\varepsilon_{H^*}(f)\underline{1_A\star 1_{H^*}} = (\mathrm{id} * S_{\underline{A\star H^*}})(\underline{a\star f})$.

Therefore, $\underline{A\star H^*}$ is a Hopf algebra. □

Remark 3.6. In Theorem 3.5, the conditions (b), (c) and (d) of the item (1) can be easily verified for the case when H^* is cocommutative (therefore, H is commutative). If a Hopf algebra H satisfies these three conditions, then H^* is not necessarily cocommutative.

A concrete counterexample is presented as follows.

Recall the definition of H_4. As a k-algebra, H_4 is generated by two symbols c and x which satisfy the relations $c^2=1$, $x^2=0$ and $xc+cx=0$. The coalgebra structure on H_4 is determined by

$$\Delta(c)=c\otimes c,\quad \Delta(x)=x\otimes 1+c\otimes x,\quad \varepsilon(c)=1,\ \varepsilon(x)=0.$$

Consequently, H_4 has the basis 1 (identity), c, x, cx. We now consider the dual H_4^* of H_4. We have $H_4\cong H_4^*$ (as Hopf algebras) via

$$1\mapsto 1^*+c^*,\quad c\mapsto 1^*+c^*,\quad x\mapsto x^*+(cx)^*,\quad cx\mapsto x^*-(cx)^*,$$

where $\{1^*,c^*,x^*,(cx)^*\}$ denotes the dual basis of $\{1,c,x,cx\}$. Then we let $T=1^*+c^*$, $P=x^*+(cx)^*$, $TP=x^*-(cx)^*$, getting another basis $\{1,T,P,TP\}$ of H_4^*. Recall from [5] that if A is the subalgebra $k[x]$ of H_4, then A is a right partial H_4-comodule algebra with the coaction $\varrho(1)=\frac{1}{2}(1\otimes 1+1\otimes c+1\otimes cx)$, $\varrho^r(x)=\frac{1}{2}(x\otimes 1+x\otimes c+x\otimes cx)$. In a similar way we can define A as a left partial H_4-comodule algebra with the coaction $\varrho(1)=\frac{1}{2}(1\otimes 1+c\otimes 1+cx\otimes 1)$, $\varrho^l(x)=\frac{1}{2}(1\otimes x+c\otimes x+cx\otimes x)$. It can be easily checked that A is a partial H_4-bicomodule algebra, hence A is a partial H_4^*-bimodule algebra via $f\rightharpoonup a=\sum\langle f,a_{[1]}\rangle a_{[0]}$ and $a\leftharpoonup g=\langle g,a_{[-1]}\rangle a_{[0]}$, for $a\in A$, $f,g\in H^*$.

We only consider the element P of H_4^* and check the condition (b) as follows:

$$\begin{aligned}
\Delta_A\Big(\sum P_1\rightharpoonup x\leftharpoonup S^*(P_2)\Big)
&=\Delta_A(P\rightharpoonup x\leftharpoonup S^*(1)+T\rightharpoonup x\leftharpoonup S^*(P))\\
&=\Delta_A\Big(\Big\langle P,\frac{1}{2}(1+c+cx)\Big\rangle x\Big\langle 1,\frac{1}{2}(1+c+cx)\Big\rangle\\
&\quad+\Big\langle T,\frac{1}{2}(1+c+cx)\Big\rangle\Big\langle P,\frac{1}{2}(1+c+cx)\Big\rangle x\Big)\\
&=\Big\langle P,\frac{1}{2}(1+c+cx)\Big\rangle(x\otimes 1+1\otimes x)\\
&\quad+\Big\langle T,\frac{1}{2}(1+c+cx)\Big\rangle\Big\langle P,\frac{1}{2}(1+c+cx)\Big\rangle(x\otimes 1+1\otimes x)\\
&=\Big\langle P,\frac{1}{2}(1+c+cx)\Big\rangle(x\otimes 1+1\otimes x),
\end{aligned}$$

