Ahmad Abbasi; Hajar Roshan-Shekalgourabi; Dawood Hassanzadeh-Lelekaami
Some results on the local cohomology of minimax modules

*Czechoslovak Mathematical Journal*, Vol. 64 (2014), No. 2, 327–333

**Czechoslovak Mathematical Journal, 64 (139) (2014), 327–333**

# SOME RESULTS ON THE LOCAL COHOMOLOGY OF MINIMAX MODULES

Ahmad Abbasi, Hajar Roshan-Shekalgourabi, Dawood Hassanzadeh-Lelekaami, Rasht

(Received November 23, 2012)

*Abstract.* Let $R$ be a commutative Noetherian ring with identity and $I$ an ideal of $R$. It is shown that, if $M$ is a non-zero minimax $R$-module such that $\dim \operatorname{Supp} H_I^i(M) \leqslant 1$ for all $i$, then the $R$-module $H_I^i(M)$ is $I$-cominimax for all $i$. In fact, $H_I^i(M)$ is $I$-cofinite for all $i \geqslant 1$. Also, we prove that for a weakly Laskerian $R$-module $M$, if $R$ is local and $t$ is a non-negative integer such that $\dim \operatorname{Supp} H_I^i(M) \leqslant 2$ for all $i < t$, then $\operatorname{Ext}_R^j(R/I, H_I^i(M))$ and $\operatorname{Hom}_R(R/I, H_I^t(M))$ are weakly Laskerian for all $i < t$ and all $j \geqslant 0$. As a consequence, the set of associated primes of $H_I^i(M)$ is finite for all $i \geqslant 0$, whenever $\dim R/I \leqslant 2$ and $M$ is weakly Laskerian.

*Keywords*: local cohomology module; Krull dimension; minimax module; cofinite module; weakly Laskerian module; associated primes

*MSC 2010*: 13D45, 13E10, 13C05

## 1. Introduction

Let $R$ be a commutative Noetherian ring with identity and $I$ an ideal of $R$. For an $R$-module $M$, the $i$th local cohomology module of $M$ with respect to $I$ is defined as

$$H_I^i(M) \cong \varinjlim_{n \in \mathbb{N}} \operatorname{Ext}_R^i(R/I^n, M).$$

We refer the reader to [5] for more details about the local cohomology. In [12] Hartshorne defined an $R$-module $M$ to be $I$-cofinite if $\operatorname{Supp} M \subseteq V(I)$ and $\operatorname{Ext}_R^j(R/I, M)$ is finite for all $j$ and he asked:

For which rings $R$ and ideals $I$ are the modules $H_I^i(M)$ $I$-cofinite for all $i$ and all finite $R$-modules $M$?

Concerning this question, Hartshorne in [12] and later Chiriacescu in [6] showed that if $R$ is a complete regular local ring and $I$ is a prime ideal such that $\dim R/I = 1$, then $H^i_I(M)$ is $I$-cofinite for any finite $R$-module $M$ (see [12], Corollary 7.7). Huneke and Koh [13], Theorem 4.1, proved that if $R$ is a complete Gorenstein local domain and $I$ is an ideal of $R$ such that $\dim R/I = 1$, then for all non-negative integers $i$ and $j$, $\operatorname{Ext}^j_R(N, H^i_I(M))$ is finite for any finite $R$-modules $M$ and $N$ such that $\operatorname{Supp} N \subseteq V(I)$. Furthermore, Delfino [7] proved that if $R$ is a complete local domain then under some mild conditions similar results hold. Also, Delfino and Marley [8], Theorem 1, and Yoshida [18], Theorem 1.1, have eliminated the completeness hypothesis entirely. Finally, Bahmanpour and Naghipour [4], Theorem 2.6, have removed the local assumption on $R$. They proved for a non-negative integer $t$ and a finite $R$-module $M$ such that $\dim \operatorname{Supp} H^i_I(M) \leqslant 1$ for all $i < t$, the $R$-modules $H^i_I(M)$ are $I$-cofinite for all $i < t$ and $\operatorname{Hom}_R(R/I, H^t_I(M))$ is a finite $R$-module. Azami, Naghipour and Vakili in [1] defined an $R$-module $M$ to be $I$-cominimax, as a generalization of $I$-cofiniteness, if $\operatorname{Supp}(M) \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, M)$ is minimax for all $j$. As one of the main results of this paper, we generalize the Bahmanpour and Naghipour's result to the class of minimax modules. More precisely, we show that if $M$ is a minimax module over an arbitrary commutative Noetherian ring $R$ such that $\dim \operatorname{Supp} H^i_I(M) \leqslant 1$ for all $i$, then $H^i_I(M)$ is $I$-cominimax for all $i$. As a consequence of this result, we prove that if $M$ is minimax with $\dim \operatorname{Supp} H^i_I(M) \leqslant 1$ for all $i$, then the Bass numbers and Betti numbers of $H^i_I(M)$ are finite for all $i \geqslant 0$.

