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GOSSET POLYTOPES IN INTEGRAL OCTONIONS

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Abstract. We study the integral quaternions and the integral octonions along the combinatorics of the 24-cell, a uniform polytope with the symmetry D_4, and the Gosset polytope 4_{21} with the symmetry E_8.

We identify the set of the unit integral octonions or quaternions as a Gosset polytope 4_{21} or a 24-cell and describe the subsets of integral numbers having small length as certain combinations of unit integral numbers according to the E_8 or D_4 actions on the 4_{21} or the 24-cell, respectively.

Moreover, we show that each level set in the unit integral numbers forms a uniform polytope, and we explain the dualities between them. In particular, the set of the pure unit integral octonions is identified as a uniform polytope 2_{31} with the symmetry E_7, and it is a dual polytope to a Gosset polytope 3_{21} with the symmetry E_7 which is the set of the unit integral octonions with $\mathrm{Re} = 1/2$.

Keywords: integral octonion; 24-cell; Gosset polytope

MSC 2010: 52B20, 06B99, 11Z05

1. Introduction

The octonions have the most complex algebra among the normed division algebras whose classification consists of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. It is one of the worst cases of algebras since its product is neither commutative nor associative. But each element in it is still invertible so that its algebraic operations are related to certain types of symmetries which drive attraction to physics and geometry. Furthermore, the octonion algebra sits on

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the bottom lines of the most complex issues in the exceptional groups including G_2 and E_8. On the other hand, the octonion algebra is an 8-dimensional algebra with the 7-degree of freedom. The number *eight* appears in the elementary particle realm with 8-fold symmetries, and the number *seven* is also related to 7-dimensional hyperspaces acquired for 10-d string theory, 11-d M-theory or 12-d F-theory in physics. In fact, one can find numerous studies utilizing the octonions in physics including the Great Unification Theory or the Theory of Everything. In fact, very often, the E_8-lattice (also called the Gosset lattice) which is a unimodular root lattice of rank 8, appears as a key player to convey the above symmetries on behalf of the octonions.

The symmetries related to the E_8-lattice can be studied via the set of the integral octonions which is an analog of integers in the real numbers. In particular, Coxeter [3] considered the integral octonions (also called integral Cayley numbers) to study the Gosset polytope 4_{21} which is an 8-dimensional uniform polytope with E_8-symmetry. Koca et al. [8], [9], [10] worked on the symmetries given by the integral octonions and the pure integral octonions, and they applied their studies to mathematical physics. Recently, Conway and Smith explained the integral octonions and their relationship to the E_8-lattice in [2]. All of these studies used the hierarchy of the normed algebras. They began with simple symmetries in the integral quaternions and reached to complicated symmetries in the integral octonions. Led by this motivation, in this article we consider the integral quaternions along with the 24-cell and explore the integral octonions along with the Gosset polytope 4_{21}.

According to [3], we reproduce the 24-cell in the integral quaternions and the Gosset polytopes 4_{21} in the integral octonions. And we consider certain shells in these integral numbers and characterize elements in each shell via the combinatorics of the polytopes. Thus we conclude integral numbers whose square of the length < 8 can be written as certain combinations of the unit integral numbers in the quaternions and the octonions.

Finally, we consider level sets in the polytopes where each of them is determined by a fixed real part of the integral numbers. We identify each level set as a uniform polytope given by the symmetry group of the integral numbers. In particular, the set of the pure integral octonions forms a uniform polytope 2_{31} with the symmetry group E_7. Furthermore, we explain the duality between the level sets such as the set of the pure unit integral octonions (which is a polytope 2_{21}) and the set of the unit integral octonions with $\mathrm{Re} = 1/2$ (which is a Gosset 3_{21}) which are dual to each other along the correspondence between the vertices in 2_{21} and the 6-crosspolytopes in 3_{21}.

This approach can be applied to algebraic geometry of rational surfaces including del Pezzo surfaces which have the classical correspondence between lines in del Pezzo surfaces and vertices in Gosset polytopes. This study will be described in another article (also see one of the author's recent works [12], [11]).

2. Integral quaternions and octonions

In this section, we reproduce Coxeter's work [3] on the integral normed division algebras using the modern treatment of the normed division algebras [1], [2].

Let A be an algebra which is a finite dimensional vector space over $\mathbb{R}$ equipped with a multiplication "$\cdot$" and its unit element 1. If the algebra is also a normed vector space with a norm $\| \ \|$ satisfying $\|a \cdot b\| = \|a\|\|b\|$ for all a and b in A, the algebra is called a *normed (division) algebra*. Each norm on the normed algebra gives a derived inner product defined by

$$(a, b) := \tfrac{1}{2}\{\|a + b\|^2 - \|a\|^2 - \|b\|^2\},$$

and moreover, each a in A satisfies the *rank equation*

$$a^2 - 2(a, 1)a + \|a\|^2 = 0.$$

It is well known that the classification of the normed algebras consists of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

A subset S of a normed algebra A is called *integral* (or a set of *integral elements*) if it satisfies the following conditions:

(1) For each element S, the coefficients of the above rank equation are integers.
(2) The set S is closed under subtraction and multiplication.
(3) $1 \in S$.
(4) S is not a proper subset of a subset in A with (1), (2) and (3).

For example, the subset $\{a_0 + a_1 \mathrm{i} \in \mathbb{C};\ a_0, a_1 \in \mathbb{Z}\}$ in $\mathbb{C}$ satisfies the above conditions. The elements of the subset are called the *Gaussian integers*. In the following subsections, we discuss the integral subsets in the quaternions and the octonions.

2.1. Quaternions. The set of the quaternions $\mathbb{H}$ is a real 4-dimensional vector space which is spanned by a basis $\{1, i, j, k\}$, and its normed algebra is given by an associative multiplication satisfying

$$i^2 = j^2 = k^2 = ijk = -1.$$

Here, $\mathbb{H}$ is an extension of complex $\mathbb{C}$, but the multiplication of $\mathbb{H}$ is not commutative.

Now, we consider a subset $\mathbb{H}_I$ in $\mathbb{H}$ consisting of the quaternions $a_0 1 + a_1 i + a_2 j + a_3 k$ whose coefficients a_0, a_1, a_2, a_3 are synchronically chosen from either $\mathbb{Z}$ or $\mathbb{Z}+1/2$, namely,

$$\mathbb{H}_I = \left\{ q \in \mathbb{H}\,;\ \begin{array}{c} q = b_0 1 + b_1 i + b_2 j + b_3 k \quad \text{or} \\ b_0 1 + b_1 i + b_2 j + b_3 k + \frac{1}{2}(1+i+j+k) \\ \text{for} \quad b_0, b_1, b_2, b_3 \in \mathbb{Z} \end{array} \right\}.$$

In fact, this subset $\mathbb{H}_I$ is integral and it is known as the Hurwitz integral quaternions which we consider in this article.