and

$$
\begin{aligned}
\sum (P_1 \rightharpoonup x_1 \leftharpoonup S^*(P_2)) \otimes (P_3 \rightharpoonup x_2 \leftharpoonup S^*(P_4))
&= \sum (P_1 \rightharpoonup x \leftharpoonup S^*(P_2)) \otimes (P_3 \rightharpoonup 1 \leftharpoonup S^*(P_4)) \\
&\quad + \sum (P_1 \rightharpoonup 1 \leftharpoonup S^*(P_2)) \otimes (P_3 \rightharpoonup x \leftharpoonup S^*(P_4)) \\
&= \sum (P \rightharpoonup x \leftharpoonup S^*(1)) \otimes (1 \rightharpoonup 1 \leftharpoonup S^*(1)) \\
&\quad + \sum (P \rightharpoonup 1 \leftharpoonup S^*(1)) \otimes (1 \rightharpoonup x \leftharpoonup S^*(1)) \\
&\quad + \sum (T \rightharpoonup x \leftharpoonup S^*(T)) \otimes (P \rightharpoonup 1 \leftharpoonup S^*(1)) \\
&\quad + \sum (T \rightharpoonup 1 \leftharpoonup S^*(T)) \otimes (P \rightharpoonup x \leftharpoonup S^*(1)) \\
&\quad + \sum (T \rightharpoonup x \leftharpoonup S^*(T)) \otimes (T \rightharpoonup 1 \leftharpoonup S^*(P)) \\
&\quad + \sum (T \rightharpoonup 1 \leftharpoonup S^*(T)) \otimes (T \rightharpoonup x \leftharpoonup S^*(P)) \\
&\quad + \sum (T \rightharpoonup x \leftharpoonup S^*(P)) \otimes (1 \rightharpoonup 1 \leftharpoonup S^*(1)) \\
&\quad + \sum (T \rightharpoonup 1 \leftharpoonup S^*(P)) \otimes (1 \rightharpoonup x \leftharpoonup S^*(1)) \\
&= \left\langle P, \frac{1}{2}(1 + c + cx) \right\rangle (x \otimes 1 + 1 \otimes x).
\end{aligned}
$$

By direct computation we can check that conditions (c) and (d) in Theorem 3.5 hold.

4. Morita context

In this section we construct a Morita context between $A^{\mathrm{bico}H}$ and $\underline{A \star H^{*\mathrm{rat}}}$, where A is a partial bicomodule algebra, generalizing M. Beattie et al.'s work [4].

In what follows, we always assume that $1_{[0]}\langle f_1, 1_1\rangle\langle f_2, S(1_{[-1]})\rangle$ lies in the center of A for each $f \in H^{*\mathrm{rat}}$.

Remark 4.1. By virtue of Remark 3.6 that A is a partial H_4-bicomodule algebra, we obtain that

$$
\begin{aligned}
1_{[0]}\langle f_1, 1_1\rangle\langle f_2, S(1_{[-1]})\rangle &= \frac{1}{2}(1\langle f_1, 1\rangle\langle f_2, S(1)\rangle + 1\langle f_1, c\rangle\langle f_2, S(c)\rangle \\
&\quad + 1\langle f_1, cx\rangle\langle f_2, S(cx)\rangle) \\
&= \frac{1}{2}(1 + \langle f_1, c\rangle\langle f_2, c\rangle + 1\langle f_1, cx\rangle\langle f_2, cx\rangle) \\
&= \frac{1}{2}(1 + 1) = 1.
\end{aligned}
$$

Proposition 4.2. *Let H be a co-Frobenius Hopf algebra and A a partial H-bicomodule algebra, define*

$$A^{\underline{\mathrm{bico}H}} = \{a \in A;\ (\varrho^l \otimes \mathrm{id}_H)\varrho^r(a) = 1_{[-1]} \otimes a1_{[0]} \otimes 1_{[1]} = 1_{[-1]} \otimes 1_{[0]}a \otimes 1_{[1]}\}.$$

Then the partial H-bicoinvariants $A^{\underline{\mathrm{bico}H}}$ is a subalgebra of A.