Also, Bahmanpour and Naghipour in [4], Theorem 3.1, proved that if $R$ is local, $M$ is a finite $R$-module and $t$ is a non-negative integer such that $\dim \operatorname{Supp} H^i_I(M) \leqslant 2$ for all $i < t$, then $\operatorname{Ext}^j_R(R/I, H^i_I(M))$ and $\operatorname{Hom}_R(R/I, H^t_I(M))$ are weakly Laskerian for all $i < t$ and all $j \geqslant 0$. As a generalization, we prove that this result holds under the more general assumption that $M$ is weakly Laskerian. As a consequence, it follows that the set of associated primes of $H^i_I(M)$ is finite for all $i \geqslant 0$, whenever $\dim R/I \leqslant 2$ and $M$ is weakly Laskerian.

Throughout the article, $R$ denotes a commutative Noetherian ring, $I$ is an ideal of $R$ and $V(I)$ is the set of all prime ideals of $R$ containing $I$.

## 2. Maim results

In [20], Zöschinger introduced the class of minimax modules and in [20] and [21] he gave many equivalent conditions for a module to be minimax. An $R$-module $M$ is called *minimax* if there is a finite submodule $N$ of $M$ such that $M/N$ is Artinian. It was shown by T. Zink [19] and by E. Enochs [11] that a module over a complete local ring is minimax if and only if it is Matlis reflexive. We first recall briefly the definitions and basic properties of minimax modules that we shall use.

**Remark 2.1.**

(i) The class of minimax modules contains all finite and all Artinian modules.

(ii) Let $0 \to L \to M \to N \to 0$, be an exact sequence of $R$-modules. Then $M$ is minimax if and only if both $L$ and $N$ are minimax (see [3], Lemma 2.1). Thus any submodule and quotient of a minimax module is minimax. Moreover, if $M$ and $N$ are two $R$-modules such that $N$ is finite and $M$ is minimax, then $\operatorname{Ext}^j_R(N, M)$ and $\operatorname{Tor}^R_j(N, M)$ are minimax for all $j \geqslant 0$

(iii) The set of associated primes of any minimax $R$-module is finite.

(iv) If $M$ is a minimax $R$-module and $\mathfrak{p}$ is a non-maximal prime ideal of $R$, then $M_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module.

**Theorem 2.2.** *Let $M$ be a minimax $R$-module such that* $\dim \operatorname{Supp} H^i_I(M) \leqslant 1$ *for all $i$ and let $N$ be a finite $R$-module with* $\operatorname{Supp} N \subseteq V(I)$. *Then* $\operatorname{Ext}^j_R(N, H^i_I(M))$ *is a minimax $R$-module for all $i$ and $j$. In fact,* $\operatorname{Ext}^j_R(N, H^i_I(M))$ *is finite for all $j$ and $i \geqslant 1$.*

Proof. By Gruson's theorem one can reduce the problem to the case where $N = R/I$. Since $H^0_I(M)$ is a submodule of $M$, the assertion holds for $i = 0$ and so we prove the claim for $i \geqslant 1$. To do this, since $M$ is a minimax $R$-module, there exists an exact sequence

$$0 \to S \to M \to T \to 0$$

where $S$ is finite and $T$ is Artinian. So, we obtain the exact sequence

$$0 \to H^0_I(S) \to H^0_I(M) \to H^0_I(T) \xrightarrow{f} H^1_I(S) \to H^1_I(M) \to 0,$$

and the isomorphism

$$H^i_I(S) \cong H^i_I(M)$$

for all $i \geqslant 2$ by the Artinianness of $T$. Put $L := \operatorname{Im} f$ and consider the exact sequence

$$0 \to L \to H^1_I(S) \to H^1_I(M) \to 0.$$

So, it is easy to see that $\dim \operatorname{Supp} H^i_I(S) \leqslant 1$ for all $i$. Hence, $H^i_I(S)$ is $I$-cofinite for all $i$ by [4], Theorem 2.6. So, $H^i_I(M)$ is $I$-cofinite for all $i \geqslant 2$. Moreover, $(0 :_L I)$ is of finite length, since $L$ is Artinian and $(0 :_{H^1_I(S)} I)$ is finite. Therefore, by [15] Proposition 4.1, $L$ is $I$-cofinite. So, in view of the long exact sequence

$$\ldots \to \operatorname{Ext}^j_R(R/I, H^1_I(S)) \to \operatorname{Ext}^j_R(R/I, H^1_I(M)) \to \operatorname{Ext}^{j+1}_R(R/I, L) \to \ldots$$

we deduce that $H^1_I(M)$ is $I$-cofinite. Hence $H^i_I(M)$ is $I$-cofinite for all $i \geqslant 1$. $\square$

