Jichun Liu; Longgang Sun
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IDEMPOTENT COMPLETION OF PRETRIANGULATED CATEGORIES

Jichun Liu, Fuzhou, Longgang Sun, Hefei

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Abstract. A pretriangulated category is an additive category with left and right triangulations such that these two triangulations are compatible. In this paper, we first show that the idempotent completion of a left triangulated category admits a unique structure of left triangulated category and dually this is true for a right triangulated category. We then prove that the idempotent completion of a pretriangulated category has a natural structure of pretriangulated category. As an application, we show that a torsion pair in a pretriangulated category extends uniquely to a torsion pair in the idempotent completion.

Keywords: idempotent completion; pretriangulated category; torsion pair

MSC 2010: 16B50, 18E05, 18E30, 18E40

1. Introduction

It is well known that stable categories of Frobenius categories have the structure of triangulated categories. So one may ask whether there exists an analogue of triangulated structure on stable categories of additive categories. The answer is a pretriangulated structure (see [6]). Recall that, for an additive category $\mathcal{C}$ and a functorially finite subcategory $\mathcal{X}$ of $\mathcal{C}$, if each $\mathcal{X}$-epic has a kernel and each $\mathcal{X}$-monic has a cokernel, then $\mathcal{C}/\mathcal{X}$ is a pretriangulated category. We should point out that $\mathcal{C}/\mathcal{X}$ may not be idempotent complete (see Example 2.10). However, we always require additive categories discussed in the representation theory with the property of being idempotent complete.

Balmer and Schlichting showed that the idempotent completion of a triangulated category is a triangulated category (see [4], Theorem 1.12). Buhler showed that the

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idempotent completion of an exact category is an exact category (see [8], Proposition 6.4).

We are interested in studying the idempotent completion of a pretriangulated category. We should point out that pretriangulated categories provide a common generalization of triangulated categories, stable categories and abelian categories. For a concrete example, the homotopy category of an additive model category is naturally a pretriangulated category (see [7]). For more details on pretriangulated categories, one can refer to [2], [5], [7].

The concept of torsion pair in an abelian category was introduced by Dickson in 1966 [9]. Since then, the concept of torsion has become a fundamental object in algebra, geometry and topology, and the use of torsion pair has become an indispensable tool for the study of localization in various categories. Due to the importance of torsion pairs in various categories, it is natural to ask whether a torsion pair $(\mathcal{X},\mathcal{Y})$ in a pretriangulated category induces a torsion pair $(\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}})$ in the idempotent completion $\widetilde{\mathcal{C}}$.

This article is organized as follows. In Section 2, we recall some basic definitions and facts, and prove that the idempotent completion of a left or right triangulated category is still a left or right triangulated category, respectively. In Section 3, we show that the idempotent completion of a pretriangulated category is a pretriangulated category (see Theorem 3.4). In Section 4, we show that any torsion pair $(\mathcal{X},\mathcal{Y})$ in a pretriangulated category $\mathcal{C}$ admits an extension to a torsion pair $(\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}})$ in its idempotent completion category $\widetilde{\mathcal{C}}$ (see Theorem 4.3).

2. Idempotent completion of left triangulated categories

In this section, we recall some basic facts and notions on the idempotent completion of additive categories and left or right triangulated categories for later use. For the background on the idempotent completion of additive categories, one can refer to [8], [12].

Definition 2.1. Let $\mathcal{C}$ be an additive category. An idempotent morphism $e\colon A\to A$ is said to be *split* if there are two morphisms $p\colon A\to B$ and $q\colon B\to A$ such that $e=q\circ p$ and $p\circ q=\mathrm{id}_B$.

An additive category $\mathcal{C}$ is said to be *idempotent complete* provided each idempotent morphism splits. Note that $\mathcal{C}$ is idempotent complete if and only if every idempotent morphism has a kernel.

Definition 2.2 ([8]). Let $\mathcal{C}$ be an additive category. The *idempotent completion of* $\mathcal{C}$ is the category $\widetilde{\mathcal{C}}$ defined as follows. Objects of $\widetilde{\mathcal{C}}$ are pairs (A,e), where A is an object of $\mathcal{C}$ and $e\colon A \to A$ is an idempotent morphism. A morphism in $\widetilde{\mathcal{C}}$ from (A,e) to (B,f) is a morphism $\alpha\colon A \to B$ in $\mathcal{C}$ such that $\alpha \circ e = f \circ \alpha = \alpha$.

The assignment $A \longmapsto (A, \mathrm{id})$ defines a functor l from $\mathcal{C}$ to $\widetilde{\mathcal{C}}$. If $e\colon A \to A$ is an idempotent morphism, then $(A, \mathrm{id}) \simeq (A,e) \oplus (A, \mathrm{id} - e)$ in $\widetilde{\mathcal{C}}$.

If $\mathcal{C}$ is already idempotent complete, then $l\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ is easily seen to be an equivalence of categories. The following result is well known.

Theorem 2.3 ([8]). *The category $\widetilde{\mathcal{C}}$ is additive, the functor $l\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ is additive, and $\widetilde{\mathcal{C}}$ is idempotent complete. Moreover, the functor l induces an equivalence*

$$\mathrm{Hom}_{\mathrm{add}}(\widetilde{\mathcal{C}}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{add}}(\mathcal{C}, \mathcal{D})$$

for each idempotent complete additive category $\mathcal{D}$, where $\mathrm{Hom}_{\mathrm{add}}$ *denotes additive functors between two additive categories.*

Since the functor l is full and faithful, we can view $\mathcal{C}$ as a full subcategory of $\widetilde{\mathcal{C}}$.

Example 2.4. Let $\mathcal{F}$ be the category of free modules over a ring R, then its idempotent completion $\widetilde{\mathcal{F}}$ is equivalent to the category of projective modules over R.

The notions of left (right) triangulated categories were introduced in [2], [6], [7] by Assem, Beligiannis and Reiten in connection with studying the structure of stable categories. Following [6], we recall the definition of left triangulated categories as follows.

Let $\mathcal{C}$ be an additive category with an additive endofunctor $\Omega\colon \mathcal{C} \to \mathcal{C}$. Consider the category $\mathcal{LT}(\mathcal{C}, \Omega)$ whose objects are diagrams of the form $\Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ and whose morphisms are indicated by the following diagram:

$$\begin{tikzcd}
\Omega(C) \arrow[r, "f"] \arrow[d, "\Omega(\gamma)"'] & A \arrow[r, "g"] \arrow[d, "\alpha"'] & B \arrow[r, "h"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\
\Omega(C') \arrow[r, "f'"] & A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C'
\end{tikzcd}$$

The composition of the morphisms of $\mathcal{LT}(\mathcal{C}, \Omega)$ is induced in the canonical way by the corresponding composition of the morphisms of $\mathcal{C}$.