Proof. Straightforward. □

Lemma 4.3. *Let A be a partial H-bicomodule algebra. Then A is a left $\underline{A \star H^{*\mathrm{rat}}}$-module and a right $\underline{A \star H^{*\mathrm{rat}}}$-module with module structure maps defined as follows: for all $a, b \in A$, $f \in H^{*\mathrm{rat}}$,*

$$(\underline{a \star f}) \triangleright b = \sum a\langle f_1, b_{[1]}\rangle, \langle f_2, S(b_{[-1]})\rangle b_{[0]},$$

and

$$b \triangleleft (\underline{a \star f}) = \sum b_{[0]}a_{[0]}\langle f_1, S^{-1}(b_{[1]}a_{[1]})\rangle\langle f_2, S^2(b_{[-1]}a_{[-1]})\rangle.$$

Proof. For all $a, b, c \in A$, $f, g \in H^{*\mathrm{rat}}$, it is easy to check that $(\underline{1_A \star 1_{H^{*\mathrm{rat}}}}) \triangleright c = c$, and we have

$$\begin{aligned}
((\underline{a \star f})(\underline{b \star g})) \triangleright c &= \sum (\underline{ab_{[0]} \star (S(b_{[-1]}) \rightarrow f \leftarrow b_{[1]})g}) \triangleright c \\
&= \sum ab_{[0]}c_{[0]}\langle f_4, S(b_{[-1]})\rangle\langle f_1, b_{[1]}\rangle\langle f_2 g_1, c_{[1]}\rangle\langle f_3 g_2, S(c_{[-1]})\rangle \\
&= \sum ab_{[0]}1_{[0]}c_{[0]}\langle f_1, b_{[1]}\rangle\langle f_2, 1_{[1]}\rangle\langle f_3 g_1, c_{[1]}\rangle \\
&\quad \times \langle f_4 g_2, S(c_{[-1]})\rangle\langle f_5, S(1_{[-1]})\rangle\langle f_6, S(b_{[-1]})\rangle \\
&= \sum ab_{[0]}1_{[0]}c_{[0]}\langle f_1, b_{[1]}1_{[1]}c_{[1]1}\rangle\langle f_2, S(b_{[-1]}1_{[-1]}c_{[-1]2})\rangle \\
&\quad \times \langle g_1, c_{[1]2}\rangle\langle g_2, S(c_{[-1]1})\rangle \\
&= \sum ab_{[0]}1_{[0]}c_{[0]}\langle f_1, b_{[1]}c_{[1]}\rangle\langle f_2, S(b_{[-1]}1_{[-1]}c_{[-1]2})\rangle \\
&\quad \times \langle g_1, c_{[1]}\rangle\langle g_2, S(c_{[0][-1]1})\rangle \\
&= \sum ab_{[0]}c_{[0]}\langle f_1, b_{[1]}c_{[1]}\rangle\langle f_2, S(b_{[-1]}c_{[-1]})\rangle \\
&\quad \times \langle g_1, c_{[1]}\rangle\langle g_2, S(c_{[-1]})\rangle \\
&= (\underline{a \star f}) \triangleright ((\underline{b \star g}) \triangleright c).
\end{aligned}$$

Hence, A is a left $\underline{A \star H^{*\mathrm{rat}}}$-module.

Now, we show A is a right $\underline{A \star H^{*\mathrm{rat}}}$-module. It is not hard to prove that $b \lhd (\underline{1_A \star 1_{H^{*\mathrm{rat}}}}) = b$, and we have