For the integral quaternions $\mathbb{H}_I$, we consider integral shells defined by

$$\mathbb{H}_I(n) := \{q \in \mathbb{H}_I\,;\ \|q\|^2 = n\}.$$

For small natural numbers $1 \leqslant n \leqslant 6$, we can write each shell explicitly as follows:

$$\mathbb{H}_I(1) = \left\{\pm 1,\ \pm i,\ \pm j,\ \pm k,\ \pm\tfrac{1}{2} \pm \tfrac{1}{2}i \pm \tfrac{1}{2}j \pm \tfrac{1}{2}k\right\},$$

$$\mathbb{H}_I(2) = \left\{ \pm a_0 \pm a_1 i \pm a_2 j \pm a_3 k \in \mathbb{H}\,;\ \begin{array}{l} (a_0, a_1, a_2, a_3) = (1,1,0,0) \\ \text{and its permutations} \end{array} \right\},$$

$$\mathbb{H}_I(3) = \left\{ \pm a_0 \pm a_1 i \pm a_2 j \pm a_3 k \in \mathbb{H}\,;\ \begin{array}{l} (a_0, a_1, a_2, a_3) = (1,1,1,0),\ \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right) \\ \text{and their permutations} \end{array} \right\},$$

$$\mathbb{H}_I(4) = \left\{ \pm a_0 \pm a_1 i \pm a_2 j \pm a_3 k \in \mathbb{H}\,;\ \begin{array}{l} (a_0, a_1, a_2, a_3) = (2,0,0,0),\ (1,1,1,1) \\ \text{and their permutations} \end{array} \right\},$$

$$\mathbb{H}_I(5) = \left\{ \pm a_0 \pm a_1 i \pm a_2 j \pm a_3 k \in \mathbb{H}\,;\ \begin{array}{l} (a_0, a_1, a_2, a_3) = (2,1,0,0),\ \left(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right) \\ \text{and their permutations} \end{array} \right\},$$

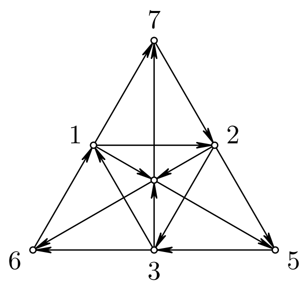
$$\mathbb{H}_I(6) = \left\{ \pm a_0 \pm a_1 i \pm a_2 j \pm a_3 k \in \mathbb{H}\,;\ \begin{array}{l} (a_0, a_1, a_2, a_3) = (2,1,1,0) \\ \text{and its permutations} \end{array} \right\}.$$

Here, $|\mathbb{H}_I(1)| = |\mathbb{H}_I(2)| = |\mathbb{H}_I(4)| = 24$, $|\mathbb{H}_I(3)| = |\mathbb{H}_I(6)| = 96$ and $|\mathbb{H}_I(5)| = 144$.

2.2. Octonions. The set of octonions $\mathbb{O}$ is a real 8-dimensional vector space spanned by $\{1, e_1, \ldots, e_7\}$ whose multiplication is given by the following table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

This table is also written as the Fano plane. Here the square of each vector e_i is -1, and the multiplication of two vectors on one side of the Fano plane produces the other vector on the side where $(+)$ sign is given if the order of the multiplication is matched with the direction of the arrow of the side and otherwise $(-)$ sign is given. Here the triangle formed by e_1, e_2 and e_3 is also considered as a side.



Fano plane.

Just like the quaternions $\mathbb{H}$, the multiplication of the octonions $\mathbb{O}$ is not commutative, and furthermore, it is not associative. This lack of associativity makes the research on the octonions $\mathbb{O}$ very difficult and complicated, but it is also the main source of anomalites in mathematical physics such as the M-theory.

We define a subset $\mathbb{O}_I$ in $\mathbb{O}$ consisting of the octonions $a_0 1 + a_1 e_1 + \ldots + a_7 e_7$ whose coefficients $a_0, a_1, \ldots, a_7$ are synchronically chosen from either $\mathbb{Z}$ or $\mathbb{Z}+1/2$ to satisfy $\sum_{i=0}^{7} a_i \in 2\mathbb{Z}$ and the relationships in the Fano plane. Each four choices in the Fano plane not containing a line in the Fano plane are called a *nonassociative block*. We also consider three choices in the Fano plane forming a line. After adding 1 to such three choices, we call such choice of four octonions an *associative block*. In fact, to define the integral octonions, we need to replace e_1 by 1 for each nonassociative block and associative block, and we denote by $\mathcal{B}$ the set of four chosen octonions given by the substitution on these blocks. Define

$$\mathbb{O}_I = \left\{ a_0 1 + a_1 e_1 + \ldots + a_7 e_7 \in \mathbb{O}; \begin{array}{l} a_i \in \mathbb{Z} \cup \left(\mathbb{Z} + \frac{1}{2}\right) \text{ with } \sum_{i=0}^{7} a_i \in 2\mathbb{Z}. \\ \text{when four } a_i \text{ are in } \mathbb{Z} + 1/2, \\ \text{the four choices are given from } \mathcal{B} \end{array} \right\}.$$

One can show that this subset $\mathbb{O}_I$ satisfies the conditions to be integral, and in fact it is known as the Cayley integral numbers or the integral octonions.

We also consider shells $\mathbb{O}_I(n)$ in $\mathbb{O}_I$ consisting of the integral octonions whose square of length is n. For example, we identify $\mathbb{O}_I(1)$ as follows:

$$\mathbb{O}_I(1) = \left\{ \begin{array}{l} \pm 1,\ \pm e_1,\ \pm e_2,\ \pm e_3,\ \pm e_4,\ \pm e_5,\ \pm e_6,\ \pm e_7, \\ \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3),\ \frac{1}{2}(\pm 1 \pm e_1 \pm e_4 \pm e_5), \\ \frac{1}{2}(\pm 1 \pm e_1 \pm e_6 \pm e_7),\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_4 \pm e_6), \\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_5 \pm e_7),\ \frac{1}{2}(\pm e_1 \pm e_3 \pm e_4 \pm e_7), \\ \frac{1}{2}(\pm e_1 \pm e_3 \pm e_5 \pm e_6),\ \frac{1}{2}(\pm 1 \pm e_2 \pm e_4 \pm e_7), \\ \frac{1}{2}(\pm 1 \pm e_2 \pm e_5 \pm e_6),\ \frac{1}{2}(\pm e_2 \pm e_3 \pm e_4 \pm e_5), \\ \frac{1}{2}(\pm 1 \pm e_3 \pm e_5 \pm e_7),\ \frac{1}{2}(\pm 1 \pm e_3 \pm e_4 \pm e_6), \\ \frac{1}{2}(\pm e_2 \pm e_3 \pm e_6 \pm e_7),\ \frac{1}{2}(\pm e_4 \pm e_5 \pm e_6 \pm e_7) \end{array} \right\}.$$

The first shell $\mathbb{O}_I(1)$ has 240 integral octonions, and the second shell $\mathbb{O}_I(2)$ contains 2160 elements in $\mathbb{O}_I$ whose square of length is 2. In general, $|\mathbb{O}_I(n)|$ is given by $240 \sum\limits_{d|n} d^3$.