**Corollary 2.3.** *Suppose that $M$ is a minimax $R$-module and*

$$\dim \operatorname{Supp} H_I^i(M) \leqslant 1$$

*for all $i$. Then for all $i \geqslant 0$ and any minimax submodule $X$ of $H_I^i(M)$, the $R$-module $H_I^i(M)/X$ is $I$-cominimax. In particular, the Bass numbers of $H_I^i(M)$ are finite.*

Proof. The first assertion follows from the short exact sequence $0 \to X \to H_I^i(M) \to H_I^i(M)/X \to 0$ and Theorem 2.2. For the last assertion, let $\mathfrak{p} \in \operatorname{Spec}(R)$ and let $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ be the residue field of $R_\mathfrak{p}$. If $I \nsubseteq \mathfrak{p}$, then $\mathfrak{p} \notin \operatorname{Supp}_R(H_I^i(M))$. So, there is nothing to prove in this case. Otherwise, $\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^i(M))$ is finite for all $j$ and $i \geqslant 1$ and $\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^0(M))$ is minimax for all $j$ by Theorem 2.2. If $\mathfrak{p}$ is a non-maximal prime ideal of $R$, it follows from Remark 2.1 (iv) that $(\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^0(M)))_\mathfrak{p}$ is finite over $R_\mathfrak{p}$ for all $j$. Also, if $\mathfrak{p}$ is a maximal ideal of $R$, since $(\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^0(M)))_\mathfrak{p}$ is also a $k(\mathfrak{p})$-vector space, the $R_\mathfrak{p}$-module must be a finite length for all $j$ by minimaxness of $H_I^0(M)$. Thus, in either case, $(\operatorname{Ext}_R^j(R/\mathfrak{p}, H_I^i(M)))_\mathfrak{p}$ is finite for all $j$ and $i \geqslant 0$ and the claim is true. □

**Theorem 2.4.** *Under the hypotheses of Corollary 2.3, $\operatorname{Tor}_j^R(R/I, H_I^i(M))$ is minimax for all $i$ and $j$. In fact, the $R$-modules $\operatorname{Tor}_j^R(R/I, H_I^i(M))$ are finite for all $j$ and $i \geqslant 1$.*

Proof. The result follows from Remark 2.1, Theorem 2.2 and [15], Theorem 2.1. □

**Corollary 2.5.** *Under the hypotheses of Corollary 2.3, the Betti numbers of $H_I^i(M)$ are finite for all $i$.*

An $R$-module $M$ is said to be *weakly Laskerian* if the set of associated primes of any quotient module of $M$ is finite. Note that in some texts the weakly Laskerian modules are called *skinny* modules. For example see [17]. Recently, in [16], Quy has introduced the class of FSF modules, i.e., modules containing some finite submodules such that the support of the quotient module is finite. Also, more recently in [2], Theorem 3.3, it has been shown by Bahmanpour that over a Noetherian ring $R$, an $R$-module $M$ is weakly Laskerian if and only if it is FSF.

Here, we prove that [4], Theorem 3.1, holds for the larger class of weakly Laskerian modules instead of the class of finite modules.

**Theorem 2.6.** *Let $R$ be a local ring, $M$ a weakly Laskerian $R$-module and $N$ a finite $R$-module with $\operatorname{Supp} N \subseteq V(I)$. Let $t$ be a non-negative integer such that $\dim \operatorname{Supp} H_I^i(M) \leqslant 2$ for all $i < t$. Then the $R$-modules $\operatorname{Ext}_R^j(N, H_I^i(M))$ for all $j$ and $i < t$, $\operatorname{Hom}_R(N, H_I^t(M))$ and $\operatorname{Ext}_R^1(N, H_I^t(M))$ are weakly Laskerian. In particular, the set of associated primes of $H_I^i(M)$ is finite for all $i \leqslant t$.*

