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NORMABILITY OF LORENTZ SPACES— AN ALTERNATIVE APPROACH

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Abstract. We study normability properties of classical Lorentz spaces. Given a certain general lattice-like structure, we first prove a general sufficient condition for its associate space to be a Banach function space. We use this result to develop an alternative approach to Sawyer's characterization of normability of a classical Lorentz space of type Λ. Furthermore, we also use this method in the weak case and characterize normability of Λ_v^∞. Finally, we characterize the linearity of the space Λ_v^∞ by a simple condition on the weight v.

Keywords: weighted Lorentz space; weighted inequality; non-increasing rearrangement; Banach function space; associate space

MSC 2010: 46E30

1. Introduction

Classical Lorentz spaces were introduced by Lorentz in 1951 in [6]. Their normability and duality properties have been intensively studied since 1990 when Sawyer in [7] determined when a classical Lorentz space of type Λ is equivalent to a Banach space. It turns out that a classical Lorentz space of type Λ need not in general be normable and even does not have to be necessarily a linear set (see [3]), similarly for the space of weak type.

In this paper we present an alternative approach to this problem, using duality methods based on properties of associate spaces to rather general structures. In

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our first main result we characterize when the set defined as an associate space to a certain structure of lattice type has the properties required by the definition of the so-called Banach function norm (definitions are given in Section 2 below). We then apply this general result to the specific case of the classical Lorentz space, obtaining thereby a new proof of Sawyer's result. We then turn our attention to the classical Lorentz space of weak type, studied for example in [2] and [4]. We give a necessary and sufficient condition for the normability of this space.

The paper is structured as follows. In the following section we give some background material and fix notation. In Section 3 we recall the results of general nature concerning Banach function spaces. In Section 4 we state and prove our main results concerning the classical Lorentz spaces. Finally, in Section 5 we state and prove our results concerning weak-type spaces.

2. Preliminaries

Throughout the paper we shall always consider a σ-finite nonatomic underlying measure space $(\mathcal{R}, \mu)$. The symbol $\mathcal{M}(\mathcal{R})$ will always be used to denote the set of all real-valued μ-measurable functions on $\mathcal{R}$. For $f \in \mathcal{M}(\mathcal{R})$ we shall consider the *distribution function* defined by

$$\lambda_f(s) := \mu(\{|f| > s\}), \quad s \in (0, \infty),$$

the *nonincreasing rearrangement* of f defined by

$$f^*(s) := \inf\{\lambda_f \leqslant s\}, \quad s \in (0, \infty),$$

and

$$f^{**}(s) := \frac{1}{s} \int_0^s f^*(t)\,\mathrm{d}t, \quad s \in (0, \infty).$$

The set of all simple functions on $\mathcal{R}$ will be denoted by

$$\mathcal{S}(\mathcal{R}) := \left\{ f\colon\ f = \sum_{i=1}^{n} a_i \chi_{A_i}\colon\ \mu(A_k) < \infty \right\}.$$

Moreover, if $\mu(\mathcal{R}) < \infty$, then we set $f^*(s) := 0$ for $s > \mu(\mathcal{R})$. The expression *weight* will always refer to a locally integrable nonnegative function defined on $(0, \infty)$, positive on $(0, \delta)$ for some $\delta > 0$ and with $v(s) = 0$ for $s \in (\mu(\mathcal{R}), \infty)$. In the following text we shall also use capitals U, V, W for functions defined as

$$U(t) := \int_0^t u(s)\,\mathrm{d}s,$$

$$V(t) := \int_0^t v(s)\,\mathrm{d}s,$$

and

$$W(t) := \int_0^t w(s)\,\mathrm{d}s, \quad t \in (0,\infty).$$

The symbol p' will always denote the associate exponent to $p \in (1,\infty)$ defined by $p' = p/(p-1)$.

Definition 2.1. Let $(\mathcal{R},\mu)$ be a nonatomic σ-finite measure space. Let us consider a functional $\|\cdot\|_X\colon \mathcal{M}(\mathcal{R}) \to [0,\infty]$ and set $X := \{f \in \mathcal{M}(\mathcal{R})\colon \|f\|_X < \infty\}$. Let us consider the following properties.

(P1) $\|\cdot\|_X$ is a norm on X.
(P2) If $|f| \geqslant |g|$ a.e., then $\|f\|_X \geqslant \|g\|_X$.
(P3) If $0 \leqslant f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$.
(P4) $\|\chi_E\|_X < \infty$, whenever $\mu(E) < \infty$.
(P5) For every set E of a finite measure, there exists a constant C_E such that

$$\|f\chi_E\|_X \geqslant C_E \int_E |f|\,\mathrm{d}\mu.$$

(P6) If $f^*(s) = g^*(s)$ for every $s \in (0,\mu(\mathcal{R}))$, then $\|f\|_X = \|g\|_X$.

We call X

(1) a *Banach function space* if (P1)–(P5) are satisfied;
(2) a *rearrangement-invariant Banach function space* if (P1)–(P6) are satisfied;
(3) a *rearrangement-invariant lattice* if $\|\cdot\|_X$ is a positively homogeneous functional and (P2), (P3) and (P6) are satisfied.

Remark 2.1. If $\|\cdot\|_X$ satisfies (P2), it easily follows that $|f| = |g|$ implies $\|f\|_X = \|g\|_X$.

Definition 2.2. Let $\|\cdot\|_X\colon \mathcal{M}(\mathcal{R}) \to [0,\infty]$ be a functional. For $f \in \mathcal{M}(\mathcal{R})$ define

$$\|f\|_{X'} := \sup_{g \in X} \frac{\int_{\mathcal{R}} fg\,\mathrm{d}\mu}{\|g\|_X}$$

and

$$\|f\|_{X''} := \sup_{g \in X'} \frac{\int_{\mathcal{R}} fg\,\mathrm{d}\mu}{\|g\|_{X'}}$$

(following the convention $0/0 = \infty/\infty = 0$).