3. 24-cell and Gosset polytope

In this article, we deal with polytopes with highly nontrivial symmetries whose symmetry groups (called *Coxeter groups*) play key roles in the corresponding Coxeter-Dynkin diagrams. In this section, we introduce the general theory of the regular and semiregular polytopes according to their symmetry groups and the corresponding Coxeter-Dynkin diagrams. In particular, we consider the family of semiregular polytopes known as the Gosset polytopes (k_{21} according to Coxeter). Here, we only present a brief introduction, and for further details the reader should look up [4], [5], [7], [6], and [12].

We consider a convex n-polytope P_n in an n-dimensional Euclidean space. For each vertex of P_n, the set of the midpoints of all the edges emanating from the vertex in P_n is called the *vertex figure* of P_n at the vertex when it forms an $(n-1)$-polytope. A *regular* polytope P_n ($n \geqslant 2$) is a polytope whose facets and the vertex figure at each vertex are regular. Naturally, the facets of a regular P_n are all congruent, and the vertex figures are all the same. A polytope P_n is called *semiregular* if its facets are regular and its vertices are equivalent, namely, the symmetry group of P_n acts transitively on the vertices of P_n.

The *Coxeter groups* are reflection groups generated by the reflections with respect to hyperplanes (called mirrors) and the relationships between them are given by the *Coxeter-Dynkin diagrams.* The Coxeter-Dynkin diagrams of Coxeter groups are labeled graphs where their nodes present indexed mirrors and the labels on edges present the order n of dihedral angle π/n between two mirrors. If two mirrors are perpendicular, namely $n = 2$, no edge joins two nodes presenting the corresponding mirrors, and this also implies that there is no interaction between the mirrors. Since the dihedral angle $\pi/3$ appears very often, we only label the edges when the corresponding order is $n > 3$. Each Coxeter-Dynkin diagram contains at least one ringed node which represents an active mirror, i.e., there is a point off the mirror, and the construction of a polytope begins with reflecting the point through the active mirror. The Coxeter-Dynkin diagrams of polytopes considered in this article have only one ringed node and no labeled edges. These are called polytopes with ADE-type reflection groups. For these cases, the following simple procedure using the Coxeter-Dynkin diagrams describes possible subpolytopes and gives the total number of them.

The Coxeter-Dynkin diagram of each subpolytope P' is a connected subgraph Γ containing the ringed node. And the subgraph obtained by taking off all the nodes joined with the subgraph Γ represents the isotropy group $G_{P'}$ of P'. Furthermore, the index between the symmetry group G of the ambient polytope and the isotropy group $G_{P'}$ gives the total number of such subpolytopes. In particular, by taking off the ringed node, we obtain the subgraph corresponding to the isotropy group of a vertex, and in fact the isotropy group is the symmetry group of the vertex figure. We present this process in detail for the 24-cell and Gosset polytope 4_{21} below. Here the orders of ADE-type of Coxeter groups are given in the following table.

Coxeter group	A_n	D_n	E_6	E_7	E_8
order	$(n+1)!$	$2^{n-1}n!$	$2^7 3^4 5$	$2^{10}3^4 5\times 7$	$2^{14}3^5 5^2 7$

Order of Weyl groups.

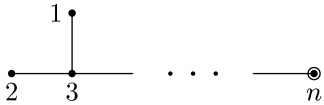
The most fundamental polytopes with ADE-type reflection groups are the following two classes of regular polytopes and one class of semiregular polytopes.

(1) (A-type) A *regular simplex* α_n is an n-dimensional simplex with equilateral edges. Then α_n is inductively constructed as a pyramid based on an $(n-1)$-dimensional simplex α_{n-1}. Thus the facets of a regular simplex α_n are regular simplexes α_{n-1}, and the vertex figure of α_n is also α_{n-1}. For a regular simplex α_n, only regular simplexes α_k, $0 \leqslant k \leqslant n-1$ appear as subpolytopes.

Coxeter-Dynkin diagram of α_n.

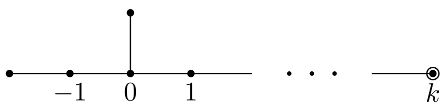
(2) (D-type) A *crosspolytope* β_n is an n-dimensional polytope whose $2n$-vertices are the intersects of an n-dimensional Cartesian coordinate frame and a sphere cen-

tered at the origin. And β_n is also inductively constructed as a bipyramid based on an $(n-1)$-dimensional crosspolytope β_{n-1}, and the n-vertices in β_n form a simplex α_{n-1} if the choice is made of one vertex from each Cartesian coordinate line. So the vertex figure of a crosspolytope β_n is also a crosspolytope β_{n-1}, and the facets of β_n are simplexes α_{n-1}. For a crosspolytope β_n, only the regular simplexes α_k, $0 \leqslant k \leqslant n-1$ appear as subpolytopes.



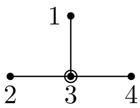
Coxeter-Dynkin diagram of β_n.

(3) (E-type) *Gosset polytopes* k_{21} ($k = -1, 0, 1, 2, 3, 4$) are semiregular polytopes discovered by Gosset which are $(k+4)$-dimensional polytopes whose symmetry groups are the Coxeter groups E_{k+4}. Here the vertex figure of k_{21} is $(k-1)_{21}$. And for $k \neq -1$ the facets of k_{21}-polytopes are the regular simplexes α_{k+3} and the crosspolytopes β_{k+3}, but all the lower dimensional subpolytopes are regular simplexes. In this article, we focus on the Gosset polytope 4_{21} with a Coxeter group E_8.



Coxeter-Dynkin diagram of k_{21}, $k \neq -1$.

3.1. 24-cell. The 24-cell is a 4-dimensional convex regular polytope bounded by 24 octahedra whose Coxeter group is known to be F_4 with order 1152 or B_4 with order 384. In fact, we observe that the Coxeter group D_4 with the Coxeter-Dynkin diagram



Coxeter-Dynkin diagram of 24-cell.

gives the 24-cell. Moreover, D_4 is the better choice for us to identify the 24-cell as the unit integral quaternions in $\mathbb{H}_I$ and to extend the story for Gosset 4_{21} in $\mathbb{O}_I$ via the Coxeter-Dynkin diagram.

The 24-cell is one of very well-known polytopes thanks to its interesting properties such as self-duality. In this article, we also consider the dualities between its subpolytopes. In the following, we describe fundamental relationships between combinatorics of the 24-cell and the Coxeter-Dynkin diagram. Similar relationships between Gosset 4_{21} and the corresponding Coxeter-Dynkin diagram will play key roles for this article.