P r o o f. By Gruson's theorem one can reduce the problem to the case where $N = R/I$. In view of [14], Theorem 3.7, it is enough to show that if $\dim \operatorname{Supp} H_I^i(M) \leqslant 2$ for all $i$, then $\operatorname{Ext}_R^j(N, H_I^i(M))$ are weakly Laskerian for all $i$ and $j$. Since $H_I^0(M)$ is a submodule of $M$, the assertion holds for $i = 0$ and so we prove the claim for $i \geqslant 1$. As we mentioned in the paragraph after Corollary 2.5, since $M$ is a weakly Laskerian $R$-module, there exists an exact sequence

$$0 \to S \to M \to T \to 0$$

where $S$ is finite and $T$ has finite support. Since any module with finite support over a Noetherian ring has dimension at most one, we obtain the exact sequence

$$0 \to H_I^0(S) \to H_I^0(M) \to H_I^0(T) \xrightarrow{f} H_I^1(S) \xrightarrow{g} H_I^1(M) \xrightarrow{h} H_I^1(T) \\ \xrightarrow{k} H_I^2(S) \to H_I^2(M) \to 0,$$

and the isomorphism

$$H_I^i(S) \cong H_I^i(M)$$

for all $i \geqslant 3$. Consider the following exact sequences:

$$0 \to \operatorname{Im} f \to H_I^1(S) \to \operatorname{Im} g \to 0;$$
$$0 \to \operatorname{Im} g \to H_I^1(M) \to \operatorname{Im} h \to 0;$$
$$0 \to \operatorname{Im} h \to H_I^1(T) \to \operatorname{Im} k \to 0;$$
$$0 \to \operatorname{Im} k \to H_I^2(S) \to H_I^2(M) \to 0.$$

Hence, it is easy to see that $\dim \operatorname{Supp} H_I^i(S) \leqslant 2$ for all $i$. So, $\operatorname{Ext}_R^j(R/I, H_I^i(S))$ is weakly Laskerian for all $i$ and $j$ by [4], Theorem 3.1. Also, the fact that $\operatorname{Supp} H_I^i(T) \subseteq \operatorname{Supp} T$ for all $i$ implies that $\operatorname{Im} f$, $\operatorname{Im} h$ and $\operatorname{Im} k$ have finite supports and so are weakly Laskerian. Therefore, in the light of the long exact sequences of Ext modules induced by the above short exact sequences we infer that $\operatorname{Ext}_R^j(R/I, H_I^i(M))$ are weakly Laskerian for all $i$ and $j$. □

An $R$-module $M$ is called *$I$-weakly cofinite* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}_R^j(R/I, M)$ is weakly Laskerian for all $j \geqslant 0$ (see [9] and [10]). Now, in view of Theorem 2.6, the proofs of Corollaries 3.2, 3.3 and 3.7 in [4] may be adapted. So, we may improve these results as follows.

**Corollary 2.7.** *Let $I$ be an ideal of the local ring $R$ such that $\dim R/I \leqslant 2$ and let $M$ be a weakly Laskerian $R$-module. Then for any $i \geqslant 0$ and any weakly Laskerian submodule $N$ of $H_I^i(M)$, the $R$-module $H_I^i(M)/N$ is $I$-weakly cofinite for all $i$. In particular, the set $\operatorname{Ass}_R H_I^i(M)$ is finite for all $i$.*

**Corollary 2.8.** *Let $R$ be a local ring and $M$ a weakly Laskerian $R$-module such that* $\dim \operatorname{Supp} H^i_I(M) \leqslant 2$ *for all* $i < \operatorname{cd}(I, M)$*, where* $\operatorname{cd}(I, M) := \max\{i \in \mathbb{Z} : H^i_I(M) \neq 0\}$ *is the cohomological dimension of $M$ with respect to $I$. Then the modules* $\operatorname{Ext}^j_R(R/I, H^i_I(M))$ *are weakly Laskerian for all $i$ and $j$.*

**Proposition 2.9.** *Let $R$ be a commutative Noetherian (not necessarily local) ring and let $I \subseteq J$ be ideals of $R$ such that* $\dim R/I = 2$*. Suppose that $M$ is an $I$-weakly cofinite $R$-module such that $H^i_J(M)$ is $I$-weakly cofinite for $i = 0, 1$. Then $H^i_J(M)$ is $J$-weakly cofinite for all $i$.*

Proof. Since $\operatorname{Ext}^j_R(R/I, M)$ is weakly Laskerian and $I \subseteq J$, we conclude that $\operatorname{Ext}^j_R(R/J, M)$ is weakly Laskerian for all $j \geqslant 0$ by [10], Lemma 2.8. Also, as $\operatorname{Supp} M \subseteq V(I)$ and $\dim R/I = 2$ we have $\dim \operatorname{Supp} M \leqslant 2$. Thus $H^i_J(M) = 0$ for all $i \geqslant 3$. Now, the result follows from [4], Proposition 3.6. □