Definition 2.3. Let $\|\cdot\|_X$ have the properties (P2), (P3) and (P6). For $t \in (0,\infty)$ we define the fundamental function by

$$\varphi_X(t) := \|\chi_E\|_X, \quad \text{where } \mu(E) = t.$$

Definition 2.4. Let $\|\cdot\|_X, \|\cdot\|_Y \colon \mathcal{M}(\mathbb{R}) \to [0,\infty]$ and let

$$X := \{f \in \mathcal{M}(\mathbb{R}) \colon \|f\|_X < \infty\}$$

and

$$Y := \{f \in \mathcal{M}(\mathbb{R}) \colon \|f\|_Y < \infty\}.$$

Define

$$\operatorname{Opt}(X,Y) := \sup_{f\in X} \frac{\|f\|_Y}{\|f\|_X}$$

(following the convention $0/0 = \infty/\infty = 0$).

3. General duality theorems

We first present a simple sufficient condition for the identity $X = X''$. This result is of independent interest but also will be very useful for the proofs in the next chapters.

Theorem 3.1. *Let $\|\cdot\|_X \colon \mathcal{M}(\mathbb{R}) \to [0,\infty]$ be a functional with the following properties.*

(1) *If $\|f\|_X = \||f|\|_X$.*
(2) *$\|\chi_E\|_X < \infty$ whenever $\mu(E) < \infty$.*
(3) *For every E of finite measure there exists $\infty > C_E > 0$ such that*

$$C_E\|f\chi_E\|_X \geqslant \int_E |f| \,\mathrm{d}\mu.$$

Then the functional $\|\cdot\|_{X'}$ is a Banach function norm.

Moreover, $\|\cdot\|_X$ is equivalent to a Banach function norm if and only if $\|\cdot\|_X \approx \|\cdot\|_{X''}$.

Proof. Let us first assume $\|\cdot\|_X \approx \|\cdot\|_{X''}$. We shall verify that $\|\cdot\|_{X'}$ is a Banach function norm. Let $f_1, f_2 \in X'$ and $g \in X$, obviously

$$\frac{\int_{\mathbb{R}}(f_1+f_2)g\,\mathrm{d}\mu}{\|g\|_X} = \frac{\int_{\mathbb{R}} f_1 g\,\mathrm{d}\mu}{\|g\|_X} + \frac{\int_{\mathbb{R}} f_2 g\,\mathrm{d}\mu}{\|g\|_X} \leqslant \sup_{g\in X}\frac{\int_{\mathbb{R}} f_1 g\,\mathrm{d}\mu}{\|g\|_X} + \sup_{g\in X}\frac{\int_{\mathbb{R}} f_2 g\,\mathrm{d}\mu}{\|g\|_X}.$$

Passing to the supremum on the left-hand side proves

$$\|f_1+f_2\|_{X'} \leqslant \|f_1\|_{X'} + \|f_2\|_{X'}.$$

If $\mu(\{|f|>0\})>0$, then there exists $\varepsilon>0$ such that $\mu(\{|f|>\varepsilon\})>0$. Let us consider $A\subset\{|f|>\varepsilon\}$ with $\mu(A)>0$. Then

$$0<\frac{\int_{\mathbb{R}}\varepsilon\chi_A\,\mathrm{d}\mu}{\|\chi_A\|_X}\leqslant\frac{\int_{\mathbb{R}}f\chi_A\cdot\operatorname{sgn}(f)\,\mathrm{d}\mu}{\|\chi_A\|_X}\leqslant\|f\|_{X'}.$$

Since the homogeneity is obvious, we have that $\|\cdot\|_{X'}$ is a norm. Now, if $|f|\geqslant|g|$ a.e., then (due to assumption (1)) for every $h\in X$ we have $\|h\|_X=\||h|\operatorname{sgn}(f)\|_X$, and therefore

$$\begin{aligned}\frac{\int_{\mathbb{R}}gh\,\mathrm{d}\mu}{\|h\|_X}\leqslant\frac{\int_{\mathbb{R}}|gh|\,\mathrm{d}\mu}{\|h\|_X}\leqslant\frac{\int_{\mathbb{R}}|f||h|\,\mathrm{d}\mu}{\|h\|_X}&=\frac{\int_{\mathbb{R}}f\operatorname{sgn}(f)|h|\,\mathrm{d}\mu}{\|h\|_X}\\&=\frac{\int_{\mathbb{R}}f\operatorname{sgn}(f)|h|\,\mathrm{d}\mu}{\||h|\operatorname{sgn}(f)\|_X}\leqslant\|f\|_{X'}.\end{aligned}$$

Passing to the supremum over h on the left-hand side gives (P2) for X'. Property (P3) is an easy consequence of the monotone convergence theorem. Let $\mu(E)<\infty$ and let C_E be the constant from property (3) of X. Then

$$\frac{\int_{\mathbb{R}}\chi_E g\,\mathrm{d}\mu}{\|g\|_X}\leqslant C_E<\infty.$$

Passing to the supremum over $g\in X$ we obtain (P4). Choose E with $\mu(E)<\infty$ and $g\in X'$ such that

$$\int_E g\,\mathrm{d}\mu=\int_{\mathbb{R}}g\chi_E\chi_E\,\mathrm{d}\mu\leqslant\|g\chi_E\|_{X'}\|\chi_E\|_X=C_E\|g\chi_E\|_{X'}$$

and that proves (P5) for X'. If X' is a BFS then X'' is also a BFS.