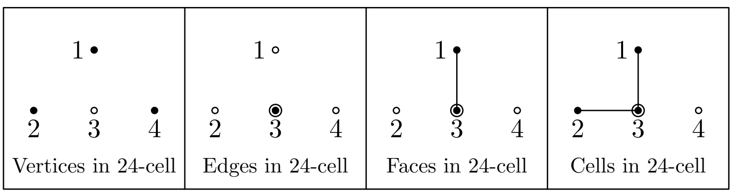
In the following we describe the subpolytopes in the 24-cell via D_4-action and obtain their total numbers.

(1) *Vertices*: The diagram of the vertex figure of the 24-cell is of $A_1 \times A_1 \times A_1$-type because the subgraph remaining after removing the ringed node represents $A_1 \times A_1 \times A_1$. Thus the total number of vertices $N^{24}_{\alpha_0}$ is obtained by $N^{24}_{\alpha_0} = [D_4\colon A_1 \times A_1 \times A_1] = 2^3 4!/(2! \cdot 2! \cdot 2!) = 24$.

(2) *Edges*: The edges in the 24-cell are A_1-type regular simplexes α_1. And the isotropy group is obtained by the subgraph of A_1 containing the ringed node and another subgraph given by taking off the subgraph of A_1 and the nodes adjacent to the subgraph of A_1. As there is no subgraph remaining after this taking off process, the isotropy group is A_1. Thus the total number of edges $N^{24}_{\alpha_1}$ is obtained by $N^{24}_{\alpha_1} = [D_4\colon A_1] = 2^3 4!/2! = 96$.

(3) *Faces* (α_2-simplexes): The faces in the 24-cell are A_2-type regular simplexes α_2, and there are three different ways to get subgraphs of A_2-type containing the ringed node in the Coxeter-Dynkin diagram of the 24-cell. And the isotropy group is A_2. Thus the total number of faces $N^{24}_{\alpha_2}$ is obtained by $N^{24}_{\alpha_1} = 3[D_4\colon A_2] = 3 2^3 4!/3! = 96$.

(4) *Cells* (β_3-crosspolytopes): The cells in the 24-cell are regular octahedra because the subgraph of the cells in the 24-cell is  which is the Coxeter-Dynkin diagram of the regular rectified tetrahedron, namely the octahedron. In fact, the regular octahedra are 3-dimensional crosspolytopes β_3. There are three different ways to get subgraphs of the rectified tetrahedron containing the ringed node in the Coxeter-Dynkin diagram of the 24-cell. And the isotropy group is A_3. Thus the total number of cells $N^{24}_{\beta_3}$ is attained by $N^{24}_{\beta_3} = 3[D_4\colon A_3] = 3 2^3 4!/4! = 24$.

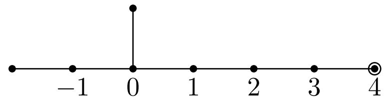


Coxeter-Dynkin diagrams of subpolytopes in 24-cell.

In the above calculation for the 24-cell, the nodes marked by empty circles represent the deleted nodes.

3.2. Gosset polytope 4_{21}. The Gosset polytope 4_{21} is a convex semiregular uniform polytope given by the Coxeter group E_8 according to the following Coxeter-

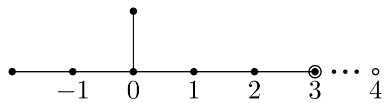
Dynkin diagram:



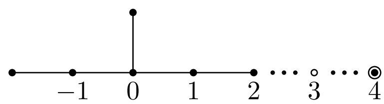
Coxeter-Dynkin diagram of 4_{21}.

As above, we can describe the subpolytopes in 4_{21} and calculate the total number of faces in 4_{21} by using the Coxeter-Dynkin diagram. For instance, to calculate the total number of vertices in 4_{21}, we remove the ringed node labelled 4 and transfer the ring to the node labelled 3 so that we obtain a subgraph of E_7-type. Here, the vertex figure of 4_{21} is 3_{21}. Since the subgraphs of A_7-type and D_7-type are all the possible biggest subgraphs in the Coxeter-Dynkin diagram of 4_{21}, there are two types of facets in 4_{21}, which are 7-simplexes and 7-crosspolytopes, respectively. And all other facets in 4_{21} are simplexes for the same reason. In the following calculation for 4_{21}, the nodes marked by empty circles represent deleted nodes.

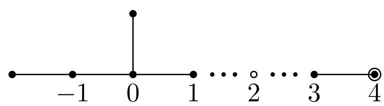
(1) Vertices in 4_{21}: $N^{4_{21}}_{\alpha_0} = [E_8 \colon E_7] = 2^{14}3^5 5^2 7/(2^{10}3^4 5 \times 7) = 240$.



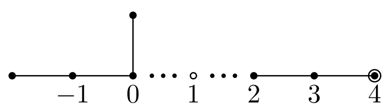
(2) 1-simplexes (edges) in 4_{21}: $N^{4_{21}}_{\alpha_1} = [E_8 \colon A_1 \times E_6] = 2^{14}3^5 5^2 7/(2! \times 2^7 3^4 5) = 6720$.



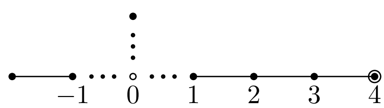
(3) 2-simplexes (facets) in 4_{21}: $N^{4_{21}}_{\alpha_2} = [E_8 \colon A_2 \times D_5] = 2^{14}3^5 5^2 7/(3! \times 2^4 5!) = 60480$.



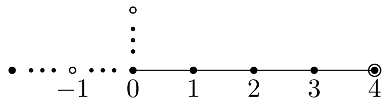
(4) 3-simplexes (cells) in 4_{21}: $N^{4_{21}}_{\alpha_3} = [E_8 \colon A_3 \times A_4] = 2^{14}3^5 5^2 7/(4! \times 5!) = 241920$.



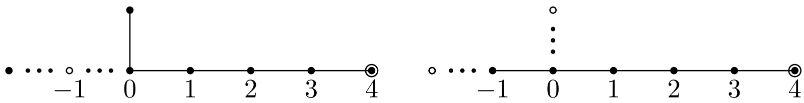
(5) 4-simplexes in 4_{21}: $N^{4_{21}}_{\alpha_4} = [E_8 \colon A_4 \times A_2 \times A_1] = 2^{14}3^5 5^2 7/(5! \times 3! \times 2!) = 483840$.