$$\begin{aligned}
b \lhd ((\underline{a \star f})(\underline{c \star g})) &= \sum b \rhd \underline{ac_{[0]} \star ((S(c_{[-1]}) \rightarrow f \leftarrow c_{[1]})g)} \\
&= \sum b_{[0]} a_{[0]} c_{[0]} \langle f_4, S(c_{[-1]})\rangle \langle f_1, c_{[1]}\rangle \langle f_2 g_1, S^{-1}(b_{[1]} a_{[1]} c_{[1]})\rangle \\
&\quad \times \langle f_3 g_2, S^2(b_{[-1]} a_{[-1]} c_{[-1]})\rangle \\
&= \sum b_{[0]} a_{[0]} 1_{[0]} c_{[0]} \langle f_4, S(c_{[-1]1})\rangle \langle f_1, c_{[1]2}\rangle \langle f_2 g_1, S^{-1}(b_{[1]} a_{[1]} 1_{[1]} c_{[1]1})\rangle \\
&\quad \times \langle f_3 g_2, S^2(b_{[-1]} a_{[-1]} 1_{[-1]} c_{[-1]2})\rangle \\
&= \sum b_{[0]} a_{[0]} c_{[0]} \langle f_4, S(c_{[0][-1]1})\rangle \langle f_1, c_{[1]3}\rangle \langle f_2, S^{-1}(b_{[1]2} a_{[1]2} c_{[1]2})\rangle \\
&\quad \times \langle g_1, S^{-1}(b_{[1]1} a_{[1]1} c_{[1]1})\rangle \langle f_3, S^2(b_{[-1]2} a_{[-1]2} c_{[-1]2})\rangle \\
&\quad \times \langle g_2, S^2(b_{[-1]1} a_{[0][-1]1} c_{[0][-1]3})\rangle \\
&= \sum b_{[0]} a_{[0]} c_{[0]} \langle f_1, S^{-1}(b_{[1]2} a_{[1]2})\rangle \langle g_1, S^{-1}(b_{[1]1} a_{[1]1} c_{[1]})\rangle \\
&\quad \times \langle f_2, S^2(b_{[-1]2} a_{[-1]2})\rangle \langle g_2, S^2(b_{[-1]1} a_{[-1]1} c_{[-1]})\rangle \\
&= \sum 1_{[0]} b_{[0]} a_{[0]} c_{[0]} \langle f_1, S^{-1}(b_{[1]2} a_{[1]2})\rangle \langle g_1, S^{-1}(1_{[1]} b_{[1]1} a_{[1]1} c_{[1]})\rangle \\
&\quad \times \langle f_2, S^2(1_{[-1]} b_{[-1]2} a_{[-1]2})\rangle \langle g_2, S^2(b_{[-1]1} a_{[-1]1} c_{[-1]})\rangle \\
&= \sum b_{[0]} a_{[0]} c_{[0]} \langle f_1, S^{-1}(b_{[1]} a_{[1]})\rangle \langle g_1, S^{-1}(b_{[1]} a_{[1]} c_{[1]})\rangle \\
&\quad \times \langle f_2, S^2(b_{[-1]} a_{[-1]})\rangle \langle g_2, S^2(b_{[-1]} a_{[-1]} c_{[-1]})\rangle \\
&= (b \lhd (\underline{a \star f})) \lhd (\underline{c \star g}).
\end{aligned}$$

□

Theorem 4.4. *With the notation as above, and a nonzero left integral t, we have a Morita context $(A^{\underline{\mathrm{bico}H}}, \underline{A \star H^{*\mathrm{rat}}}, [,], (,))$ where the connecting maps are given by*

$$[,]\colon A \otimes_{A^{\underline{\mathrm{bico}H}}} A \rightarrow \underline{A \star H^{*\mathrm{rat}}}, \quad [a, b] = \sum \underline{ab_{[0]} \star S(b_{[-1]}) \rightarrow t \leftarrow b_{[1]}},$$

$$(,)\colon A \otimes_{\underline{A \star H^{*\mathrm{rat}}}} A \rightarrow A^{\underline{\mathrm{bico}H}}, \quad (a, b) = \sum a_{[0]} b_{[0]} \langle t_1, a_{[1]} b_{[1]}\rangle \langle t_2, S(a_{[-1]} b_{[-1]})\rangle.$$

P r o o f. (1) We will check that $[,], (,)$ are well defined, i.e., $[,]$ is $A^{\underline{\mathrm{bico}H}}$-balanced and $(,)$ is $\underline{A \star H^{*\mathrm{rat}}}$-balanced.

First, for the map $[,]$, if $a, b \in A$ and $c \in A^{\underline{\mathrm{bico}H}}$, then we have:

$$\begin{aligned}
[ac, b] &= \sum \underline{acb_{[0]} \star S(b_{[-1]}) \rightarrow t \leftarrow b_{[1]}} \\
&= \sum \underline{ac1_{[0]} b_{[0]} \star S(1_{[-1]} b_{[-1]}) \rightarrow t \leftarrow 1_{[1]} b_{[1]}} \\
&= \sum \underline{a(cb)_{[0]} \star S((cb)_{[-1]}) \rightarrow t \leftarrow (cb)_{[1]}} = [a, cb].
\end{aligned}$$

Hence, $[,]$ is $A^{\underline{\text{bico}H}}$-balanced.