Let us now assume that $\|\cdot\|_X$ is equivalent to some Banach function norm $\|\cdot\|_Y$. Then, obviously $\|\cdot\|_{X'}\approx\|\cdot\|_{Y'}$. And hence $\|\cdot\|_{X''}\approx\|\cdot\|_{Y''}=\|\cdot\|_Y\approx\|\cdot\|_X$. The proof is complete. □

Lemma 3.1. *Define $X:=\{f\in\mathcal{M}(\mathbb{R})\colon\ \|f\|_X<\infty\}$, where $\|\cdot\|_X$ satisfies the conditions of Theorem* 3.1. *Then $X\hookrightarrow X''$.*

P r o o f. The proof is analogous to the one in [1], Theorem 2.7. □

Lemma 3.2. *Let X_0, X_1, Y be rearrangement invariant lattices. Let (X_0, X_1) be a compatible couple. Then*

$$\operatorname{Opt}(X_0 + X_1, Y) \approx \operatorname{Opt}(X_0, Y) + \operatorname{Opt}(X_1, Y).$$

Proof.

$$\operatorname{Opt}(X_0 + X_1, Y) = \sup_f \frac{\|f\|_Y}{\inf\limits_{f=f_1+f_2} (\|f_1\|_{X_0} + \|f_2\|_{X_1})}.$$

We search for the optimal constant of the embedding

$$\|f\|_Y \leqslant C(\|f_1\|_{X_1} + \|f - f_1\|_{X_0}), \tag{3.1}$$

where f, f_1 are arbitrary measurable functions. Since we have the assumption (P2), the following holds

$$\|f\|_Y \leqslant \|(|f_1| + |f - f_1|)\|_Y.$$

Therefore, to prove (3.1) it is enough, in fact, to prove

$$\|(|f_1| + |f - f_1|)\|_Y \leqslant C(\||f_1|\|_{X_1} + \||f - f_1|\|_{X_0}).$$

Thus we may suppose $f \geqslant 0$, $f_1 \geqslant 0$, and $f - f_1 \geqslant 0$. We have

$$\frac{1}{2}(\|f_1\|_Y + \|f - f_1\|_Y) \leqslant \|f\|_Y \leqslant \|f_1\|_Y + \|f - f_1\|_Y.$$

Since

$$\sup_{f_1, f_2 \geqslant 0} \frac{\|f_1\|_Y + \|f_2\|_Y}{\|f_1\|_{X_0} + \|f_2\|_{X_1}} \approx \sup_{f \geqslant 0} \frac{\|f\|_Y}{\|f\|_{X_0}} + \sup_{f \geqslant 0} \frac{\|f\|_Y}{\|f\|_{X_1}},$$

the inequality $\gtrsim$ is obtained immediately, since the sum of the two suprema on the right-hand side is equivalent to its maximum, which is attained if we set $f_1 = 0$ or $f_2 = 0$. Since the other inequality is obvious, we have

$$\begin{aligned}
\operatorname{Opt}(X_0 + X_1, Y) &= \sup_{f \geqslant g \geqslant 0} \frac{\|g\|_Y + \|f - g\|_Y}{(\|g\|_{X_0} + \|f - g\|_{X_1})} \approx \sup_{f_1, f_2 \geqslant 0} \frac{\|f_1\|_Y + \|f_2\|_Y}{\|f_1\|_{X_0} + \|f_2\|_{X_1}} \\
&\approx \sup_{f \geqslant 0} \frac{\|f\|_Y}{\|f\|_{X_0}} + \sup_{f \geqslant 0} \frac{\|f\|_Y}{\|f\|_{X_1}} = \operatorname{Opt}(X_0, Y) + \operatorname{Opt}(X_1, Y).
\end{aligned}$$

$\square$

4. Normability of lambda spaces, case $1 < p < \infty$

Definition 4.1. Let $p \in (1,\infty)$, $\mu(\mathbb{R}) = \infty$ and let v be a weight. Define

$$\Lambda_v^p := \Big\{ f \in \mathcal{M} \colon \ \|f\|_{\Lambda_v^p} := \Big(\int_0^\infty f^*(s)^p v(s) \, \mathrm{d}s \Big)^{1/p} < \infty \Big\},$$
$$\Gamma_v^p := \Big\{ f \in \mathcal{M} \colon \ \|f\|_{\Gamma_v^p} := \Big(\int_0^\infty f^{**}(s)^p v(s) \, \mathrm{d}s \Big)^{1/p} < \infty \Big\},$$
$$\Lambda_v^\infty := \Big\{ f \in \mathcal{M}(\mathbb{R}) \colon \ \|f\|_{\Lambda_v^\infty} := \operatorname*{ess\,sup}_{s>0} f^*(s) v(s) < \infty \Big\},$$

and

$$\Gamma_v^\infty := \Big\{ f \in \mathcal{M}(\mathbb{R}) \colon \ \|f\|_{\Gamma_v^\infty} := \operatorname*{ess\,sup}_{s>0} f^{**}(s) v(s) < \infty \Big\}.$$