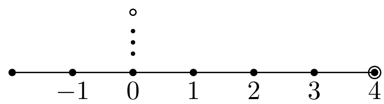
(6) 5-simplexes in 4_{21}: $N^{4_{21}}_{\alpha_5} = [E_8\colon\ A_5 \times A_1] = 2^{14}3^5 5^2 7/(6! \times 2!) = 483840.$



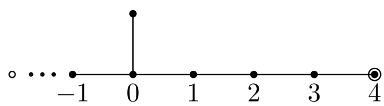
(7) 6-simplexes in 4_{21}: $N^{4_{21}}_{\alpha_6} = [E_8\colon\ A_6 \times A_1] + [E_8\colon\ A_6] = 2^{14}3^5 5^2 7/(7! \times 2!) + 2^{14}3^5 5^2 7/7! = 69120 + 138240 = 207360.$



(8) 7-simplexes in 4_{21}: $N^{4_{21}}_{\alpha_7} = [E_8\colon\ A_7] = 2^{14}3^5 5^2 7/8! = 17280.$



(9) 7-crosspolytopes in 4_{21}: $N^{4_{21}}_{\beta_7} = [E_8\colon\ D_7] = 2^{14}3^5 5^2 7/(2^6 \times 7!) = 2160.$



4. Polytopes in integral normed algebras

In this section, we consider the reflections defined for the integral normed algebras and study polytopes in the algebras constructed by the reflections as in the work of Coxeter [3]. We characterize Coxeter's reflections producing Coxeter groups and recover properties of polytopes in [3] along with the Coxeter groups. Moreover, we discover hidden dualities in the polytopes.

4.1. Reflections on normed algebra. For each element a in a normed algebra A, we define the conjugate of a by $\bar{a} := -a + 2(a,1)1$. We get the following useful and well-known lemmas by direct calculations.

Lemma 1. *For elements a, x, y in a normed algebra A, we have*

$$(x \cdot y, a) = (y, \bar{x} \cdot a) = (x, a \cdot \overline{y}).$$

P r o o f. From the definition of the normed algebra, we get $(a{\cdot}x, b{\cdot}y)+(a{\cdot}y, b{\cdot}x) = 2(a,b)(x,y)$ and apply $\bar{x} = -x + 2(x,1)1$ to get the lemma. $\square$

In [3], the reflection for the normed algebra A is defined by

$$\sigma_a(x) := -a \cdot \bar{x} \cdot a$$

for $a \in A$ with $\|a\| = 1$. When the normed algebra is the octonions $\mathbb{O}$, even though it is not associative, the map is well-defined since $(a \cdot \bar{x}) \cdot a = a \cdot (\bar{x} \cdot a)$ by the Artin theorem. The following lemma shows that this map is a reflection indeed.

Lemma 2. *For each a in a normed algebra A with $\|a\| = 1$,*

$$\sigma_a(x) = x - 2(x,a)a.$$

P r o o f. By applying the above lemma we obtain

$$\begin{aligned}\sigma_a(x) &= -(a \cdot \bar{x}) \cdot a = -(-x \cdot \bar{a} + 2(a \cdot \bar{x}, 1)1) \cdot a \\ &= (x \cdot \bar{a}) \cdot a + 2(a,x)a = x \cdot (\bar{a} \cdot a) + 2(a,x)a \\ &= x - 2(x,a)a.\end{aligned}$$

Here we use $x + \bar{x} = 2(x,1)1$ on the first line and the above lemma on the second line. $\square$

Remark 3. By this lemma, the reflection hyperplane of the reflection σ_a is the linear hyperplane perpendicular to the unit vector a. Moreover, the reflection σ_a preserves each affine hyperplane perpendicular to a in A. In particular, if σ_a fixes 1 in A, we have $\operatorname{Re} x = \operatorname{Re} \sigma_a(x)$.

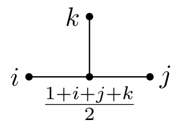
Since each reflection is an orthogonal transform, we have $\|\sigma_a(x)\| = \|x\|$. Moreover, it preserves the set of integral elements A_I by virtue of the following lemma. Thus each shell $A_I(n)$ consisting of elements in A_I with the square of length $= n$ is preserved by the reflection.

Lemma 4. *For $b \in A_I$ with $\|b\| = 1$, the corresponding reflections σ_b acts on the set A_I of the integral elements in A.*

P r o o f. We consider an integral element x in A_I. The conjugate $\bar{x}$ is also integral because $(x,1)$ is an integer. Since A_I is closed under multiplication, $\sigma_b(x) = -b\cdot\bar{x}\cdot b$ is integral. This gives the lemma. $\square$

For a, $b \in A$ with $\|a\| = \|b\| = 1$, the corresponding reflections σ_a and σ_b are related via the angle between a and b which is $\cos^{-1}(a,b) = \cos^{-1}\frac{1}{2}(\bar{a}b+\bar{b}a)$. Thus the proper choice of the unit elements in A satisfying some relations produces a Coxeter group. If we choose the unit elements to be integral and satisfy the relationships given by the above Dynkin diagrams, naturally the obtained Coxeter groups act on A_I. In particular, we consider the Coxeter groups for $\mathbb{H}_I$ and $\mathbb{O}_I$ as follows.

4.2. 24**-cell in integral quaternions.** We consider the integral quaternions $\mathbb{H}_I$ and a Coxeter group generated by the following Dynkin diagram where the reflections are given by the unit integral quaternions in the diagram.



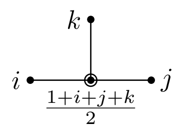
Dynkin diagram of D_4.

Here, the Coxeter group presented by the Dynkin diagram is a Weyl group of type D_4.

Theorem 5. *The convex hull of* $\mathbb{H}_I(1)$ *in* $\mathbb{H}$ *is a* 24*-cell.*

P r o o f. As the Coxeter group D_4 acts on $\mathbb{H}_I(1)$, it is enough to show that $\mathbb{H}_I(1)$ is one D_4-orbit and check if its Coxeter-Dynkin diagram presents the 24-cell.

If we consider $1 \in \mathbb{H}_I(1)$, then we have $\sigma_i(1) = \sigma_j(1) = \sigma_k(1) = 1$ but $\sigma_{(1+i+j+k)/2}(1) = (1-i-j-k)/2$. Thus the D_4-orbit of 1 in $\mathbb{H}_I(1)$ is the set of vertices of a 24-cell given by a Coxeter-Dynkin diagram



Coxeter-Dynkin diagram of 24-cell.

Since $|\mathbb{H}_I(1)| = 24 =$ the number of vertices of a 24-cell, we show that $\mathbb{H}_I(1)$ transitively acts by D_4 and it is the set of vertices of a 24-cell. This proves the theorem. $\square$

Remark 6. In fact, the shell $\mathbb{H}_I(1)$ is a finite group of order 24, but it is not isomorphic to $\mathrm{SL}(2,\mathbb{Z}_3)$.