Now, for the second map $(,)$, if $a, b, c \in A$ and $f \in H^{*\text{rat}}$, then we have:

$$\begin{aligned}
(a \lhd (\underline{c \star f}), b) &= \sum (a_{[0]} c_{[0]} \langle (S^*)^{-1}(f_1), (a_{[1]} c_{[1]}) \rangle \langle (S^*)^2(f_2), a_{[-1]} c_{[-1]} \rangle, b) \\
&= \sum \langle (S^*)^{-1}(f_1), (a_{[1]} c_{[1]}) \rangle \langle (S^*)^2(f_2), a_{[-1]} c_{[-1]} \rangle a_{[0]} c_{[0]} b_{[0]} \\
&\quad \times \langle t_1, a_{[1]} c_{[1]} b_{[1]} \rangle \langle S^*(t_2), a_{[-1]} c_{[-1]} b_{[-1]} \rangle \\
&= \sum \langle (S^*)^{-1}(f_1), (a_{[1]2} c_{[1]2}) \rangle \langle (S^*)^2(f_2), a_{[-1]1} c_{[-1]1} \rangle 1_{[0]} a_{[0]} c_{[0]} b_{[0]} \\
&\quad \times \langle t_1, 1_{[1]} a_{[1]1} c_{[1]1} b_{[1]} \rangle \langle S^*(t_2), 1_{[-1]} a_{[-1]2} c_{[-1]2} b_{[-1]} \rangle \\
&= \sum \langle (S^*)^{-1}(f_1), (a_{[1]2} c_{[1]2}) \rangle \langle (S^*)^2(f_2), a_{[-1]1} c_{[-1]1} \rangle a_{[0]} c_{[0]} b_{[0]} \\
&\quad \times \langle t_1, a_{[1]1} c_{[1]1} b_{[1]} \rangle \langle S^*(t_2), a_{[-1]2} c_{[-1]2} b_{[-1]} \rangle \\
&= \sum \langle (S^*)^{-1}(f_1), (a_{[1]2} c_{[1]2}) \rangle \langle (S^*)^2(f_2), a_{[-1]1} c_{[-1]1} \rangle a_{[0]} c_{[0]} b_{[0]} \\
&\quad \times \langle t_1, a_{[1]1} c_{[1]1} \rangle \langle t_2, b_{[1]} \rangle \langle S^*(t_4), a_{[-1]2} c_{[-1]2} \rangle \langle S^*(t_3), b_{[-1]} \rangle \\
&= \sum \langle t_1 (S^*)^{-1}(f_1), (a_{[1]} c_{[1]}) \rangle \langle (S^*)^2(f_2) S^*(t_4), a_{[-1]} c_{[-1]} \rangle a_{[0]} c_{[0]} b_{[0]} \\
&\quad \times \langle t_2, b_{[1]} \rangle \langle S^*(t_3), b_{[0][-1]} \rangle \\
&= \sum \langle t_1 f_2 (S^*)^{-1}(f_1), (a_{[1]} c_{[1]}) \rangle \langle (S^*)^2(f_2) S^*(t_4 f_5), a_{[-1]} c_{[-1]} \rangle a_{[0]} c_{[0]} b_{[0]} \\
&\quad \times \langle t_2 f_3, b_{[1]} \rangle \langle S^*(t_3 f_4), b_{[-1]} \rangle \\
&= \sum \langle t_1, (a_{[1]} c_{[1]}) \rangle \langle S^*(t_4), a_{[-1]} c_{[-1]} \rangle a_{[0]} c_{[0]} b_{[0]} \\
&\quad \times \langle t_2 f_1, b_{[1]} \rangle \langle S^*(t_3 f_2), b_{[-1]} \rangle \\
&= \sum \langle t_1, (a_{[1]} c_{[1]}) \rangle \langle S^*(t_4), a_{[-1]} c_{[-1]} \rangle a_{[0]} c_{[0]} 1_{[0]} b_{[0]} \\
&\quad \times \langle t_2, 1_{[1]} b_{[1]1} \rangle \langle f_1, b_{[1]2} \rangle \langle S^*(f_2), b_{[-1]1} \rangle \langle S^*(t_3), 1_{[-1]} b_{[-1]2} \rangle \\
&= \sum \langle t_1, (a_{[1]} c_{[1]}) \rangle \langle S^*(t_4), a_{[-1]} c_{[-1]} \rangle a_{[0]} c_{[0]} b_{[0]} \\
&\quad \times \langle t_2, b_{[1]} \rangle \langle f_1, b_{[1]} \rangle \langle S^*(f_2), b_{[-1]} \rangle \langle S^*(t_3), b_{[-1]} \rangle \\
&= (a, (\underline{c \star f}) \rhd b).
\end{aligned}$$

Hence $(,)$ is well defined.