Remark 4.1. Note that the spaces Λ_v^∞ and Γ_v^∞ generalize the spaces of type $\Lambda_v^{p,\infty}$ and $\Gamma_v^{p,\infty}$ (see [2] for the definition). Indeed, we have

$$\|f\|_{\Lambda_{V^p}^\infty} = \|f\|_{\Lambda_v^{p,\infty}}$$

and

$$\|f\|_{\Gamma_{V^p}^\infty} = \|f\|_{\Gamma_v^{p,\infty}}.$$

Remark 4.2. Usually, X may be called rearrangement invariant laticce only if, in addition to our assumptions on this structure, it is a linear set. But that would cause certain troubles in this case, because for an arbitrary weight v, Λ_v^p and $\Lambda_v^{p,\infty}$ do not have to be linear sets (for an equivalent condition on weight for which Λ_v^p is a linear space see [3]). Consider for instance the case of $\mathbb{R} = (-\infty,\infty)$ Λ_v^p with $\mu(\mathbb{R}) = \infty$, where

$$v(s) := \sum_{n=1}^\infty n! \, \chi_{(n,n+1)}(s), \quad s \in (0,\infty),$$

and functions f, g with $\operatorname{supt} f \cap \operatorname{supt} g = \emptyset$. For instance, set

$$f(s) := \sum_{n=1}^\infty \frac{1}{n^2 n!} \chi_{(n,n+1)}(s), \quad g(s) := \sum_{n=1}^\infty \frac{1}{n^2 n!} \chi_{(-n-1,-n)}(s).$$

Then clearly

$$f^*(s) = g^*(s) = \sum_{n=1}^\infty \frac{1}{n^2 n!} \chi_{(n,n+1)}(s).$$

Therefore $f, g \in \Lambda_v^p$, but $f + g \notin \Lambda_v^p$. But in the following text by abuse of language we shall call the Λ_v^p spaces.

For a weight v, $p\in(1,\infty)$ and $t\in(0,\infty)$, let us recall the fundamental function for spaces Λ and Γ. We have

$$\varphi_{\Lambda^p_v}(t)=\biggl(\int_0^\infty \chi_{(0,t)}v(s)\,\mathrm{d}s\biggr)^{1/p}=V(t)^{1/p}.$$

As for Γ^p_v, let $\mu(E)=t$. We have

$$(\chi_E)^{**}(s)=\frac{1}{s}\int_0^s \chi_{(0,t)}(\xi)\,\mathrm{d}\xi=\min\Bigl\{1,\frac{t}{s}\Bigr\}.$$

Therefore

$$\varphi_{\Gamma^p_v}(t)=\biggl(\int_0^t v(s)\,\mathrm{d}s+t^p\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\biggr)^{1/p}\approx V^{1/p}(t)+t\biggl(\int_t^\infty \frac{v(s)}{s^p}\,\mathrm{d}s\biggr)^{1/p}.$$

Similarly

$$\varphi_{\Lambda^{p,\infty}_v}(t)=\sup_{s>0}\chi_{(0,t)}(s)V(s)^{1/p}=V(t)^{1/p}$$

and

$$\varphi_{\Gamma^{p,\infty}_v}=\sup_{s>0}(\chi_{(0,t)})^{**}(s)V(s)^{1/p}\approx V(t)^{1/p}+t\sup_{s>t}\frac{V(s)^{1/p}}{s}.$$

Remark 4.3. Let us check that $\|\cdot\|_{\Lambda^p_v}$ satisfies the assumptions of Theorem 3.1. The assumption (1) is obviously satisfied. The assumption (2) demands the fundamental function to be finite. For this it is sufficient to have $v\in L^1_{\mathrm{loc}}$. The characterization of the assumption (3) is described in the next proposition.

Proposition 4.1. *The functional* $\|\cdot\|_{\Lambda^p_v}$ *satisfies assumption* (3) *from Theorem* 3.1 *if and only if*

$$\int_0^{\min\{1,\mu(\mathbb{R})\}}\frac{t^{p'-1}}{V(t)^{p'-1}}\,\mathrm{d}t<\infty.$$

Proof. Choose $E\subset\mathbb{R}$ with $\mu(E)<\infty$. We need to show

$$\biggl(\int_0^\infty (\chi_E f)^*(s)^p v(s)\,\mathrm{d}s\biggr)^{1/p}\geqslant C\int_{\mathbb{R}}\chi_E f\,\mathrm{d}\mu.$$

This inequality holds if and only if there exists $0<C<\infty$ such that

$$\biggl(\int_0^{\mu(E)} f^*(s)^p v(s)\,\mathrm{d}s\biggr)^{1/p}\geqslant C\int_0^{\mu(E)} f^*(s)\,\mathrm{d}s,$$

for every $f \in \mathcal{M}(\mathbb{R})$. This is equivalent to the embedding

$$\Lambda_v^p \hookrightarrow \Lambda_1^1,$$

which holds (see [2]) if and only if

$$\int_0^{\min\{1,\mu(\mathbb{R})\}} \frac{t^{p'-1}}{V(t)^{p'-1}}\,\mathrm{d}t < \infty.$$

$\square$

Now for a weight v define

$$v_a(s) = s^{p'}\frac{v(s)}{V(s)^{p'}}. \tag{4.1}$$

Then, since the embedding of type $\Gamma \hookrightarrow \Lambda$ has already been characterized in [5], Theorem 4.2, we have

$$\|f\|_{(\Gamma_{v_a}^{p'})'} = \operatorname{Opt}(\Gamma_{v_a}^{p'}, \Lambda_f^1) \approx \left(\int_0^\infty f^{**}(t)^p v_{aa}(t)\,\mathrm{d}t\right)^{1/p},$$

where

$$v_{aa}(t) = \frac{t^{p+p'+1}\int_0^t s^{p'}v(s)V(s)^{-p}\,\mathrm{d}s\big[V(t)^{1-p'} - V(\infty)^{1-p'}\big]}{\big(\int_0^t s^{p'}v(s)V^{-p'}(s)\,\mathrm{d}s + t^p\big[V(t)^{1-p'} - V(\infty)^{1-p'}\big]\big)^{p'+1}}. \tag{4.2}$$

Lemma 4.1. *Let $1 < p \leqslant q < \infty$. Then the following holds:*

$$\operatorname{Opt}(\Gamma_v^p, \Lambda_w^q) := \sup_{f\in\Gamma_p^v}\frac{\|f\|_{\Lambda_w^q}}{\|f\|_{\Gamma_v^p}} \approx \sup_{t>0}\frac{\varphi_{\Lambda_w^q}(t)}{\varphi_{\Gamma_v^p}(t)}.$$

P r o o f. Obviously

$$\sup_{f\in\Gamma_p^v}\frac{\|f\|_{\Lambda_w^q}}{\|f\|_{\Gamma_v^p}} \geqslant \sup_{t>0}\frac{\varphi_{\Lambda_w^q}(t)}{\varphi_{\Gamma_v^p}(t)}.$$

It is enough to realize that on the right-hand side we are taking the supremum over the characteristic functions of sets of finite measure.