Definition 2.5 ([6]). A full subcategory Δ of $\mathcal{LT}(\mathcal{C}, \Omega)$ is said to be a *left triangulation* of $(\mathcal{C}, \Omega)$ if it is closed under isomorphisms and satisfies the following four axioms:

(LT1 a) For any object A of $\mathcal{C}$, the left triangle $0 \xrightarrow{0} A \xrightarrow{\mathrm{id}_A} A \xrightarrow{0} 0$ belongs to Δ.

(LT1 b) For any morphism $h\colon B \to C$, there is a left triangle in Δ of the form $\Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$.

(LT2) For any left triangle $\Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ in Δ, the left triangle $\Omega(B) \xrightarrow{-\Omega h} \Omega(C) \xrightarrow{f} A \xrightarrow{g} B$ is also in Δ.

(LT3) For any two left triangles $\Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$, $\Omega(C') \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ in Δ and any two morphisms $\beta\colon B \to B'$ and $\gamma\colon C \to C'$ of $\mathcal{C}$ with $\gamma \circ h = h' \circ \beta$, there is a morphism $\alpha\colon A \to A'$ of $\mathcal{C}$ such that the triple (α, β, γ) is a morphism from the first triangle to the second.

(LT4) For any two left triangles $\Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$, $\Omega(D) \xrightarrow{i} E \xrightarrow{l} C \xrightarrow{k} D$ in Δ, there is a third left triangle $\Omega(D) \xrightarrow{j} F \xrightarrow{m} B \xrightarrow{k \circ h} D$ in Δ and two morphisms $\alpha\colon A \to F$ and $\beta\colon F \to E$ of $\mathcal{C}$ such that the diagram below is commutative, where the second column from the left is a left triangle in Δ:

$$\begin{tikzcd}
 & \Omega E \arrow[d, "f \circ \Omega l"] & & \\
\Omega(C) \arrow[r, "f"] \arrow[d, "\Omega(k)"] & A \arrow[r, "g"] \arrow[d, "\alpha"] & B \arrow[r, "h"] \arrow[d, "\mathrm{id}_B"] & C \arrow[d, "k"] \\
\Omega(D) \arrow[r, "j"] \arrow[d, "\mathrm{id}_{\Omega(D)}"] & F \arrow[r, "m"] \arrow[d, "\beta"] & B \arrow[r, "k \circ h"] \arrow[d, "h"] & D \arrow[d, "\mathrm{id}_D"] \\
\Omega(D) \arrow[r, "i"] & E \arrow[r, "l"] & C \arrow[r, "k"] & D
\end{tikzcd}$$

(i.e., the triples $(\alpha, \mathrm{id}_B, k)$ and $(\beta, h, \mathrm{id}_D)$ are morphisms of Δ and $\Omega(E) \xrightarrow{f \circ \Omega l} F \xrightarrow{\alpha} F \xrightarrow{\beta} E$ is in Δ).

Definition 2.6 ([6])**.** The triple $(\mathcal{C}, \Omega, \Delta)$ is called a *left triangulated category*, the functor Ω is called a *loop functor* and the diagrams in Δ are called *left triangles*.

Definition 2.7 ([7])**.** Let $(\mathcal{C}, \Omega, \Delta)$ and $(\mathcal{C}', \Omega', \Delta')$ be two left triangulated categories. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is called *left exact* if there exists a natural isomorphism $\xi\colon F\Omega \to \Omega' F$ such that for any left triangle $\Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ in $\mathcal{C}$, the diagram

$$\Omega'(F(C)) \xrightarrow{F(f) \circ \xi_C^{-1}} F(A) \xrightarrow{F(g)} F(B) \xrightarrow{F(h)} F(C)$$

is a left triangle in $\mathcal{C}'$.

Dually, if $\mathcal{C}$ is an additive category with an additive endofunctor $\Sigma\colon \mathcal{C} \to \mathcal{C}$, consider the category $\mathcal{RT}(\mathcal{C}, \Sigma)$ whose objects are diagrams of the form $A \xrightarrow{d_1} B \xrightarrow{d_2}$

$C \xrightarrow{d_3} \Sigma(A)$ and whose morphisms are indicated by the following diagram:

$$\begin{tikzcd}
A \arrow[r, "d_1"] \arrow[d, "u_1"'] & B \arrow[r, "d_2"] \arrow[d, "u_2"'] & C \arrow[r, "d_3"] \arrow[d, "u_3"'] & \Sigma(A) \arrow[d, "\Sigma(u_1)"'] \\
A' \arrow[r, "d_1'"] & B' \arrow[r, "d_2'"] & C' \arrow[r, "d_3'"] & \Sigma(A')
\end{tikzcd}$$

The composition of the morphisms of $\mathcal{RT}(\mathcal{C}, \Sigma)$ is induced in the canonical way by the corresponding composition of the morphisms of $\mathcal{C}$.

Similarly, we can define the *right triangulated category* $(\mathcal{C}, \Sigma, \nabla)$ and the *right exact functor* between two right triangulated categories. The functor Σ is called a *suspension functor* and the diagrams in ∇ are called *right triangles*.

Following [7], we recall the definition of pretriangulated categories.

Definition 2.8 ([7]). Let $\mathcal{C}$ be an additive category. A *pretriangulation* of $\mathcal{C}$ consists of the following data:

(1) An adjoint pair (Σ, Ω) of additive endofunctors $\Sigma, \Omega\colon \mathcal{C} \to \mathcal{C}$. Let $\varepsilon\colon \Sigma\Omega \to \mathrm{id}_{\mathcal{C}}$ the counit and $\delta\colon \mathrm{id}_{\mathcal{C}} \to \Omega\Sigma$ be the unit of the adjoint pair.

(2) A collection of diagrams Δ in $\mathcal{C}$ of the form $\Omega(C) \to A \to B \to C$ such that the triple $(\mathcal{C}, \Omega, \Delta)$ is a left triangulated category.

(3) A collection of diagrams ∇ in $\mathcal{C}$ of the form $A \to B \to C \to \Sigma(A)$ such that the triple $(\mathcal{C}, \Sigma, \nabla)$ is a right triangulated category.

(4) For any diagram in $\mathcal{C}$ with commutative left square:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\exists\gamma"'] & \Sigma(A) \arrow[d, "\varepsilon_{C'}\Sigma(\alpha)"'] \\
\Omega(C') \arrow[r, "f'"] & A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C'
\end{tikzcd}$$

where the upper row is in ∇ and the lower row is in Δ, there exists a morphism $\gamma\colon C \to B'$ making the diagram commutative.

(5) For any diagram in $\mathcal{C}$ with commutative right square:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\Omega(\alpha)\circ\delta_A"'] & B \arrow[r, "g"] \arrow[d, "\exists\gamma"'] & C \arrow[r, "h"] \arrow[d, "\beta"'] & \Sigma(A) \arrow[d, "\alpha"'] \\
\Omega(C') \arrow[r, "f'"] & A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C'
\end{tikzcd}$$

where the upper row is in ∇ and the lower row in Δ, there exists a morphism $\gamma\colon B \to A'$ making the diagram commutative.