As in subsection 3.1, the subpolytopes in a 24-cell are regular polytopes α_0 (vertex), α_1 (edges), α_2 (regular 2-simplex) and β_3 (3-crosspolytope). Each subpolytope in a 24-cell is identified by its barycenter. As we want to stick to integral elements in the normed algebra, we take alternate barycenters as follows. The barycenter of a regular simplex α_n is the sum of all the vertices, and the barycenter of a crosspolytope β_n is the sum of an antipodal pair of vertices in it. Therefore, by the above theorem, the barycenter of a regular simplex α_n is the sum of $(n+1)$-unit integral elements in $\mathbb{H}_I(1)$ whose inner product with each other is $1/2$, and the barycenter of a crosspolytope β_n is the sum of two unit integral elements in $\mathbb{H}_I(1)$ whose inner product is 0.

Now we describe the relationship between a few first shells in $\mathbb{H}_I$ and the subpolytopes in the 24-cell.

(1) For $b \in \mathbb{H}_I(2)$, by simple calculation we know that $b = a_1 + a_2$ for $a_i \in \mathbb{H}_I(1)$ with $(a_1, a_2) = 0$. Equivalently, b represents a center of a 3-crosspolytope. Thus $\mathbb{H}_I(2)$ corresponds to the set of 3-crosspolytopes (β_3) in the 24-cell. Note that each $b \in \mathbb{H}_I(2)$ can be written by three pairs of elements with $(a_1, a_2) = 0$ in $\mathbb{H}_I(1)$ corresponding to three antipodal pairs in the 3-crosspolytope.

(2) For $c \in \mathbb{H}_I(3)$, by simple calculation and comparison, we get $c = a_1 + a_2$ for $a_i \in \mathbb{H}_I(1)$ with $(a_1, a_2) = 1/2$. Equivalently, c represents an edge in the 24-cell. Thus $\mathbb{H}_I(3)$ corresponds to the set of edges (α_1) in the 24-cell. Note that each $c \in \mathbb{H}_I(3)$ can be written by only one pair of elements with $(a_1, a_2) = 1/2$ in $\mathbb{H}_I(1)$.

(3) For $d \in \mathbb{H}_I(4)$, we get $d = 2a_1$ for some $a_1 \in \mathbb{H}_I(1)$. Thus $\mathbb{H}_I(4)$ corresponds to the set of vertices in the 24-cell, and each d is uniquely determined by $\mathbb{H}_I(1)$.

(4) Each element $e \in \mathbb{H}_I(5)$ can be written as $e = 2a_1 + a_2$ for $a_i \in \mathbb{H}_I(1)$ with $(a_1, a_2) = 0$. Since $2a_1 + a_2 = (a_1 + a_2) + a_1$, the element e corresponds to a 3-crosspolytope β_3 represented by $a_1 + a_2$ and a vertex a_1 in it. In fact, since the 3-crosspolytope β_3 contains three pairs of antipodal vertices producing the common center $a_1 + a_2$, we obtain $|\mathbb{H}_I(5)| = 24 \times 6 = 144$.

(5) For $f \in \mathbb{H}_I(6)$, we observe $f = a_1 + a_2 + a_3$ for $a_i \in \mathbb{H}_I(1)$ with $(a_i, a_j) = 1/2$ and obtain the correspondence between $\mathbb{H}_I(6)$ and the set of 2-simplexes in the 24-cell. Thus, there is only one triple of elements in $\mathbb{H}_I(1)$ presenting f.

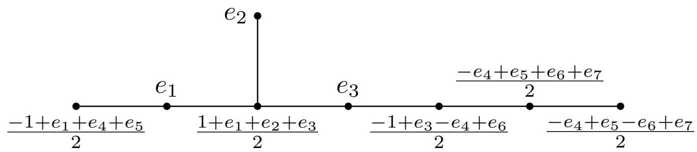
In summary, we obtain the following identifications for a few first shells in $\mathbb{H}_I$.

$\mathbb{H}_I(i)$	$\lvert\mathbb{H}_I(i)\rvert$	24-cell	$a_i \in \mathbb{H}_I(1)$
$\mathbb{H}_I(1)$	24	vertex (α_0)	a_1
$\mathbb{H}_I(2)$	24	cell (β_3)	$a_1 + a_2$ with $(a_1, a_2) = 0$
$\mathbb{H}_I(3)$	96	edge (α_1)	$a_1 + a_2$ with $(a_1, a_2) = 1/2$
$\mathbb{H}_I(4)$	24	vertex (α_0)	$2a_1$
$\mathbb{H}_I(5)$	144	vertex in each β_3	$2a_1 + a_2$ with $(a_1, a_2) = 0$
$\mathbb{H}_I(6)$	96	face α_2	$a_1 + a_2 + a_3$ with $(a_i, a_j) = 1/2$

Table 1: Small shells in $\mathbb{H}_I$.

Remark 7. For each b in $\mathbb{H}_I(2)$, the map $\varphi_b(x) := b \cdot x$ gives correspondences among $\mathbb{H}_I(1)$, $\mathbb{H}_I(2)$ and $\mathbb{H}_I(4)$. This also produces a correspondence between $\mathbb{H}_I(3)$ and $\mathbb{H}_I(6)$. Therefrom, we obtain the well-known self duality of the 24-cell.

4.3. Gosset polytope 4_{21} in integral octonions. Now, we consider the integral octonions $\mathbb{O}_I$ and the Coxeter group generated by the following Dynkin diagram where the reflections are given by the unit integral octonions in the diagram:



Coxeter-Dynkin diagram of E_8.

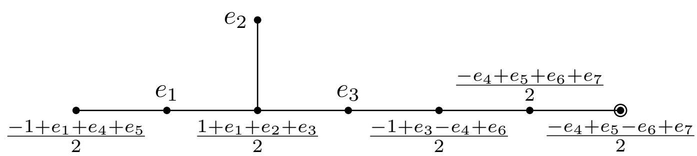
Here, the Coxeter group presented by the Dynkin diagram is a Weyl group of type E_8, and the Dynkin diagram is written as an extension of the diagram of D_4 for $\mathbb{H}_I$.

As the convex hull of $\mathbb{H}_I(1)$ in $\mathbb{H}$ gives a 24-cell, we have a similar conclusion for $\mathbb{O}_I(1)$ as follows.

Theorem 8. *The convex hull of $\mathbb{O}_I(1)$ in $\mathbb{O}$ is a Gosset polytope 4_{21}.*

Proof. We consider $b = (-e_4 + e_5 - e_6 + e_7)/2 \in \mathbb{O}_I(1)$ and its E_8-orbit.

The reflection $\sigma_{(-e_4+e_5-e_6+e_7)/2}$ is the only active reflection moving b among the reflections in the above Dynkin diagram. Thus the E_8-orbit of b in $\mathbb{O}_I(1)$ is the set of vertices of the Gosset polytope 4_{21} obtained by the Coxeter-Dynkin diagram



Coxeter-Dynkin diagram of 4_{21}.