From [5], page 24, we obtain

$$\operatorname{Opt}(\Gamma_{v,u}^p, \Lambda_w^q) \approx \sup_{t>0}\frac{W(t)^{1/q}}{\big(V(t) + U(t)^p\int_t^\infty U(s)^{-p}v(s)\,\mathrm{d}s\big)^{1/p}},$$

in this particular case with $u(t) = 1$ and $U(t) = t$. Hence we get

$$\operatorname{Opt}(\Gamma_v^p, \Lambda_w^q) \approx \sup_{t>0}\frac{W(t)^{1/q}}{\big(V(t) + t^p\int_t^\infty s^{-p}v(s)\,\mathrm{d}s\big)^{1/p}} \approx \sup_{t>0}\frac{\varphi_{\Lambda_w^q}(t)}{\varphi_{\Gamma_v^p}(t)}.$$

$\square$

Lemma 4.2. *Let v be a weight. If we set $X := \Lambda^p_v$, then the following conditions are equivalent.*

(1) $\operatorname{Opt}(X'', X) < \infty$.
(2) $\int_0^t s^{p'} v(s) V^{-p'}(s)\,\mathrm{d}s \lesssim t^{p'} V^{1-p'}(t)$.

P r o o f. According to [7], Theorem 1, we have:

$$\|f\|_{(\Lambda^p_v)'} \approx \|f\|_{\Gamma^{p'}_{v_a}} + \frac{\|f\|_1}{\|v\|_1}.$$

In the case of $v \notin L^1$, we have

$$(\Lambda^p_v)' = \Gamma^{p'}_{v_a},$$

where v_a is defined by (4.1). If $v \in L^1$, then

$$(\Lambda^p_v)' = \Gamma^{p'}_{v_a} \cap L^1.$$

In the case of $v \notin L^1$, (1) is satisfied if and only if

$$(\Gamma^{p'}_{v_a})' = \Gamma^p_{v_{aa}} \hookrightarrow \Lambda^p_v$$

holds (where v_{aa} is defined by (4.2)). In the case of $v \in L^1$, this occurs if and only if

$$(\Gamma^{p'}_{v_a} \cap L^1)' = (\Gamma^p_{v_{aa}} + L^\infty) \hookrightarrow \Lambda^p_v.$$

For $v \notin L^1$ we therefore need to check if

$$\operatorname{Opt}(\Gamma^p_{v_{aa}}, \Lambda^p_v) < \infty.$$

In the case of $v \in L^1$, we need to verify whether

$$\operatorname{Opt}(\Gamma^p_{v_{aa}} + L^\infty, \Lambda^p_v) \approx \operatorname{Opt}(\Gamma^p_{v_{aa}}, \Lambda^p_v) + \operatorname{Opt}(L^\infty, \Lambda^p_v) < \infty.$$

But $v \in L^1$ implies

$$\operatorname{Opt}(L^\infty, \Lambda^p_v) = \sup_{t>0} V(t)^{1/p} = \|v\|_1^{1/p} < \infty,$$

therefore in both cases it is necessary and sufficient to check that

$$\operatorname{Opt}(\Gamma^p_{v_{aa}}, \Lambda^p_v) < \infty.$$

Due to a well known theorem (see [1], Theorem 5.2, 66) we have

$$\varphi_{X''}(t) = \frac{t}{\varphi_{X'}(t)}.$$

From [5], Theorem 4.2, we know it is enough to show that

$$\varphi_{\Lambda_v^p}(t) \lesssim \varphi_{\Gamma^p_{v_{aa}}}(t) \approx \frac{t}{\varphi_{\Gamma^{p'}_{v_a}}(t)}.$$

Therefore, we need the following inequality

$$\varphi_{\Lambda_v^p}(t) \lesssim \frac{t}{\varphi_{\Gamma^{p'}_{v_a}}(t)}.$$

We have

$$\varphi_{\Lambda_v^p}(t) = V(t)^{1/p},$$

and

$$\varphi^{p'}_{\Gamma^{p'}_{v_a}}(t) = V_a(t) + t^{p'}\int_t^\infty \frac{v_a(s)}{s^{p'}}\,\mathrm{d}s.$$

This implies

$$\varphi_{X'}(t) \approx V_a(t)^{1/p'} + t\Big(\int_t^\infty \frac{v_a(s)}{s^{p'}}\,\mathrm{d}s\Big)^{1/p'}.$$

Hence we have

$$\begin{aligned}\varphi_{X'}(t) &\approx \Big(\int_0^t s^{p'}\frac{v(s)}{V^{p'}(s)}\,\mathrm{d}s\Big)^{1/p'} + t\Big(\int_t^\infty \frac{v(s)}{V^{p'}(s)}\,\mathrm{d}s\Big)^{1/p'}\\ &\approx \Big(\int_0^t s^{p'}\frac{v(s)}{V^{p'}(s)}\,\mathrm{d}s\Big)^{1/p'} + tV^{-1/p}(t) - tV^{-1/p}(\infty).\end{aligned}$$