A *pretriangulated category* is an additive category together with a pretriangulation, and is denoted by $(\mathcal{C}, \Sigma, \Omega, \nabla, \Delta, \varepsilon, \delta)$.

Let $H\colon \mathcal{C}_1 \to \mathcal{C}_2$ be a functor between two pretriangulated categories. Then H is called *exact* if H is both left and right exact.

There are some important examples of pretriangulated categories.

Example 2.9. (1) Triangulated categories are pretriangulated. Here $\nabla = \Delta$ and the notions of right and left triangulation coincide.

(2) Any additive category with kernels and cokernels (in particular, any abelian category) is pretriangulated, with $\Omega = \Sigma = 0$ and Δ the class of left exact sequences and ∇ the class of right exact sequences.

(3) Let $\mathcal{C}$ be an additive category, and $\mathcal{X}$ a contravariantly finite subcategory of $\mathcal{C}$. Assume that any $\mathcal{X}$-epic has a kernel. Then the stable category $\mathcal{C}/\mathcal{X}$ always has a left triangulated structure. Dually, we can define a right triangulated structure on stable categories (see [6]).

The idempotent completion of a triangulated category was discussed in [4]. Any abelian category is idempotent complete. However, there exist many pretriangulated categories which are not idempotent complete. This implies that it is an interesting work to study the idempotent completion of pretriangulated categories. The following example is a concrete example of Example 2.9 (3) which is not idempotent complete.

Example 2.10. Let R be the localization of the algebra $\mathbb{C}[x, y]/(y^2 - x^2 - x^3)$ with respect to the ideal corresponding to the origin. Let $\mathcal{C} = R$-mod be the abelian category of finitely generated R-modules and $\mathcal{X}$ the full subcategory of projective modules. Then by combining Example 5.5 and Proposition 5.1 in [10] (also see the example given before in [3], Proposition 2.9), one infers that the left triangulated category $\mathcal{C}/\mathcal{X}$ is not idempotent complete.

Following [2], Lemma 1.3, Corollary 1.4, we recall some properties of left triangulated categories. Let $(\mathcal{C}, \Omega, \Delta)$ be a left triangulated category. A *pseudokernel* of a morphism $h\colon B \to C$ in $\mathcal{C}$ is a morphism $g\colon A \to B$ such that $h \circ g = 0$ and, if $g'\colon A' \to B$ is a morphism in $\mathcal{C}$ with $h \circ g' = 0$, there exists $\alpha\colon A' \to A$ such that $g' = g \circ \alpha$. *Pseudocokernels* are dually defined. For a left exact triangle $\Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ in $(\mathcal{C}, \Omega, \Delta)$, f is a pseudokernel of g and g is a pseudokernel of h. Thus for any object X in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to Ab$ is homological. But in general g is not a pseudocokernel of f and h is not a pseudocokernel of g. Thus $\mathrm{Hom}_{\mathcal{C}}(-, X)\colon \mathcal{C} \to Ab$ is not homological.

It is well known that a monomorphism in a triangulated category is a section, that is, it admits a left inverse. Dually, an epimorphism in a triangulated category is a retraction, that is, it admits a right inverse (see [13], Lemma 2.2, or [14], Proposition 1.5.1). However, the above results usually fail in a left triangulated category. From these properties we can see that there exist some differences between triangulated categories and left triangulated categories.

Lemma 2.11. *Let $(\mathcal{C}, \Omega, \Delta)$ be a left triangulated category. Assume that $\Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ and $\Omega(C') \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ are two left triangles in Δ, and $\beta\colon B \to B'$, $\gamma\colon C \to C'$ are two isomorphisms such that $\gamma \circ h = h' \circ \beta$. Then there exists an isomorphism $\alpha\colon A \to A'$ such that the following diagram is commutative:*

$$\begin{tikzcd}
\Omega(C) \arrow[r, "f"] \arrow[d, "\Omega(\gamma)"'] & A \arrow[r, "g"] \arrow[d, dotted, "\alpha"] & B \arrow[r, "h"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
\Omega(C') \arrow[r, "f'"] & A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C'
\end{tikzcd}$$

P r o o f. The proof follows directly from the dual of [2], Corollary 1.5. □

Note that for a triple $(\mathcal{C}, \Omega, \Delta)$ satisfying (LT1)–(LT3), Lemma 2.11 also holds.

Lemma 2.12. *Let $(\mathcal{C}, \Omega, \Delta)$ be a left triangulated category. A diagram*

$$\Omega(C) \oplus \Omega(C') \xrightarrow{f \oplus f'} A \oplus A' \xrightarrow{g \oplus g'} B \oplus B' \xrightarrow{h \oplus h'} C \oplus C'$$

is a left triangle if and only if both $\Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$ and $\Omega(C') \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ are left triangles.

P r o o f. The proof is analogous to those of [15], Corollary 1.2.5; and [4], Lemma 1.6. □

Let $(\mathcal{C}, \Omega, \Delta)$ be a left triangulated category, and $\widetilde{\mathcal{C}}$ the idempotent completion of $\mathcal{C}$. Define $\widetilde{\Omega}\colon \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ by $\widetilde{\Omega}(A, e) = (\Omega(A), \Omega(e))$. For convenience, we usually write $\widetilde{\Omega}$ as Ω. Define a diagram in $\widetilde{\mathcal{C}}$

$$(\diamond) \qquad \Omega(C) \xrightarrow{h} A \xrightarrow{g} B \xrightarrow{f} C$$

to be a left triangle when it is a direct summand of a left triangle in Δ, that is, there exists a left triangle $\diamond'$ of Δ and left triangle maps $s\colon \diamond \to \diamond'$ and $r\colon \diamond' \to \diamond$ with $rs = \mathrm{id}_\diamond$; equivalently, when there is a left triangle $\diamond''$ in $\widetilde{\mathcal{C}}$ such that $\diamond \oplus \diamond''$ is isomorphic to a left triangle in Δ. Denote by $\widetilde{\Delta}$ the class of left triangles in $\widetilde{\mathcal{C}}$.

Lemma 2.13. *Let*

$$\begin{tikzcd} \Omega(C) \arrow[r, "f"] \arrow[d, "\Omega(\gamma)"'] & A \arrow[r, "g"] & B \arrow[r, "h"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\ \Omega(C) \arrow[r, "f"] & A \arrow[r, "g"] & B \arrow[r, "h"] & C \end{tikzcd}$$

be a commutative diagram in a left triangulated category $(\mathcal{C}, \Omega, \Delta)$*. Assume that* β *and* γ *are idempotent morphisms. Then there exists an idempotent morphism* $\alpha = \alpha^2\colon A \to A$ *such that the diagram*

(2.1)
$$\begin{tikzcd} \Omega(C) \arrow[r, "f"] \arrow[d, "\Omega(\gamma)"'] & A \arrow[r, "g"] \arrow[d, dotted, "\alpha"'] & B \arrow[r, "h"] \arrow[d, "\beta"'] & C \arrow[d, "\gamma"'] \\ \Omega(C) \arrow[r, "f"] & A \arrow[r, "g"] & B \arrow[r, "h"] & C \end{tikzcd}$$

commutes.