Since $|\mathbb{O}_I(1)| = 240 =$ the number of vertices of 4_{21}, we show that $\mathbb{O}_I(1)$ transitively acts by E_8 and it is the set of vertices of 4_{21}. This gives the theorem. $\square$

As in a 24-cell, the subpolytopes in 4_{21} can be identified by its barycenter. Again, by the above theorem, the barycenter of a regular simplex α_n is the sum of $(n+1)$-unit integral elements in $\mathbb{O}_I(1)$ whose inner product with each other is $1/2$, and the barycenter of the crosspolytope β_7 is the sum of two unit integral elements in $\mathbb{O}_I(1)$ whose inner product is 0.

Now we describe the relationships between a few first shells in $\mathbb{O}_I$ and the subpolytopes in 4_{21}.

Note the following description may not be obtained by simple calculation. As a matter of fact, it is motivated by the comparison of the combinatorics of the Gosset polytope k_{21} and special divisors in del Pezzo surfaces in [12], [11] via the representation theory. However, as we consider 4_{21} in $\mathbb{O}_I$, the comparison can be done by studying algebraic relationships in $\mathbb{O}_I$ via the representation theory. This is the major benefit of considering 4_{21} in $\mathbb{O}_I$.

(1) For $b \in \mathbb{O}_I(2)$, by simple calculation we know $b = a_1 + a_2$ for $a_i \in \mathbb{O}_I(1)$ with $(a_1, a_2) = 0$. Equivalently, b represents a center of a 7-crosspolytope. Thus $\mathbb{O}_I(2)$ corresponds to the set of 7-crosspolytopes (β_3) in 4_{21}. Moreover each $b \in \mathbb{O}_I(2)$ can be written by seven pairs of elements with $(a_1, a_2) = 0$ in $\mathbb{O}_I(1)$ corresponding to seven antipodal pairs in the 7-crosspolytope.

(2) For $c \in \mathbb{O}_I(3)$, by simple calculation and comparison, we get $c = a_1 + a_2$ for $a_i \in \mathbb{O}_I(1)$ with $(a_1, a_2) = 1/2$. Equivalently, c represents an edge in 4_{21}. Thus $\mathbb{O}_I(3)$ corresponds to the set of edges (α_1) in 4_{21}. Therefore each $c \in \mathbb{O}_I(3)$ can be written by only one pair of elements with $(a_1, a_2) = 1/2$ in $\mathbb{O}_I(1)$.

(3) A shell $\mathbb{O}_I(4)$ consists of two E_8-orbits which correspond to a set of vertices in 4_{21} and a set of 7-simplexes which is one of the two types of 7-faces in 4_{21}. To see the correspondence between the set of 7-simplexes and one of the E_8-orbits in $\mathbb{O}_I(4)$, we consider a 7-simplex given by

$$\left\{\begin{array}{l} e_1, \frac{1}{2}(1+e_1+e_2+e_3), \frac{1}{2}(1+e_1+e_4+e_5), \frac{1}{2}(1+e_1+e_6+e_7), \\ \frac{1}{2}(e_1+e_2+e_4+e_6), \frac{1}{2}(e_1+e_2+e_5+e_7), \frac{1}{2}(e_1+e_3+e_4+e_7), \\ \frac{1}{2}(e_1+e_3+e_5+e_6) \end{array}\right\}.$$

Here the corresponding barycenter is the sum of all the vertices

$$\frac{9e_1 + 3(1+e_2+\ldots+e_7)}{2} = 3\Big(e_1 + \frac{1+e_1+e_2+e_3}{2} + \frac{e_4+e_5+e_6+e_7}{2}\Big) \in 3\mathbb{O}_I$$

and the barycenter is divisible by 3. Therefore, each $d \in \mathbb{O}_I(4)$ can be written as (i) $2a_1$ for some $a_1 \in \mathbb{O}_I(1)$ just like for some $\mathbb{H}_I(4)$ and (ii) $d = \frac{1}{3}\sum_{i=1}^{8} a_i$ for $a_i \in \mathbb{O}_I(1)$ with $(a_i, a_j) = 1/2$. Thus $\mathbb{O}_I(4)$ is the union of two Weyl orbits corresponding to the set of vertices in 4_{21} (case (i)) and the set of 7-simplexes in 4_{21} (case (ii)). Moreover, since each center of 7-simplexes in 4_{21} is uniquely determined, each $d \in \mathbb{O}_I(4)$ can be uniquely written by the seven $a_i \in \mathbb{O}_I(1)$ with $(a_i, a_j) = 1/2$.

(4) By applying the method of $\mathbb{H}_I(5)$ similarly to $\mathbb{O}_I(5)$, we obtain that $e \in \mathbb{O}_I(5)$ can be written as $e = 2a_1 + a_2$ for $a_i \in \mathbb{O}_I(1)$ with $(a_1, a_2) = 0$. And the element e corresponds to a 7-crosspolytope β_7 represented by $a_1 + a_2$ and a vertex a_1 in it.

(5) Again, by applying the method of $\mathbb{H}_I(6)$ similarly to $\mathbb{O}_I(6)$, we obtain that each $f \in \mathbb{O}_I(6)$ can be written as $f = a_1 + a_2 + a_3$ for $a_i \in \mathbb{O}_I(1)$ with $(a_i, a_j) = 1/2$ and get the correspondence between $\mathbb{O}_I(6)$ and the set of 2-simplexes in 4_{21}. Thus, there is only one triple of elements in $\mathbb{O}_I(1)$ presenting f.

(6) By reasoning similar to that used in $\mathbb{O}_I(4)$, one can get the description for $\mathbb{O}_I(7)$. Note that the 6-faces in 4_{21} consist of two E_8-orbits where one of them is contained in $\mathbb{O}_I(7)$.

Summarizing, we obtain the following identifications for shells in $\mathbb{O}_I$.

$\mathbb{O}_I(i)$	$\lvert\mathbb{O}_I(i)\rvert$	4_{21}	$a_i \in \mathbb{O}_I(1)$, $b \in \mathbb{O}_I(2)$
$\mathbb{O}_I(1)$	240	vertex (α_0)	a_1
$\mathbb{O}_I(2)$	2160	7-face (β_7)	$a_1 + a_2$, $(a_1, a_2) = 0$
$\mathbb{O}_I(3)$	6720	edge (α_1)	$a_1 + a_2$, $(a_1, a_2) = 1/2$
$\mathbb{O}_I(4)$	$240 + 17280 = 17520$	vertex (α_0) 7-face (α_7)	240; $2a_1$ 17280; $\frac{1}{3}\sum_{i=1}^{8} a_i$, $(a_i, a_j) = 1/2$
$\mathbb{O}_I(5)$	30240	vertex $\in \beta_7$	$2a_1 + a_2$, $(a_1, a_2) = 0$
$\mathbb{O}_I(6)$	60480	α_2	$a_1 + a_2 + a_3$, $(a_i, a_j) = 1/2$
$\mathbb{O}_I(7)$	$13440 + 69120 = 82560$	vertex $\in \alpha_1$ 6-face (α_6)	13440; $2a_1 + a_2$, $(a_1, a_2) = 1/2$ 69120; $\frac{1}{2}\sum_{i=1}^{7} a_i$, $(a_i, a_j) = 1/2$

Table 2: Small shells in $\mathbb{O}_I$.