This occurs if and only if

$$V^{1/p}(t) \lesssim \frac{t}{\big(\int_0^t s^{p'}v(s)V^{-p'}(s)\,\mathrm{d}s\big)^{1/p'} + tV^{-1/p}(t) - tV^{-1/p}(\infty)},$$

which is equivalent to

$$\Big(\int_0^t s^{p'}\frac{v(s)}{V^{p'}(s)}\,\mathrm{d}s\Big) + t^{p'}(V^{1-p'}(t) - V^{1-p'}(\infty)) \lesssim t^{p'}V^{1-p'}(t),$$

and the latter holds if and only if

$$\int_0^t s^{p'}\frac{v(s)}{V^{p'}(s)}\,\mathrm{d}s \lesssim t^{p'}V^{1-p'}(t).$$

The proof is complete. □

Theorem 4.1. *The following conditions are equivalent.*

(1) *Functional* $\|\cdot\|_{\Lambda^p_v}$ *is equivalent to a Banach function norm.*

(2) $\int_0^t s^{p'} v(s) V^{-p'}(s)\,\mathrm{d}s \lesssim t^{p'} V^{1-p'}(t)$, $t \in (0,\infty)$.

(3) $\int_0^t s^{p'-1} V^{-p'+1}(s)\,\mathrm{d}s \lesssim t^{p'} V^{1-p'}(t)$, $t \in (0,\infty)$.

P r o o f. Let us first show the equivalence of the second and the third condition. (2) $\Leftrightarrow$ (3): Clearly

$$\begin{aligned}\int_0^t \frac{s^{p'-1}}{V^{p'-1}(s)}\,\mathrm{d}s &\approx \int_0^t s^{p'-1}\Big(\int_s^\infty \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z + V^{1-p'}(\infty)\Big)\,\mathrm{d}s \\ &\approx \int_0^t\int_0^z s^{p'-1}\,\mathrm{d}s \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z + \int_t^\infty\int_0^t s^{p'-1}\,\mathrm{d}s \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z \\ &\quad + t^{p'} V^{1-p'}(\infty) =: I + II + III.\end{aligned}$$

Now, since all three terms on the right-hand side are nonnegative, we have

$$I \approx \int_0^t s^{p'} \frac{v(s)}{V^{p'}(s)}\,\mathrm{d}s \leqslant \int_0^t \frac{s^{p'-1}}{V^{p'-1}(s)}\,\mathrm{d}s.$$

Therefore (3) $\Rightarrow$ (2). For the converse implication, let us recall that since V is increasing, we have

$$III = t^{p'} V^{1-p'}(\infty) \leqslant t^{p'} V^{1-p'}(t).$$

Furthermore, we have

$$\begin{aligned}II = \int_0^t s^{p'-1}\,\mathrm{d}s \int_t^\infty \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z &\approx t^{p'} \int_t^\infty \frac{v(z)}{V^{p'}(z)}\,\mathrm{d}z \\ &\approx t^{p'}(V^{1-p'}(t) - V^{1-p'}(\infty)) \lesssim t^{p'} V^{1-p'}(t).\end{aligned}$$

Therefore, if (2) is satisfied, we have $I \lesssim t^{p'} V^{1-p'}(t)$ and also $II + III \lesssim t^{p'} V^{1-p'}(t)$ and that implies $I + II + III \lesssim t^{p'} V^{1-p'}(t)$, which is nothing else but (3).

Now let us show the implication (2) $\Rightarrow$ (1). First note that if (2) is satisfied, then (3) is satisfied as well and hence also

$$\int_0^1 \frac{s^{p'-1}}{V^{p'-1}(s)}\,\mathrm{d}s < \infty.$$

Therefore by Proposition 4.1, the assumption (3) in Theorem 3.1 is satisfied in the case of $X = \Lambda^p_v$ (we shall use this identity till the end of the proof). Since all weights are defined as locally integrable positive functions, we also have the assumption (2) in Theorem 3.1, and as the reader can easily check, the assumption (1) in Theorem 3.1

is satisfied as well. Theorem 3.1 claims that $\|\cdot\|_{\Lambda^p_v}$ is equivalent to a BFN if and only if $\|\cdot\|_X \approx \|\cdot\|_{X''}$. Let us first recall that the inequality $\|\cdot\|_{X''} \lesssim \|\cdot\|_X$ is trivially satisfied. It remains to investigate when $\|\cdot\|_X \lesssim \|\cdot\|_{X''}$ occurs. If the condition (2) is satisfied, we only use Lemma 4.2 and obtain $\operatorname{Opt}(X'', X) < \infty$, which gives the desired inequality.

Now let $\|\cdot\|_X$ be equivalent to a Banach function norm. Therefore the assumptions (2) and (3) in Theorem 3.1 have to be satisfied. And hence we have $\operatorname{Opt}(X'', X) < \infty$. If we use Lemma 4.2, we obtain (2). This completes the proof.

□

5. Normability of lambda spaces, case $p = \infty$

In order to meet the assumption (2) in Theorem 3.1, we need the weight function v to be essentialy bounded on every finite interval $(0, t)$. This follows from the fact that for E, with $\mu(E) = t < \infty$, we demand

$$\tilde{v}(t) := \|\chi_E\|_{\Lambda^\infty_v} = \operatorname*{ess\,sup}_{s>0} \chi_{(0,t)}(s)v(s) = \operatorname*{ess\,sup}_{0<s<t} v(s) < \infty. \tag{5.1}$$

In the following text we shall assume that this assumption is satisfied. And the weight $\tilde{v}$ will always be defined by (5.1).

Lemma 5.1. *Let v be a weight. Then*

$$\operatorname*{ess\,sup}_{s>0} f^*(s)v(s) = \operatorname*{ess\,sup}_{s>0} \tilde{v}(s)f^*(s),$$

for every measurable f.