Proof. The proof is similar to that of [4], Lemma 1.13. $\square$

Theorem 2.14. *Let* $(\mathcal{C}, \Omega, \Delta)$ *be a left triangulated category. Then with the collection of left triangles in* $\widetilde{\Delta}$, $(\widetilde{\mathcal{C}}, \widetilde{\Omega}, \widetilde{\Delta})$ *is a left triangulated category.*

Proof. To prove that $(\widetilde{\mathcal{C}}, \widetilde{\Omega}, \widetilde{\Delta})$ is a left triangulated category, we only need to show that left triangles in $\widetilde{\Delta}$ satisfy four axioms of Definition 2.5.

(LT1 a) For any object A of $\widetilde{\mathcal{C}}$, there exists A' in $\widetilde{\mathcal{C}}$ such that $A \oplus A' \in \mathcal{C}$ (in fact, if $A = (M, e)$, take $A' = (M, \mathrm{id}_M - e)$ and we have $A \oplus A' \simeq l(M)$). By Lemma 2.12, the left triangle

$$0 \to A \oplus A' \xrightarrow{\mathrm{id}} A \oplus A' \to 0$$

in Δ guarantees that $0 \to A \xrightarrow{\mathrm{id}} A \to 0$ is a left triangle in $\widetilde{\Delta}$.

(LT1 b) Let $h\colon B \to C$ be a morphism in $\widetilde{\mathcal{C}}$, then there exist two objects B' and C' in $\widetilde{\mathcal{C}}$ such that $B \oplus B' \in \mathcal{C}$ and $C \oplus C' \in \mathcal{C}$. Let

$$(*) \qquad \Omega(C \oplus C') \xrightarrow{a} D \xrightarrow{b} B \oplus B' \xrightarrow{h'} C \oplus C'$$

be a left triangle in Δ with $h' = \left(\begin{smallmatrix} h & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By Lemma 2.13, there exists an idempotent morphism $d\colon D \to D$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} \Omega(C \oplus C') \arrow[r, "a"] \arrow[d, "\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & D \arrow[r, "b"] \arrow[d, "d"] & B \oplus B' \arrow[r, "h'"] \arrow[d, "\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix}\right)"] & C \oplus C' \arrow[d, "\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix}\right)"] \\ \Omega(C \oplus C') \arrow[r, "a"] & D \arrow[r, "b"] & B \oplus B' \arrow[r, "h'"] & C \oplus C'. \end{tikzcd}$$

commutes. By the definition of left triangles in $\widetilde{\mathcal{C}}$, we have a left triangle $\Omega(C) \xrightarrow{da} (D,d) \xrightarrow{bd} B \xrightarrow{h} C$ in $\widetilde{\Delta}$.

(LT2) It is immediate from the definition of left triangles.

(LT3) Given two left triangles

$$(\Sigma_1) \qquad \Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C,$$

$$(\Sigma_2) \qquad \Omega(C') \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$$

in $\widetilde{\Delta}$ with morphisms $\beta\colon B \to B', \gamma\colon C \to C'$ such that $\gamma \circ h = h' \circ \beta$. By the definition of left triangles in $\widetilde{\Delta}$, there exist left triangle morphisms $i\colon \Sigma_1 \to \Sigma_3$, $p\colon \Sigma_3 \to \Sigma_1$, $j\colon \Sigma_2 \to \Sigma_4$ and $q\colon \Sigma_4 \to \Sigma_2$ such that $p \circ i = \mathrm{id}_{\Sigma_1}$, $q \circ j = \mathrm{id}_{\Sigma_2}$, where Σ_3, $\Sigma_4 \in \Delta$. The partial map of the left triangle (β, γ) induces a partial map of left triangles $j \circ (\beta, \gamma) \circ p\colon \Sigma_3 \to \Sigma_4$. Since Σ_3 and Σ_4 are left triangles in Δ, by (LT3) we get a left triangle map $\eta\colon \Sigma_3 \to \Sigma_4$ in Δ, which induces a left triangle morphism $q \circ \eta \circ i\colon \Sigma_1 \to \Sigma_2$ extending (β, γ) in $\widetilde{\Delta}$.

(LT4) For any two left triangles

$$(1) \qquad \Omega(C) \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C$$

$$(2) \qquad \Omega(D) \xrightarrow{i} E \xrightarrow{l} C \xrightarrow{k} D$$

in $\widetilde{\Delta}$ choose X, Y and Z in $\widetilde{\mathcal{C}}$ such that $B \oplus X$, $C \oplus Y$, $D \oplus Z$ are in $\mathcal{C}$. Clearly,

$$(3) \qquad 0 \longrightarrow X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0$$

and

$$(4) \qquad \Omega(Y) \xrightarrow{\Omega(\mathrm{id}_Y)} \Omega(Y) \longrightarrow 0 \longrightarrow Y$$

are left triangles in $\widetilde{\Delta}$. Taking the direct sum of left triangles (1), (3) and (4), we get the following left triangle:

$$(5) \qquad \Omega(C) \oplus \Omega(Y) \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}} A \oplus X \oplus \Omega(Y) \xrightarrow{\begin{pmatrix} g & 0 & 0 \\ 0 & \mathrm{id} & 0 \end{pmatrix}} B \oplus X \xrightarrow{\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}} C \oplus Y.$$

Observe that the third morphism of (5) is in $\mathcal{C}$. Thus $B \oplus X \xrightarrow{\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}} C \oplus Y$ can be extended to a left triangle $(\square)$ in $\mathcal{C}$. Since Lemma 2.11 also holds for a triple $(\mathcal{C}, \Omega, \Delta)$ satisfying (LT1)–(LT3), by Lemma 2.11, (5) is isomorphic to $(\square)$ in Δ.

Similarly, the following left triangle is isomorphic to a left triangle in Δ:

$$\Omega(D)\oplus\Omega(Z) \xrightarrow{\begin{pmatrix} i & 0\\ 0 & 0\\ 0 & \mathrm{id}\end{pmatrix}} E\oplus Y\oplus\Omega(Z) \xrightarrow{\begin{pmatrix} l & 0 & 0\\ 0 & \mathrm{id} & 0\end{pmatrix}} C\oplus Y \xrightarrow{\begin{pmatrix} k & 0\\ 0 & 0\end{pmatrix}} D\oplus Z. \tag{6}$$

Since $k\circ h\colon\ B\to D$ is a morphism in $\widetilde{\mathcal{C}}$, by (LT1) $k\circ h\colon\ B\to D$ can be embedded into a left triangle in $\widetilde{\Delta}$

$$\Omega(D) \xrightarrow{j} F \xrightarrow{m} B \xrightarrow{k\circ h} D. \tag{7}$$

Similarly, we have the left triangle

$$\Omega(D)\oplus\Omega(Z) \xrightarrow{\begin{pmatrix} j & 0\\ 0 & 0\\ 0 & \mathrm{id}\end{pmatrix}} F\oplus X\oplus\Omega(Z) \xrightarrow{\begin{pmatrix} m & 0 & 0\\ 0 & \mathrm{id} & 0\end{pmatrix}} B\oplus X \xrightarrow{\begin{pmatrix} k\circ h & 0\\ 0 & 0\end{pmatrix}} D\oplus Z \tag{8}$$

in Δ.