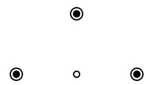
4.4. Hidden duality. In this subsection, we introduce a method of analyzing each shell in the integral normed algebras via reflections. We consider level sets $\mathbb{H}_{I,x}(1)$ or $\mathbb{O}_{I,x}(1)$ in $\mathbb{H}_I(1)$ or $\mathbb{O}_I(1)$ defined as subsets consisting of elements whose real part is x. Thus there are 5-level sets in $\mathbb{H}_I(1)$ or $\mathbb{O}_I(1)$, respectively. And we show that there is a duality between the level set of $\mathrm{Re}\, a = 1/2$ and the set of $\mathrm{Re}\, a = 0$ via the Coxeter group. In fact, this study can be naturally extended to interesting questions for bigger shells of $\mathbb{H}_I$ or $\mathbb{O}_I$ which have more levels. We will discuss it in another paper.

Hidden duality in 24-cell. First, we recall that $\mathbb{H}_I(1) = \{\pm 1, \pm i, \pm j, \pm k, \pm\frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k\}$, and its level sets are

$$\begin{aligned}
\mathbb{H}_{I,1}(1) &= \{1\}, \\
\mathbb{H}_{I,1/2}(1) &= \{\tfrac{1}{2} \pm \tfrac{1}{2}i \pm \tfrac{1}{2}j \pm \tfrac{1}{2}k\}, \\
\mathbb{H}_{I,0}(1) &= \{\pm i, \pm j, \pm k\}, \\
\mathbb{H}_{I,-1/2}(1) &= \{-\tfrac{1}{2} \pm \tfrac{1}{2}i \pm \tfrac{1}{2}j \pm \tfrac{1}{2}k\}, \\
\mathbb{H}_{I,-1}(1) &= \{-1\}.
\end{aligned}$$

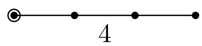
Here we observe that the isotropy subgroup of $1 \in \mathbb{H}_{I,1}(1)$ in a Coxeter group D_4 is $A_1 \times A_1 \times A_1$ (the isotropy group of a vertex) generated by $S_1 = \{\sigma_i, \sigma_j, \sigma_k\}$. And the isotropy group acts on each level set.

For $\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k \in \mathbb{H}_{I,1/2}(1)$, none of the elements in S_1 fixes $\frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$. This choice of an element in $\mathbb{H}_{I,1/2}(1)$ via $A_1 \times A_1 \times A_1$-action produces a cube with a Coxeter-Dynkin diagram



Moreover, $\mathbb{H}_{I,1/2}(1)$ is the set of vertices of a cube which is the vertex figure of a 24-cell.

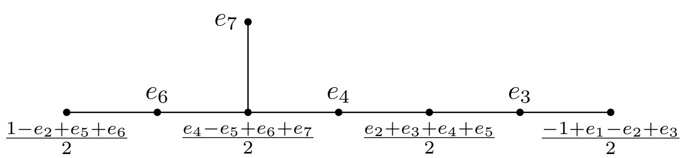
For $\mathbb{H}_{I,0}(1)$, the $A_1 \times A_1 \times A_1$-action on $\mathbb{H}_{I,0}(1)$ produces three orbits $\{i, -i\}$, $\{j, -j\}$ and $\{k, -k\}$ in $\mathbb{H}_{I,0}(1)$. Remark: to get a better understanding of $\mathbb{H}_{I,0}(1)$, one can use another construction of the 24-cell given by F_4 with a Coxeter-Dynkin diagram



Coxeter-Dynkin diagram of 24-cell.

If we consider an isotropy group of $1 \in \mathbb{H}_I(1)$ as above, then we observe that $\mathbb{H}_{I,0}(1)$ is in fact an octahedron. According to the famous duality between the cube and the octahedron, we get a *hidden duality* between $\mathbb{H}_{I,1/2}(1)$ and $\mathbb{H}_{I,0}(1)$ in $\mathbb{H}_I(1)$.

Hidden duality in 4_{21}. As in $\mathbb{H}_I(1)$, we consider $1 \in \mathbb{O}_I(1)$ and its isotropy group in E_8 which is the Coxeter group E_7. And one can show that $\mathbb{O}_{I,1/2}(1)$ is a Gosset polytope 3_{21} given by the action of the Coxeter group E_7. In fact, it is the vertex figure of 4_{21}. Furthermore, the isotropy Coxeter group E_7 also acts on $\mathbb{O}_{I,0}(1)$ which is the set of the unit pure imaginary integral octonions, and in the following theorem we show a polytope given by the Coxeter group E_7 which has a duality with 3_{21}. For better calculation, we use another set of generators for the Coxeter group E_8 given by

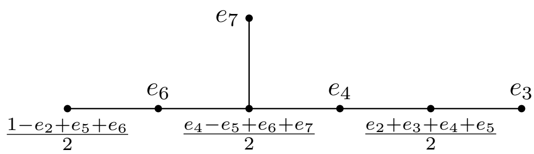


Coxeter-Dynkin diagram of E_8.

Theorem 9. *A subset $\mathbb{O}_{I,0}(1)$ in $\mathbb{O}_I(1)$ is the set of vertices of a uniform polytope 2_{31} with the Coxeter group E_7. Moreover, there is a duality between $\mathbb{O}_{I,1/2}(1)$ and $\mathbb{O}_{I,0}(1)$ via the crosspolytopes in $\mathbb{O}_{I,1/2}(1)$ and the vertices in $\mathbb{O}_{I,0}(1)$.*

P r o o f. We consider $e_1 \in \mathbb{O}_{I,0}(1)$ and observe the isotropy group of $e_1 \in \mathbb{O}_I(1)$ generated by the above Dynkin diagram is a Coxeter group E_7.

Then the Coxeter group E_7 gives a polytope in $\mathbb{O}_{I,0}(1)$ with a Coxeter-Dynkin diagram



Coxeter-Dynkin diagram of 2_{31}.

This polytope is known as a uniform polytope 2_{31} which has 128-vertices. Since $|\mathbb{O}_{I,0}(1)| = 128$, the convex hull of $\mathbb{O}_{I,0}(1)$ in $\operatorname{Im}\mathbb{O}$ is the polytope 2_{31}.

From the above description of $\mathbb{O}_I(2)$ in table 2, each 6-crosspolytope in $\mathbb{O}_{I,1/2}(1)$ (which is a Gosset polytope 3_{21}) can be written as $a_1 + a_2$ with $(a_1, a_2) = 0$ for $a_1, a_2 \in \mathbb{O}_{I,1/2}(1)$. Then we have $a_1 + a_2 - 1 \in \mathbb{O}_{I,0}(1)$ and this gives the second part of the theorem. $\square$

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