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AN ALGORITHM BASED ON ROLLING TO GENERATE SMOOTH INTERPOLATING CURVES ON ELLIPSOIDS

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We present an algorithm to generate a smooth curve interpolating a set of data on an n-dimensional ellipsoid, which is given in closed form. This is inspired by an algorithm based on a rolling and wrapping technique, described in [11] for data on a general manifold embedded in Euclidean space. Since the ellipsoid can be embedded in an Euclidean space, this algorithm can be implemented, at least theoretically. However, one of the basic steps of that algorithm consists in rolling the ellipsoid, over its affine tangent space at a point, along a curve. This would allow to project data from the ellipsoid to a space where interpolation problems can be easily solved. However, even if one chooses to roll along a geodesic, the fact that explicit forms for Euclidean geodesics on the ellipsoid are not known, would be a major obstacle to implement the rolling part of the algorithm. To overcome this problem and achieve our goal, we embed the ellipsoid and its affine tangent space in $\mathbb{R}^{n+1}$ equipped with an appropriate Riemannian metric, so that geodesics are given in explicit form and, consequently, the kinematics of the rolling motion are easy to solve. By doing so, we can rewrite the algorithm to generate a smooth interpolating curve on the ellipsoid which is given in closed form.

Keywords: rolling, group of isometries, ellipsoid, kinematic equations, interpolation

Classification: 65D05, 65D07, 65D10, 53B21, 53C22, 70B10

1. INTRODUCTION

There are several classical methods to generate smooth interpolating curves in Euclidean spaces. Cubic splines are possibly the most interesting from the point of view of applications, since they also minimize changes in velocity. However, if one requires the curve and data points to live on a curved space, the classical methods do not produce a reasonable answer. Interpolation problems on manifolds have been studied by several authors, starting with the pioneer work of Noakes, Heinzinger and Paden in [18]. Following this, other authors further developed the theory of geometric splines on manifolds using a variational approach (see, for instance, [2, 4] and [5]). A more general variational problem, that of fitting a curve to data points on a Riemannian manifold, was presented and studied in [17]. For the particular case when these curves solve a first or second order problem, a Palais-based steepest descent algorithm that solves the problem numerically was presented in [21] and complemented with illustrations in the

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2-plane and the 2-sphere. An analytical approach was proposed and studied in [7], but, as in the former approach, the results, although theoretically very interesting, are very difficult to implement except in trivial cases. To overcome this problem, a geometric algorithm, which generalises the classical De Casteljau algorithm, was developed in [20] and [3]. However, the algorithm on non Euclidean spaces produces interpolating curves defined implicitly, which makes its implementation very hard. The main drawback in all these approaches is that they do not produce interpolating curves in closed form.

In this paper we present an algorithm that generates interpolating curves on ellipsoids given in explicit form. This algorithm is based on a procedure to generate interpolating curves on manifolds embedded in Euclidean space, first described in [12] for the 2-sphere, generalised in [10] for the n-sphere and in [11] for the rotation group and Grassmann manifolds. The algorithm is based on a rolling/unrolling and wrapping/unwrapping technique that will be fully described in the last two sections.

Rolling a manifold upon another manifold of the same dimension, with the constraints of no-slip and no-twist, is a non-holonomic problem that has kept much attention. The classical rolling sphere problems had an increasing interest based on the numerous applications in physics and engineering. The mathematical formulation of rolling motions due to [19] and [22] led to a kinematic interpretation of Levi-Civita and normal connections of submanifolds. These were followed by the work of several authors. Without being exhaustive, we refer the work about rolling bodies in [1], the work on optimality properties of rolling motions of spheres and other spaces of constant curvature studied in [23] and [13], the kinematics of rolling orthogonal groups, Grassmann manifolds and Stiefel manifolds in [11] and [8], and, more recently, generalisations of rolling in general Riemannian manifolds presented in [9].

The present paper, involving rolling motions of ellipsoids, is an extended version of our previous work in [14]. The ellipsoids are embedded in $\mathbb{R}^{n+1}$ equipped with an appropriate Riemannian metric, so that geodesics are given in explicit form and the kinematics of rolling are easy to solve. By doing so, the algorithm presented here to generate a smooth interpolating curve on the ellipsoid is given in closed form.