Since $\mathcal{C}$ is a pretriangulated category, there exist two morphisms $a\colon\ A\oplus X\oplus\Omega(Y)\to F\oplus X\oplus\Omega(Z)$ and $b\colon\ F\oplus X\oplus\Omega(Z)\to E\oplus Y\oplus\Omega(Z)$ such that the diagram below is fully commutative:

$$\begin{tikzcd}
 & \Omega E\oplus\Omega Y\oplus\Omega^2 Z \arrow[d, "\overline{f}\circ\Omega(\overline{l})"] & & \\
\Omega(C)\oplus\Omega(Y) \arrow[r, "\overline{f}=\begin{pmatrix} f & 0\\ 0 & 0\\ 0 & \mathrm{id}\end{pmatrix}"] \arrow[d, "\Omega(\overline{k})"] & A\oplus X\oplus\Omega(Y) \arrow[r, "\begin{pmatrix} g & 0 & 0\\ 0 & \mathrm{id} & 0\end{pmatrix}"] \arrow[d, "a"] & B\oplus X \arrow[r, "\begin{pmatrix} h & 0\\ 0 & 0\end{pmatrix}"] \arrow[d, "\mathrm{id}"] & C\oplus Y \arrow[d, "\overline{k}=\begin{pmatrix} k & 0\\ 0 & 0\end{pmatrix}"] \\
\Omega(D)\oplus\Omega(Z) \arrow[r, "\begin{pmatrix} j & 0\\ 0 & 0\\ 0 & \mathrm{id}\end{pmatrix}"] \arrow[d, "\Omega(\mathrm{id})"] & F\oplus X\oplus\Omega(Z) \arrow[r, "\begin{pmatrix} m & 0 & 0\\ 0 & \mathrm{id} & 0\end{pmatrix}"] \arrow[d, "b"] & B\oplus X \arrow[r, "\begin{pmatrix} k\circ h & 0\\ 0 & 0\end{pmatrix}"] \arrow[d, "\begin{pmatrix} h & 0\\ 0 & 0\end{pmatrix}"] & D\oplus Z \arrow[d, "\mathrm{id}"] \\
\Omega(D)\oplus\Omega(Z) \arrow[r, "\begin{pmatrix} i & 0\\ 0 & 0\\ 0 & \mathrm{id}\end{pmatrix}"] & E\oplus Y\oplus\Omega(Z) \arrow[r, "\overline{l}=\begin{pmatrix} l & 0 & 0\\ 0 & \mathrm{id} & 0\end{pmatrix}"] & C\oplus Y \arrow[r, "\begin{pmatrix} k & 0\\ 0 & 0\end{pmatrix}"] & D\oplus Z
\end{tikzcd}$$

where

$$a=\begin{pmatrix} a_{11} & a_{12} & 0\\ 0 & \mathrm{id} & 0\\ a_{31} & a_{32} & 0\end{pmatrix} \quad\text{and}\quad b=\begin{pmatrix} b_{11} & b_{12} & 0\\ 0 & 0 & 0\\ b_{31} & b_{32} & \mathrm{id}\end{pmatrix}.$$

The 0's and id's appearing in a and b come from the commutativity property of the above diagram. Furthermore, we have

$$a_{11}\circ f=j\circ\Omega(k),\quad b_{11}\circ j=i,\quad g=m\circ a_{11},\quad l\circ b_{11}=h\circ m.$$

Define two morphisms

$$\mu = \begin{pmatrix} \mathrm{id} & -a_{12} & 0 \\ 0 & \mathrm{id} & 0 \\ b_{31} & b_{32} & \mathrm{id} \end{pmatrix} \colon F \oplus X \oplus \Omega(Z) \to F \oplus X \oplus \Omega(Z)$$

and

$$\nu = \begin{pmatrix} \mathrm{id} & a_{12} & 0 \\ 0 & \mathrm{id} & 0 \\ -b_{31} & a_{32} & \mathrm{id} \end{pmatrix} \colon F \oplus X \oplus \Omega(Z) \to F \oplus X \oplus \Omega(Z).$$

Due to $\mu \circ \nu = \mathrm{id}$, $\nu \circ \mu = \mathrm{id}$, μ is an automorphism of $F \oplus X \oplus \Omega(Z)$. Considering the compositions

$$\mu \circ a = \begin{pmatrix} \mathrm{id} & -a_{12} & 0 \\ 0 & \mathrm{id} & 0 \\ b_{31} & b_{32} & \mathrm{id} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & \mathrm{id} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$b \circ \nu = \begin{pmatrix} b_{11} & b_{12} & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & a_{12} & 0 \\ 0 & \mathrm{id} & 0 \\ -b_{31} & a_{32} & \mathrm{id} \end{pmatrix} = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix},$$

we have the following two isomorphic left triangles in Δ:

$$\begin{tikzcd}
\Omega(E) \oplus \Omega(Y) \oplus \Omega^2(Z) \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"] & A \oplus X \oplus \Omega(Y) \arrow[r, "\mu \circ a"] \arrow[d, "\mathrm{id}"] & F \oplus X \oplus \Omega(Z) \arrow[r, "b \circ \nu"] \arrow[d, "\nu"] & E \oplus Y \oplus \Omega(Z) \arrow[d, "\mathrm{id}"] \\
\Omega(E) \oplus \Omega(Y) \oplus \Omega^2(Z) \arrow[r, "\overline{f} \circ \Omega(\bar{l})"] & A \oplus X \oplus \Omega(Y) \arrow[r, "a"] & F \oplus X \oplus \Omega(Z) \arrow[r, "b"] & E \oplus Y \oplus \Omega(Z)
\end{tikzcd}$$

where

$$\varphi = \begin{pmatrix} f \circ \Omega(l) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mathrm{id} & 0 \end{pmatrix}, \quad \mu \circ a = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b \circ \nu = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}.$$

Thus, by the definition of $\widetilde{\Delta}$, we obtain a left triangle $\Omega(E) \xrightarrow{f\circ\Omega(l)} A \xrightarrow{a_{11}} F \xrightarrow{b_{11}} E$ in $\widetilde{\Delta}$ such that the following diagram is fully commutative in $\widetilde{\mathcal{C}}$:

$$\begin{tikzcd}
& \Omega E \arrow[d, "f\circ\Omega l"] & & \\
\Omega(C) \arrow[r, "f"] \arrow[d, "\Omega(k)"] & A \arrow[r, "g"] \arrow[d, "a_{11}"] & B \arrow[r, "h"] \arrow[d, "\mathrm{id}_B"] & C \arrow[d, "k"] \\
\Omega(D) \arrow[r, "j"] \arrow[d, "\mathrm{id}_{\Omega(D)}"] & F \arrow[r, "m"] \arrow[d, "b_{11}"] & B \arrow[r, "k\circ h"] \arrow[d, "h"] & D \arrow[d, "\mathrm{id}_D"] \\
\Omega(D) \arrow[r, "i"] & E \arrow[r, "l"] & C \arrow[r, "k"] & D
\end{tikzcd}$$

This completes the proof. $\square$

The following example shows the idempotent completion of $\mathcal{C}/\mathcal{X}$ in Example 2.10.