(2) A is an $\underline{A \star H^{*\text{rat}}}$-$A^{\underline{\text{bico}H}}$-bimodule.

Since A has a canonical $A^{\underline{\text{bico}H}}$-bimodule structures on A, we only need to check the compatibility condition as follows.

For all $a \in A$, $b \in A^{\underline{\text{bico}H}}$, and $\underline{c \star f} \in \underline{A \star H^{*\text{rat}}}$, we have

$$\begin{aligned}
(\underline{c \star f}) \rhd (ab) &= \sum c a_{[0]} b_{[0]} \langle f_1, a_{[1]} b_{[1]} \rangle \langle f_2, a_{[-1]} b_{[-1]} \rangle \\
&= \sum c a_{[0]} \langle f_1, a_{[1]} \rangle \langle f_2, a_{[-1]} \rangle b \\
&= ((\underline{c \star f}) \rhd a) b.
\end{aligned}$$

(3) A is an $A^{\underline{\mathrm{bico}H}}$-$\underline{A\star H^{*\mathrm{rat}}}$-bimodule.

For all $a\in A$, $b\in A^{\underline{\mathrm{bico}H}}$ and $\underline{c\star f}\in\underline{A\star H^{*\mathrm{rat}}}$, we have

$$\begin{aligned}(ba)\triangleleft(\underline{c\star f}) &= \sum b_{[0]}a_{[0]}c_{[0]}\langle f_1,S^{-1}(b_{[1]}a_{[1]}c_{[1]})\rangle\langle f_2,S^2(b_{[-1]}a_{[-1]}c_{[-1]})\rangle\\ &= \sum b_{[0]}a_{[0]}c_{[0]}\langle f_1,S^{-1}(a_{[1]}c_{[1]})\rangle\langle f_2,S^{-1}(b_{[1]})\rangle\\ &\quad\times\langle f_3,S^2(b_{[-1]})\rangle\langle f_4,S^2(a_{[-1]}c_{[-1]})\rangle\\ &= \sum ba_{[0]}c_{[0]}\langle f_1,S^{-1}(a_{[1]}c_{[1]})\rangle\langle f_2,S^2(a_{[-1]}c_{[-1]})\rangle\\ &= b(a\triangleleft(\underline{c\star f})).\end{aligned}$$

(4) $[,]$ is an $\underline{A\star H^{*\mathrm{rat}}}$-bimodule map, so we only check $[,]$ is a left $\underline{A\star H^{*\mathrm{rat}}}$-module map.

For all $a\in A$, $b\in A^{\underline{\mathrm{bico}H}}$, $\underline{c\star h}\in\underline{A\star H^{*\mathrm{rat}}}$, we have

$$\begin{aligned}(\underline{c\star f})\cdot[a,b] &= \sum(\underline{c\star f})(\underline{ab_{[0]}\star(S(b_{[-1]})\rightarrow t\leftarrow b_{[1]})})\\ &= \sum ca_{[0]}b_{[0]}\langle f_1,a_{[1]}b_{[1]}\rangle\langle f_3,S(a_{[-1]}b_{[-1]})\rangle\\ &\quad\times\langle t_1,b_{[1]}\rangle\langle t_3,S(b_{[0][-1]})\rangle f_2t_2\\ &= \sum ca_{[0]}1_{[0]}b_{[0]}\langle f_1,a_{[1]}1_{[1]}b_{[1]}\rangle\langle f_3,S(a_{[-1]}1_{[-1]}b_{[-1]})\rangle\\ &\quad\times\langle t_1,b_{[1]}\rangle\langle t_3,S(b_{[-1]})\rangle f_2t_2\\ &= \sum ca_{[0]}b_{[0]}\langle f_1,a_{[1]}b_{[1]1}\rangle\langle f_3,S(a_{[-1]}b_{[-1]2})\rangle\\ &\quad\times\langle t_1,b_{[1]2}\rangle\langle t_3,S(b_{[-1]1})\rangle f_2t_2\\ &= \sum ca_{[0]}b_{[0]}\langle f_1,a_{[1]}\rangle\langle f_2,b_{[1]1}\rangle\langle f_4,S(b_{[-1]2})\rangle\langle f_5,S(a_{[-1]})\rangle\\ &\quad\times\langle t_1,b_{[1]2}\rangle\langle t_3,S(b_{[-1]1})\rangle f_3t_2\\ &= \sum ca_{[0]}b_{[0]}\langle f_1,a_{[1]}\rangle\langle f_2t_1,b_{[1]}\rangle\langle f_4t_3,S(b_{[-1]})\rangle\langle f_5,S(a_{[-1]})\rangle f_3t_2\\ &= \sum ca_{[0]}b_{[0]}\langle f_1,a_{[1]}\rangle\langle t_1,b_{[1]}\rangle\langle t_3,S(b_{[-1]})\rangle\langle f_2,S(a_{[-1]})\rangle t_2\\ &= ((\underline{c\star h})\triangleright a)b_{[0]}\star(S(b_{[-1]})\rightarrow t\leftarrow b_{[1]})\\ &= [(\underline{c\star h})\triangleright a,b].\end{aligned}$$