To achieve our goal, we organise the paper as follows. The interpolating problem is formulated in Section 2, the general description of rolling maps appears in Section 3, the appropriate geometry of the ellipsoid is presented in Section 4, the kinematic equations for the rolling motion of an ellipsoid on its affine tangent space at a point is the content of Section 5, and, finally, the last two Sections include the algorithm to solve the interpolating problem and some simulations. Other considerations about rolling along piecewise smooth curves are also included.

2. SMOOTH INTERPOLATION ON THE ELLIPSOID $\mathcal{E}^N$

Let $d_1, d_2, \ldots, d_{n+1}$ be positive real numbers. The n-dimensional ellipsoid is defined as

$$\mathcal{E}^n := \left\{ \big(x_1, x_2, \ldots, x_{n+1}\big) \in \mathbb{R}^{n+1} \ : \ \frac{x_1^2}{d_1^2} + \frac{x_2^2}{d_2^2} + \cdots + \frac{x_{n+1}^2}{d_{n+1}^2} = 1 \right\}.$$

2.1. Statement of the problem

Given a set of $k+1$ distinct points $p_i \in \mathcal{E}^n$, $i = 0, 1, \ldots, k$, vectors V_0 and V_k tangent to $\mathcal{E}^n$ at p_0 and p_k respectively, and fixed times t_i, where

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = \tau,$$

we aim to solve the following problem:

Problem 2.1. Find a C^2-smooth curve

$$\gamma \colon [0, \tau] \to \mathcal{E}^n \tag{1}$$

satisfying interpolation conditions:

$$\gamma(t_i) = p_i, \qquad 1 \leq i \leq k-1, \tag{2}$$

and boundary conditions:

$$\begin{aligned} \gamma(0) &= p_0, \qquad & \gamma(\tau) &= p_k, \\ \dot\gamma(0) &= V_0, \qquad & \dot\gamma(\tau) &= V_k. \end{aligned} \tag{3}$$

In the last section of this paper we present an algorithm that solves this problem and is an adaptation of the algorithm presented in [11] for generating interpolating curves on manifolds embedded in Euclidean space. One important step in this algorithm is based on a rolling technique that consists fn rolling the given manifold over its affine tangent space at a point, along geodesic curves. The main drawback when trying to use the algorithm in [11] is that if the ellipsoid is embedded in Euclidean space, the corresponding geodesics have no explicit closed form. The problem of deriving geodesics on the ellipsoid goes back to Jacobi. A modern treatment of geodesics on quadratics through studies of geodesic flows has been given in [6]. The solution can be obtained in ellipsoidal coordinates. However, it is possible to embed the ellipsoid in another Riemannian manifold where geodesics can be expressed in closed form. This is the approach taken here. Moreover, rolling motions along these geodesics can be described easily, following the ideas in [9] to describe rolling motions of manifolds embedded in arbitrary Riemannian manifolds. So, in order that we can follow the implementation of the algorithm, we dedicate the next section to rolling maps. After presenting the general definition, we derive the kinematic equations for rolling the ellipsoid $\mathcal{E}^n$ over its affine tangent space at a point.

3. ROLLING MAPS

We use an extended Sharpe's definition [22] of a *rolling map* which is applicable to Riemannian manifolds and can be found in [9]. Hereafter $I \subset \mathbb{R}$ denotes a closed interval.

Definition 3.1. Let M_0 and M_1 be two n-manifolds isometrically embedded in an m-dimensional Riemannian manifold M and $\sigma_1 \colon I \to \mathrm{M}_1$ a smooth curve in M_1. A *rolling map* of M_1 on M_0 along the curve σ_1, without slipping or twisting, is a map $\boldsymbol{\chi} \colon I \to \mathsf{Isom}(\mathrm{M})$, a Lie group of isometries of M, satisfying the following conditions, for all $t \in I$:

Rolling

(a) $\boldsymbol{\chi}(t)(\sigma_1(t)) \in \mathrm{M}_0$;

(b) $\mathrm{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}(\boldsymbol{\chi}(t)(\mathrm{M}_1)) = \mathrm{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}\mathrm{M}_0$.