Example 2.15. Let R be the algebra as described in Example 2.10 and $\widehat{R}$ the completion of R. Let $\mathcal{C}' = \widehat{R}$-mod the abelian category of finitely generated $\widehat{R}$-modules and $\mathcal{X}'$ the full subcategory of projective $\widehat{R}$-modules. Then the idempotent completion of $\mathcal{C}/\mathcal{X}$ is equivalent to $\mathcal{C}'/\mathcal{X}'$.

Theorem 2.16. *Let $(\mathcal{C}, \Omega, \Delta)$ be a left triangulated category. Then its idempotent completion $(\widetilde{\mathcal{C}}, \widetilde{\Omega}, \widetilde{\Delta})$ admits a unique structure of left triangulated category such that the canonical functor $\iota\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ becomes left exact. Moreover, if $\widetilde{\mathcal{C}}$ is endowed with this structure, then for each idempotent complete left triangulated category $\mathcal{D}$, the functor ι induces an equivalence*

$$\mathrm{Hom}_{\mathrm{exact}}(\widetilde{\mathcal{C}}, \mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathrm{exact}}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Hom}_{\mathrm{exact}}$ *denotes left exact functors between two left triangulated categories.*

P r o o f. By (LT1)–(LT3), we see that there exists a unique left triangulation in $\widetilde{\mathcal{C}}$. From the definition of left triangles in $\widetilde{\mathcal{C}}$, the functor $l\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ is left exact. Clearly, the remaining part of this theorem holds by Theorem 2.3. $\square$

Note that, in this section, all notions and results for left triangulated categories can be given dually for right triangulated categories.

3. Idempotent completion of pretriangulated categories

In this section, we show the main result of this paper. First, we need some lemmas for preparation.

Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories (or especially pretriangulated categories), and (F, G) be two functors from $\mathcal{A}$ to $\mathcal{B}$. For any object (X, e) in $\widetilde{\mathcal{A}}$, define $\widetilde{F}$, $\widetilde{G}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ by $\widetilde{F}(X, e) = (F(X), F(e))$ and $\widetilde{G}(X, e) = (G(X), G(e))$, respectively. Clearly, both $\widetilde{F}$ and $\widetilde{G}$ are functors from $\widetilde{\mathcal{A}}$ to $\widetilde{\mathcal{B}}$. As in Section 2, we usually simply write $\widetilde{F}$ and $\widetilde{G}$ as F and G, respectively.

Lemma 3.1. *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ be an adjoint pair of functors. Then the extensions $\widetilde{F}\colon \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ and $\widetilde{G}\colon \widetilde{\mathcal{B}} \to \widetilde{\mathcal{A}}$ to the idempotent completions are still adjoint.*

Proof. Let $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{B}}$ be the counit and $\delta\colon \mathrm{id}_{\mathcal{A}} \to GF$ the unit of the adjoint pair (F, G). According to [8], Remark 6.7, we can define the extensions of natural transformations $\tilde{\varepsilon}$ and $\tilde{\delta}$ respectively. By uniqueness of the extensions of functors and natural transformations to the idempotent completion, one can easily see that the counit-unit equations $\varepsilon F \circ F\delta = \mathrm{id}_F$ and $G\varepsilon \circ \delta G = \mathrm{id}_G$ yield equations $\tilde{\varepsilon}\widetilde{F} \circ \widetilde{F}\tilde{\delta} = \mathrm{id}_{\widetilde{F}}$ and $\widetilde{G}\tilde{\varepsilon} \circ \tilde{\delta}\widetilde{G} = \mathrm{id}_{\widetilde{G}}$. Hence $(\widetilde{F}, \widetilde{G})$ is still an adjoint pair of functors with counit $\tilde{\varepsilon}$ and unit $\tilde{\delta}$. $\square$

Lemma 3.2. *Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ be any right triangle in $\widetilde{\nabla}$, and let $\Omega(C') \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ be any left triangle in $\widetilde{\Delta}$. If there are two morphisms $\alpha\colon A \to \Omega(C')$, $\beta\colon B \to A'$ with $f' \circ \alpha = \beta \circ f$, then there exists a morphism $\gamma\colon C \to B'$ that makes the following diagram commutative:*

(3.1)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] & \Sigma(A) \arrow[d, "\varepsilon_{C'}\Sigma(\alpha)"] \\
\Omega(C') \arrow[r, "f'"] & A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C'
\end{tikzcd}$$

Proof. Denote the right triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ as $\widetilde{\nabla}_1$ and the left triangle $\Omega(C') \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ as $\widetilde{\Delta}_1$. By the definition of right triangles in $\widetilde{\nabla}$, there exists a right triangle

$$(\nabla_1) \qquad A_1 \xrightarrow{f_1} B_1 \xrightarrow{g_1} C_1 \xrightarrow{h_1} \Sigma(A_1)$$

in ∇ and right triangle maps $s = (s_1, s_2, s_3)\colon \widetilde{\nabla}_1 \to \nabla_1$, $u = (u_1, u_2, u_3)\colon \nabla_1 \to \widetilde{\nabla}_1$ such that $u \circ s = 1_{\widetilde{\nabla}_1}$.

Also, by the definition of a left triangle in $\widetilde{\Delta}$, there exists a left triangle

$$(\Delta_1) \qquad \Omega(C_1') \xrightarrow{f_1'} A_1' \xrightarrow{g_1'} B_1' \xrightarrow{h_1'} C_1'$$

in Δ and left triangle maps $t = (t_1, t_2, t_3)\colon \widetilde{\Delta}_1 \to \Delta_1$, $v = (v_1, v_2, v_3)\colon \Delta_1 \to \widetilde{\Delta}_1$ such that $v \circ t = 1_{\widetilde{\Delta}_1}$. Consider the following diagram with left commutative square:

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "\Omega(t_3)\circ\alpha\circ u_1"'] & B_1 \arrow[r, "g_1"] \arrow[d, "t_1\circ\beta\circ u_2"'] & C_1 \arrow[r, "h_1"] & \Sigma(A_1) \arrow[d, "t_3\circ(\varepsilon_{C'}\Sigma(\alpha))\circ\Sigma(u_1)"] \\
\Omega(C_1') \arrow[r, "f_1'"] & A_1' \arrow[r, "g_1'"] & B_1' \arrow[r, "h_1'"] & C_1',
\end{tikzcd}$$

where the upper row is in ∇ and the lower row is in Δ. According to the definition of pretriangulation, there exists a morphism $\gamma'\colon C_1 \to B_1'$ that makes the diagram commutative. Set $\gamma = v_2 \circ \gamma' \circ s_3\colon C \to B'$. Obviously, (1.2) commutes. □