Proof. This proposition can be found in [4], but for the sake of completeness let us present a short proof. We have

$$\begin{aligned}\operatorname*{ess\,sup}_{s>0} f^*(s)\tilde{v}(s) &= \operatorname*{ess\,sup}_{s>0} f^*(s) \operatorname*{ess\,sup}_{t<s} v(t)\\ &\leqslant \operatorname*{ess\,sup}_{s>0} \operatorname*{ess\,sup}_{s>t>0} v(t)f^*(t) = \operatorname*{ess\,sup}_{s>0} v(s)f^*(s).\end{aligned}$$

Since the opposite inequality is trivially satisfied, the proof is complete. □

Theorem 5.1. *Let v be a weight. Then the following conditions are equivalent.*

(1) *Functional $\|\cdot\|_{\Lambda_v^\infty}$ is equivalent to a Banach function norm.*

(2) $\sup\limits_{t>0} \tilde{v}(t)t^{-1}\int_0^t \mathrm{d}z/\tilde{v}(z) < \infty$.

(3) $\Lambda_v^\infty = \Gamma_v^\infty$ *(in the sense of equivalent norms).*

P r o o f. Let us show the equivalence of (1) and (2). Denote $X := \Lambda_v^\infty$. By Lemma 5.1 we have

$$\|f\|_{\Lambda_v^\infty} = \|f\|_{\Lambda_{\tilde{v}}^\infty}.$$

Since the space $(\mathbb{R},\mu)$ is nonatomic and therefore resonant, we may express the dual norm as

$$\|f\|_{X'} = \sup_{g\in X}\frac{\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s}{\|g\|_X} = \sup_{g\in\Lambda_{\tilde{v}}^\infty}\frac{\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s}{\|g\|_{\Lambda_{\tilde{v}}^\infty}}.$$

We claim that

$$\sup_{g\in X}\frac{\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s}{\|g\|_X} = \int_0^\infty f^*(s)\frac{\mathrm{d}s}{\tilde{v}(s)}. \tag{5.2}$$

For the inequality $\geqslant$ we may just choose $g \in \mathcal{M}(\mathbb{R})$ such that $g^* = 1/\tilde{v}$. For the opposite inequality, just realize that

$$\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s = \int_0^\infty \frac{f^*(s)}{\tilde{v}(s)}g^*(s)\tilde{v}(s)\,\mathrm{d}s \leqslant \int_0^\infty \frac{f^*(s)}{\tilde{v}(s)}\,\mathrm{d}s\operatorname*{ess\,sup}_{s>0} g^*(s)\tilde{v}(s).$$

Let us compute the functional $\|\cdot\|_{X''}$. We have

$$\|f\|_{X''} = \sup_{g\in X'}\frac{\int_0^\infty f^*(s)g^*(s)\,\mathrm{d}s}{\int_0^\infty g^*(s)\,\mathrm{d}s/\tilde{v}(s)}. \tag{5.3}$$

We claim that

$$\|f\|_{X''} \approx \sup_{t>0}\frac{\int_0^t f^*(s)\,\mathrm{d}s}{\int_0^t \mathrm{d}s/\tilde{v}(s)} = \sup_{t>0} f^{**}(t)\frac{t}{\int_0^t \mathrm{d}z/\tilde{v}(z)}. \tag{5.4}$$

Indeed, the inequality $\lesssim$ is an immediate consequence of Hardy's lemma (see [1], Proposition 3.6). The opposite inequality trivially follows by taking $g = \chi_{(0,t)}$ in (5.3).

Now according to Lemma 3.1, we need to show $\|\cdot\|_X \lesssim \|\cdot\|_{X''}$. This holds if and only if the optimal constant of the inequality

$$\operatorname*{ess\,sup}_{t>0} f^*(t)\tilde{v}(t) \leqslant C\sup_{t>0} f^{**}(t)\frac{t}{\int_0^t \mathrm{d}z/\tilde{v}(z)} \tag{5.5}$$

is finite. Testing this inequality on the set of simple functions yields

$$\operatorname*{ess\,sup}_{s>0} \chi_{(0,t)}(s)\tilde{v}(s) = \tilde{v}(t) \lesssim \sup_{s>0} \frac{\min(s,t)}{\int_0^s \mathrm{d}z/\tilde{v}(z)}.$$

We have

$$\begin{aligned}\sup_{s>0} \frac{\min(t,s)}{\int_0^s \mathrm{d}z/\tilde{v}(z)} &= \max\biggl(\sup_{s<t} \frac{s}{\int_0^s \mathrm{d}z/\tilde{v}(z)}, \sup_{s>t} \frac{t}{\int_0^s \mathrm{d}z/\tilde{v}(z)}\biggr) \\ &=: \max\Bigl(\sup_{0<s<t} G(s), \sup_{s>t} H(s)\Bigr).\end{aligned}$$

Fix t. The function $H(s)$ is clearly decreasing. We also claim that $G(s)$ is nondecreasing. Indeed, we have

$$G(s) = \biggl(\frac{1}{s}\int_0^s \frac{\mathrm{d}z}{\tilde{v}(z)}\biggr)^{-1}$$

and since the mean value of a nonincreasing function is also nonincreasing, we obtain the claim. From the monotonicity of these functions, we may conclude

$$\sup_{s>0} \frac{\min(s,t)}{\int_0^s \mathrm{d}z/\tilde{v}(z)} = \frac{t}{\int_0^t \mathrm{d}z/\tilde{v}(z)}.$$

Now, using these facts in (5.5), we obtain that the condition (2) is necessary.