(5) $(,)$ is an $A^{\underline{\mathrm{bico}H}}$-bimodule map, so for all $a,b\in A$, $c\in A^{\underline{\mathrm{bico}H}}$ we have

$$\begin{aligned}(ca,b) &= \sum c_{[0]}a_{[0]}b_{[0]}\langle t_1,c_{[1]}a_{[1]}b_{[1]}\rangle\langle t_2,S(c_{[-1]}a_{[-1]}b_{[-1]})\rangle\\ &= \sum c_{[0]}a_{[0]}b_{[0]}\langle t_1,c_{[1]}\rangle\langle t_2,a_{[1]}b_{[1]}\rangle\langle t_3,S(a_{[-1]}b_{[-1]})\rangle\langle t_4,S(c_{[-1]})\rangle\\ &= \sum ca_{[0]}b_{[0]}\langle t_1,a_{[1]}b_{[1]}\rangle\langle t_2,S(a_{[-1]}b_{[-1]})\rangle = c(a,b),\\ (a,bc) &= \sum a_{[0]}b_{[0]}c_{[0]}\langle t_1,a_{[1]}b_{[1]}c_{[1]}\rangle\langle t_2,S(a_{[-1]}b_{[-1]}c_{[-1]})\rangle\\ &= \sum a_{[0]}b_{[0]}\langle t_1,a_{[1]}b_{[1]}\rangle\langle t_2,S(a_{[-1]}b_{[-1]})\rangle c = (a,b)c.\end{aligned}$$

(6) Finally, it is easy to verify that $[,]$ and $(,)$ satisfy associativity, so we omit the proof. □

References

[1] *M. M. S. Alves, E. Batista*: Enveloping actions for partial Hopf actions. Commun. Algebra *38* (2010), 2872–2902.
[2] *M. M. S. Alves, E. Batista*: Globalization theorems for partial Hopf (co)actions, and some of their applications. Groups, Algebras and Applications. Proceedings of XVIII Latin American algebra colloquium, São Pedro, Brazil, 2009 (C. Polcino Milies, ed.). Contemporary Mathematics 537, American Mathematical Society, Providence, 2011, pp. 13–30.
[3] *M. M. S. Alves, E. Batista*: Partial Hopf actions, partial invariants and a Morita context. Algebra Discrete Math. *2009* (2009), 1–19.
[4] *M. Beattie, S. Dăscălescu, Ş. Raianu*: Galois extensions for co-Frobenius Hopf algebras. J. Algebra *198* (1997), 164–183.
[5] *S. Caenepeel, K. Janssen*: Partial (co)actions of Hopf algebras and partial Hopf-Galois theory. Commun. Algebra *36* (2008), 2923–2946.
[6] *M. Dokuchaev, R. Exel*: Associativity of crossed products by partial actions, enveloping actions and partial representations. Trans. Am. Math. Soc. *357* (2005), 1931–1952.
[7] *M. Dokuchaev, M. Ferrero, A. Paques*: Partial actions and Galois theory. J. Pure Appl. Algebra *208* (2007), 77–87.
[8] *R. Exel*: Circle actions on C^*-algebras, partial automorphisms, and a generalized Pimsner-Voiculescu exact sequence. J. Funct. Anal. *122* (1994), 361–401.
[9] *C. Lomp*: Duality for partial group actions. Int. Electron. J. Algebra (electronic only) *4* (2008), 53–62.
[10] *M. E. Sweedler*: Hopf Algebras. Mathematics Lecture Note Series, W. A. Benjamin, New York, 1969.

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