Lemma 3.3. *Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma(A)$ be any right triangle in $\widetilde{\nabla}$, and let $\Omega(C') \xrightarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} C'$ be any left triangle in $\widetilde{\Delta}$. If there are two morphisms $\alpha\colon \Sigma(A) \to C'$, $\beta\colon C \to B'$ with $\alpha \circ h = h' \circ \beta$, then there exists a morphism $\gamma\colon B \to A'$ that makes the following diagram: commutative:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\Omega(\alpha)\circ\delta_A"'] & B \arrow[r, "g"] & C \arrow[r, "h"] \arrow[d, "\beta"'] & \Sigma(A) \arrow[d, "\alpha"] \\
\Omega(C') \arrow[r, "f'"] & A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C'
\end{tikzcd} .$$

P r o o f. The proof is analogous to that of Lemma 3.2. □

Theorem 3.4. *Let $(\mathcal{C}, \Sigma, \Omega, \nabla, \Delta, \varepsilon, \delta)$ be a pretriangulated category. Then its idempotent completion $(\widetilde{\mathcal{C}}, \widetilde{\Sigma}, \widetilde{\Omega}, \widetilde{\nabla}, \widetilde{\Delta}, \tilde{\varepsilon}, \tilde{\delta})$ admits a unique structure of pretriangulated category such that the canonical functor $\iota\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ becomes exact. If $\widetilde{\mathcal{C}}$ is endowed with this structure, then for each idempotent complete pretriangulated category $\mathcal{D}$, the functor ι induces an equivalence*

$$\mathrm{Hom}_{\mathrm{exact}}(\widetilde{\mathcal{C}}, \mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathrm{exact}}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Hom}_{\mathrm{exact}}$ denotes exact functors between two pretriangulated categories.

P r o o f. Clearly, this theorem follows from Definition 2.8, Theorem 2.16 and all lemmas in this section. □

Since a triangulated category is a special pretriangulated category, by Theorem 3.4 we have the following corollary.

Corollary 3.5 ([4]). *Let $\mathcal{C}$ be a triangulated category. Then its idempotent completion $\widetilde{\mathcal{C}}$ admits a unique structure of triangulated category such that the canonical functor $\iota\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ becomes exact. If $\widetilde{\mathcal{C}}$ is endowed with this structure, then for each idempotent complete triangulated category $\mathcal{D}$, the functor ι induces an equivalence*

$$\mathrm{Hom}_{\mathrm{exact}}(\widetilde{\mathcal{C}}, \mathcal{D}) \longrightarrow \mathrm{Hom}_{\mathrm{exact}}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Hom}_{\mathrm{exact}}$ *denotes exact functors between two triangulated categories.*

P r o o f. By Theorem 3.4, we get that $\widetilde{\mathcal{C}}$ is a pretriangulated category. Assume that T is the translation functor in $\mathcal{C}$, then T is an autoequivalence functor in $\mathcal{C}$, which implies that $\widetilde{T}$ is also an autoequivalence in $\widetilde{\mathcal{C}}$. Recall that for a right triangulated category $\mathcal{D}$ with a suspension functor T_1, if T_1 is an equivalence, then $\mathcal{D}$ is a triangulated category (see [2]). Thus $\widetilde{\mathcal{C}}$ is a triangulated category. $\square$

4. Extension of torsion pairs

In this section, we show that a torsion pair in a pretriangulated category is compatible with taking idempotent completion.

Let $(\mathcal{C}, \Sigma, \Omega, \nabla, \Delta, \varepsilon, \delta)$ be a pretriangulated category, and $\mathcal{X}$, $\mathcal{Y}$ two full additive subcategories of $\mathcal{C}$. Following [7], we recall the definition of a torsion pair in a pretriangulated category.

Definition 4.1 ([7]). The pair $(\mathcal{X}, \mathcal{Y})$ is called a *torsion pair* in $\mathcal{C}$ if
(1) $\mathcal{C}(\mathcal{X}, \mathcal{Y}) = 0$.
(2) $\Sigma(\mathcal{X}) \subseteq \mathcal{X}$ and $\Omega(\mathcal{Y}) \subseteq \mathcal{Y}$.
(3) [*The glueing condition*]: $\forall\, C \in \mathcal{C}$, there are triangles

$$\Delta(C)\colon\ \Omega(Y^C) \xrightarrow{g_C} X_C \xrightarrow{f_C} C \xrightarrow{g^C} Y^C \in \Delta$$
$$\nabla(C)\colon\ X_C \xrightarrow{f_C} C \xrightarrow{g^C} Y^C \xrightarrow{f^C} \Sigma(X^C) \in \nabla$$

with $X_C \in \mathcal{X}$, $Y^C \in \mathcal{Y}$.

We begin with the construction of truncation functors $\tau_{\mathcal{X}}$, $\tau_{\mathcal{Y}}$ corresponding to a given torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{C}$.

Lemma 4.2. *Let $(\mathcal{X},\mathcal{Y})$ be a torsion pair in a pretriangulated category $(\mathcal{C}, \Sigma, \Omega, \nabla, \Delta, \varepsilon, \delta)$. Then there exist functors $\tau_{\mathcal{X}}\colon \mathcal{C}\to\mathcal{X}$ and $\tau_{\mathcal{Y}}\colon \mathcal{C}\to\mathcal{Y}$ such that for any $X\in\mathcal{C}$ there exist a left triangle and a right triangle of the forms*

$$\Omega(\tau_{\mathcal{Y}}C) \xrightarrow{g_C} \tau_{\mathcal{X}}C \xrightarrow{f_C} C \xrightarrow{g^C} \tau_{\mathcal{Y}}C \in \Delta$$

and

$$\tau_{\mathcal{X}}C \xrightarrow{f_C} C \xrightarrow{g^C} \tau_{\mathcal{Y}}C \xrightarrow{f^C} \Sigma(\tau_{\mathcal{X}}C) \in \nabla.$$