Concerning the sufficiency, we have

$$\operatorname*{ess\,sup}_{t>0} f^*(t)\tilde{v}(t) \leqslant \operatorname*{ess\,sup}_{t>0} f^{**}(t)\tilde{v}(t) \lesssim \operatorname*{ess\,sup}_{t>0} f^{**}(t)\frac{t}{\int_0^t \mathrm{d}z/\tilde{v}(z)}. \tag{5.6}$$

It remains to show that (P5) holds. Let $E \subset \mathbb{R}$ be a measurable set, such that $\mu(E) < \infty$. By Hardy-Littlewood-Polya and Hölder inequality, we have

$$\int_E |f| \,\mathrm{d}\mu = \int_0^{\mu(E)} (f\chi_E)^*(s) \leqslant \int_0^{\mu(E)} \frac{\mathrm{d}s}{\tilde{v}(s)} \operatorname*{ess\,sup}_{t>0} (f\chi_E)^*(t)\tilde{v}(t).$$

Now set

$$C_E := \int_0^{\mu(E)} \frac{\mathrm{d}s}{\tilde{v}(s)}.$$

Since the condition (2) holds, the constant C_E is finite. Thus the assumption (3) in Theorem 3.1 is satisfied and therefore the condition (2) is sufficient. The equivalence of (1) and (2) is proved.

Let us now assume (2) is satisfied. From (5.6) we have

$$\|f\|_{\Gamma_v^\infty} \lesssim \|f\|_{X''} \leqslant \|f\|_X.$$

Since the opposite inequality is trivially satisfied and the condition (2) implies

$$\int_0^t \frac{\mathrm{d}z}{\tilde{v}(z)} < \infty, \quad \text{for every } t \in (0,\infty), \tag{5.7}$$

the condition (3) holds.

For the implication (3) $\Rightarrow$ (1), it suffices to verify that Γ_v^∞ is a BFS. The only axiom that is not obvious is (P5). In order to see that (P5) holds, just realize that the function $f \in \mathcal{M}(\mathbb{R})$ such that $f^*(s) = 1/\tilde{v}(s)$ belongs to the space Γ_v^∞. $\square$

Let us remind that according to Remark 4.2, Λ_v^p does not have to be a linear set. A characterization of weight for which Λ_v^p is a linear set was given in [3] for $1 \leqslant p < \infty$. The authors also gave an equivalent condition on weight for which $\Lambda_v^{1,\infty}$ is a linear set. Let us present now similar characterization for the case of Λ_v^∞.

Theorem 5.2. *Let v be a weight. Then the set Λ_v^∞ is linear if and only if*

$$\tilde{v}(2s) \lesssim \tilde{v}(s), \quad s \in (0,\infty). \tag{5.8}$$

P r o o f. Denote $X := \Lambda_v^\infty$. Due to Lemma 5.1 we have $X = \Lambda_{\tilde{v}}^\infty$. Let us first suppose that (5.8) is violated. Then there exists a sequence t_n such that

$$\tilde{v}(2t_n) \geqslant 2^n \tilde{v}(t_n). \tag{5.9}$$

We may, without loss of generality, suppose that t_n is either increasing or decreasing. And also without loss of generality suppose that $t_1 < \mu(\mathbb{R})/2$. In the case of $\mu(\mathbb{R}) = \infty$ it is trivial, otherwise, if $\mu(\mathbb{R}) < \infty$ one can see that $t_n \to 0$, so for certain n_0 we have $t_n < \mu(\mathbb{R})/2$ for all $n > n_0$. Now, taking t_{n+n_0} instead of t_n does the job. Let us first suppose t_n is increasing. Because we have $t_1 \leqslant \mu(\mathbb{R})/2$, we may choose $f, g \in \mathcal{M}(\mathbb{R})$ such that

$$\operatorname{supt}(f) \cap \operatorname{supt}(g) = \emptyset$$

and

$$f^*(s) = g^*(s) = \sum_{n=1}^{\infty} \frac{1}{\tilde{v}(t_n)} \chi_{(t_{n-1},t_n]}(s). \tag{5.10}$$

Then clearly $\|f\|_X = \|g\|_X = 1$. Choose $n \in \mathbb{N}$. We have

$$\begin{aligned}\|f+g\|_X &= \operatorname*{ess\,sup}_{s>0} \sum_{k=1}^{\infty} \frac{1}{\tilde{v}(t_k)} \chi_{(2t_{k-1},2t_k]} \tilde{v}(s) \\ &\geqslant \operatorname*{ess\,sup}_{s>0} \frac{1}{\tilde{v}(t_n)} \chi_{(2t_{n-1},2t_n]}(s) \tilde{v}(s) = \frac{\tilde{v}(2t_n)}{\tilde{v}(t_n)} \geqslant 2^n.\end{aligned}$$

Since n is an arbitrary natural number, we obtain $f+g \notin \Lambda_v^\infty$. Now, let the sequence t_n be decreasing. Since we have $t_1 \leqslant \mathbb{R}/2$, we can find f, g with disjoint supports such that

$$f^*(s) = g^*(s) = \sum_{j=1}^{\infty} \frac{1}{\tilde{v}(t_j)} \chi_{(t_j,t_{j-1})}(s).$$

If we use the similar calculation as in the first case, we obtain that $f+g \notin \Lambda_v^\infty$.

Now, let us suppose (5.8) holds. Choose $f, g \in X$. We have

$$\begin{aligned}\|f+g\|_X &\leqslant \operatorname*{ess\,sup}_{s>0} \Big(f^*\Big(\frac{s}{2}\Big) + g^*\Big(\frac{s}{2}\Big)\Big) \tilde{v}(s) \\ &= \operatorname*{ess\,sup}_{s>0} (f^*(s) + g^*(s)) \tilde{v}(2s) \lesssim \|f\|_X + \|g\|_X.\end{aligned}$$

Therefore $f+g \in \Lambda_v^\infty$. The proof is complete. □

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