**Corollary 2.10.** *Let $R$ be a local ring and let $I \subseteq J$ be ideals of $R$ such that* $\dim R/I = 2$*. Suppose that $M$ is a weakly Laskerian $R$-module such that $H^j_J(H^i_I(M))$ is $I$-weakly cofinite for all $i$ and $j = 0, 1$. Then $H^j_J(H^i_I(M))$ is $J$-weakly cofinite for all $i$ and $j$.*

Proof. It follows from Corollary 2.7 and Proposition 2.9. □

**Acknowledgment.** The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

*References*


[1] *J. Azami, R. Naghipour, B. Vakili*: Finiteness properties of local cohomology modules for $\mathfrak{a}$-minimax modules. Proc. Am. Math. Soc. *137* (2009), 439–448.

[2] *K. Bahmanpour*: On the category of weakly Laskerian cofinite modules. Math. Scand. *115* (2014), 62–68.

[3] *K. Bahmanpour, R. Naghipour*: On the cofiniteness of local cohohomology modules. Proc. Am. Math. Soc. *136* (2008), 2359–2363.

[4] *K. Bahmanpour, R. Naghipour*: Cofiniteness of local cohomology modules for ideals of small dimension. J. Algebra *321* (2009), 1997–2011.

[5] *M. P. Brodmann, R. Y. Sharp*: Local Cohomology: An Algebraic Introduction with Geometric Applications. Cambridge Studies in Advanced Mathematics 60, Cambridge University Press, Cambridge, 1998.

[6] *G. Chiriacescu*: Cofiniteness of local cohomology modules over regular local rings. Bull. Lond. Math. Soc. *32* (2000), 1–7.

[7] *D. Delfino*: On the cofiniteness of local cohomology modules. Math. Proc. Camb. Philos. Soc. *115* (1994), 79–84.

[8] *D. Delfino, T. Marley*: Cofinite modules and local cohomology. J. Pure Appl. Algebra *121* (1997), 45–52.

[9] *K. Divaani-Aazar, A. Mafi*: Associated primes of local cohomology modules. Proc. Am. Math. Soc. (electronic) *133* (2005), 655–660.
[10] *K. Divaani-Aazar, A. Mafi*: Associated primes of local cohomology modules of weakly Laskerian modules. Commun. Algebra *34* (2006), 681–690.
[11] *E. Enochs*: Flat covers and flat cotorsion modules. Proc. Am. Math. Soc. *92* (1984), 179–184.
[12] *R. Hartshorne*: Affine duality and cofiniteness. Invent. Math. *9* (1970), 145–164.
[13] *C. Huneke, J. Koh*: Cofiniteness and vanishing of local cohomology modules. Math. Proc. Camb. Philos. Soc. *110* (1991), 421–429.
[14] *A. Mafi*: A generalization of the finiteness problem in local cohomology modules. Proc. Indian Acad. Sci., Math. Sci. *119* (2009), 159–164.
[15] *L. Melkersson*: Modules cofinite with respect to an ideal. J. Algebra *285* (2005), 649–668.
[16] *P. H. Quy*: On the finiteness of associated primes of local cohomology modules. Proc. Am. Math. Soc. *138* (2010), 1965–1968.
[17] *H. R. Robbins*: Finiteness of Associated Primes of Local Cohomology Modules. Ph.D. Thesis, University of Michigan, 2008.
[18] *K.-I. Yoshida*: Cofiniteness of local cohomology modules for ideals of dimension one. Nagoya Math. J. *147* (1997), 179–191.
[19] *T. Zink*: Endlichkeitsbedingungen für Moduln über einem Noetherschen Ring. Math. Nachr. *64* (1974), 239–252. (In German.)
[20] *H. Zöschinger*: Minimax modules. J. Algebra *102* (1986), 1–32. (In German.)
[21] *H. Zöschinger*: On the maximality condition for radically full submodules. Hokkaido Math. J. *17* (1988), 101–116. (In German.)

*Authors' addresses*: Ahmad Abbasi, Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran, e-mail: `aabbasi@guilan.ac.ir`; Hajar Roshan-Shekalgourabi (corresponding author), Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran, e-mail: `hroshan@guilan.ac.ir & hrsmath@gmail.com`; Dawood Hassanzadeh-Lelekaami, Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran, e-mail: `dhmath@guilan.ac.ir & lelekaami@gmail.com`.