P r o o f. First, we show the existence of $\tau_{\mathcal{X}}$ and $\tau_{\mathcal{Y}}$. For each object X in $\mathcal{C}$, choose a left triangle $\Omega(Y^C) \xrightarrow{g_C} X_C \xrightarrow{f_C} C \xrightarrow{g^C} Y^C$ with $X_C\in\mathcal{X}$, $Y^C\in\mathcal{Y}$, and define $\tau_{\mathcal{X}}(C)=X_C$ and $\tau_{\mathcal{Y}}(C)=Y^C$. Let $f\colon C\to D$ be a morphism in $\mathcal{C}$, and $\Omega(Y^D) \xrightarrow{g_D} X_D \xrightarrow{f_D} D \xrightarrow{g^D} Y^D$ a left triangle corresponding to the object D. Applying the cohomological functor $\mathrm{Hom}(\tau_{\mathcal{X}}(\mathrm{C}),-)$ to the above left triangle $\Omega(Y^D) \xrightarrow{g_D} X_D \xrightarrow{f_D} D \xrightarrow{g^D} Y^D$, we obtain the exact sequence

$$\mathrm{Hom}(\tau_{\mathcal{X}}(C),\Omega(Y^D)) \to \mathrm{Hom}(\tau_{\mathcal{X}}(C),X_D) \to \mathrm{Hom}(\tau_{\mathcal{X}}(C),D) \to \mathrm{Hom}(\tau_{\mathcal{X}}(C),Y^D).$$

By Definition 4.1, we see that the left and right groups in this sequence vanish. Thus $\tau_{\mathcal{X}}C \xrightarrow{f_C} C \xrightarrow{f} D$ yields a unique morphism $\tau_{\mathcal{X}}(f)\colon \tau_{\mathcal{X}}(C)\to\tau_{\mathcal{X}}(D)=X_D$ and the family of these morphisms for all f's complete τ_X to a functor. Similarly, one can establish the functoriality of $\tau_{\mathcal{Y}}$. □

Theorem 4.3. *Assume that $(\mathcal{C},\Sigma,\Omega,\nabla,\Delta,\varepsilon,\delta)$ is a pretriangulated category. Then a torsion pair $(\mathcal{X},\mathcal{Y})$ in $\mathcal{C}$ induces a torsion pair $(\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}})$ in $\widetilde{\mathcal{C}}$, where*

$$\widetilde{\mathcal{X}}=\{(C,e);\ C\in\mathcal{X},\ e\in\mathrm{Hom}_{\mathcal{C}}(C,C) \textit{ is an idempotent morphism}\},$$

and

$$\widetilde{\mathcal{Y}}=\{(D,f);\ D\in\mathcal{Y},\ f\in\mathrm{Hom}_{\mathcal{C}}(D,D) \textit{ is an idempotent morphism}\}.$$

P r o o f. We should verify the three conditions in the definition of torsion pairs.

(1) For any two objects (C,e) in $\widetilde{\mathcal{X}}$ and (D,f) in $\widetilde{\mathcal{Y}}$, we have $C\in\mathcal{X}$ and $D\in\mathcal{Y}$. Noting that $\mathrm{Hom}_{\widetilde{\mathcal{C}}}((C,e),(D,f))\subset\mathrm{Hom}_{\mathcal{C}}(C,D)$ and $(\mathcal{X},\mathcal{Y})$ is a torsion pair, we get that $\mathrm{Hom}_{\widetilde{\mathcal{C}}}((C,e),(D,f))=0$.

(2) For each object (C,e) in $\widetilde{\mathcal{X}}$, we have $C\in\mathcal{X}$. Due to $\Sigma(\mathcal{X})\subseteq\mathcal{X}$, it follows that $\Sigma(C)\in\mathcal{X}$. Thus $\widetilde{\Sigma}(C,e)=(\Sigma(C),\Sigma(e))\in\widetilde{\mathcal{X}}$. This means that $\Sigma(\widetilde{\mathcal{X}})\subseteq\widetilde{\mathcal{X}}$. Similarly, we have $\Omega(\widetilde{\mathcal{Y}})\subseteq\widetilde{\mathcal{Y}}$.

(3) For each object (C,e) in $\widetilde{\mathcal{C}}$, there exists a left triangle $\Omega(\tau_Y C) \xrightarrow{g_C} \tau_X C \xrightarrow{f_C} C \xrightarrow{g^C} \tau_Y C$ in $\mathcal{C}$. Set $C_1 = (C,e)$, $C_2 = (C, \mathrm{id}_C - e)$. There is an isomorphism $(C, \mathrm{id}_C) \simeq (C,e) \oplus (C, \mathrm{id}_C - e)$ in $\widetilde{\mathcal{C}}$ and we have the commutative diagram

$$\begin{tikzcd}
\Omega(\tau_Y(C)) \arrow[r, "g_C"] \arrow[d] & \tau_X(C) \arrow[r, "f_C"] \arrow[d] & C \arrow[r, "g^C"] \arrow[d] & \tau_Y(C) \arrow[d] \\
\widetilde{\Omega}(\tilde\tau_Y(C_1)) \oplus \widetilde{\Omega}(\tilde\tau_Y(C_2)) \arrow[r, "\begin{pmatrix} \tilde g_{C_1} & 0 \\ 0 & \tilde g_{C_2} \end{pmatrix}"'] & \tilde\tau_X(C_1) \oplus \tilde\tau_X(C_2) \arrow[r, "\begin{pmatrix} \tilde f_{C_1} & 0 \\ 0 & \tilde f_{C_2} \end{pmatrix}"'] & C_1 \oplus C_2 \arrow[r, "\begin{pmatrix} \tilde g^{C_1} & 0 \\ 0 & \tilde g^{C_2} \end{pmatrix}"'] & \tilde\tau_Y(C_1) \oplus \tilde\tau_Y(C_2).
\end{tikzcd}$$

This implies that

$$\widetilde{\Omega}(\tilde\tau_Y C_1) \xrightarrow{\tilde g_{C_1}} \tilde\tau_X(C_1) \xrightarrow{\tilde f_{C_1}} C_1 \xrightarrow{\tilde g^{C_1}} \tilde\tau_Y(C_1)$$

is a left triangle in $\widetilde{C}$, where $\tilde\tau_X(C_1) = (\tau_X(C), \tau_X(e)) \in \widetilde{\mathcal{X}}$ and $\tilde\tau_Y(C_1) = (\tau_Y(C), \tau_Y(e)) \in \widetilde{\mathcal{Y}}$.

Similarly, we can obtain a right triangle $\tilde\tau_X(C_1) \xrightarrow{\tilde f_{C_1}} C_1 \xrightarrow{\tilde g^{C_1}} \tilde\tau_Y(C_1) \xrightarrow{\tilde f^{C_1}} \widetilde{\Sigma}(\tilde\tau_X C_1)$ in $\widetilde{C}$, where $\tilde\tau_X(C_1) = (\tau_X(C), \tau_X(e)) \in \widetilde{\mathcal{X}}$, $\tilde\tau_Y(C_1) = (\tau_Y(C), \tau_Y(e)) \in \widetilde{\mathcal{Y}}$.

This completes the proof. $\square$

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Authors' addresses: Jichun Liu, Department of Mathematics and Physics, Fujian Jiangxia University, No. 2, Xiyuangong Road, Fuzhou, Fujian Province 350 108, China, e-mail: `liujc1982@126.com`; Longgang Sun, School of Mathematical Sciences, University of Science and Technology of China, No 96, Jinzhai Road, Hefei, Anhui Province 230 026, China, e-mail: `sunlg@ustc.edu.cn`.