The curve $\sigma_0 \colon I \to \mathrm{M}_0$ defined by $\sigma_0(t) := \boldsymbol{\chi}(t)(\sigma_1(t))$ is called the *development curve* of σ_1.

No-slip $\dot{\sigma}_0(t) = \boldsymbol{\chi}_*(t)(\dot{\sigma}_1(t))$, where $\boldsymbol{\chi}_*$ is the push-forward of $\boldsymbol{\chi}$.

No-twist

***tangential*:** $\big(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}^{-1}(t)\big)_*(\mathrm{T}_{\sigma_0(t)}\mathrm{M}_0) \subset \mathrm{T}_{\sigma_0(t)}\mathrm{M}_0^{\perp}$,

***normal*:** $\big(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}^{-1}(t)\big)_*(\mathrm{T}_{\sigma_0(t)}\mathrm{M}_0^{\perp}) \subset \mathrm{T}_{\sigma_0(t)}\mathrm{M}_0$, where $\mathrm{T}_p\mathrm{M}_0^{\perp}$ denotes the *normal* space at $p \in \mathrm{M}_0$.

This definition can be extended to the situation when σ_1 is only piecewise smooth. In this case $\boldsymbol{\chi}$ is also piecewise smooth and the constraints of *no-slip* and *no-twist* are valid for almost all t.

It is worth to recall that $\dot{\boldsymbol{\chi}}$ is a mapping from I to $\mathrm{T}\mathsf{Isom}(\mathrm{M})$ and $\dot{\boldsymbol{\chi}}(t) \colon \mathrm{M} \to \mathrm{TM}$ acts on M according to

$$\dot{\boldsymbol{\chi}}(t)(p) := \lim_{s \to 0} \frac{\boldsymbol{\chi}(t+s)(p) - \boldsymbol{\chi}(t)(p)}{s}.$$

The curve $\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1} \colon I \to \mathfrak{isom}(\mathrm{M})$ lies in the Lie algebra of the Lie group of isometries $\mathsf{Isom}(\mathrm{M})$ of the manifold M. The tangent mapping $\boldsymbol{\chi}_* \colon I \to \mathsf{Isom}(\mathrm{M})$ maps interval I to a subgroup of $\mathsf{Isom}(\mathrm{M})$ and consequently $\big(\dot{\boldsymbol{\chi}} \circ \boldsymbol{\chi}^{-1}\big)_* \colon I \to \mathfrak{isom}(\mathrm{M})$ lies in a sub-algebra, it is a generalisation of "spatial angular velocity" in $\mathbb{R}^3$, cf. Figure 1.

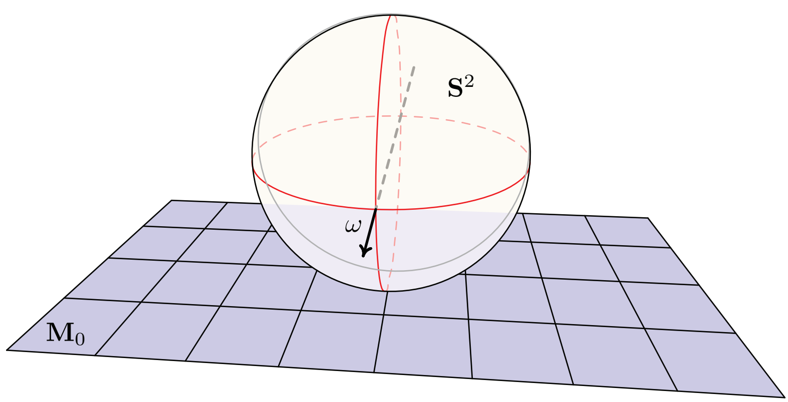

Fig. 1. Spatial angular velocity of the rolling sphere.

Remark 3.2. The *"no-slip"* condition is equivalent to

$$\dot{\boldsymbol{\chi}}(t)(\sigma_1(t)) = \big(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}^{-1}(t)\big)(\sigma_0(t)) = \mathbf{0}. \tag{4}$$

The *"no-twist"* conditions are equivalent to

***tangential*:** for any vector field V_1 along $\sigma_1 \colon I \to \mathrm{M}_1$ let V_0 be the vector field along the development curve $\sigma_0 \colon I \to \mathrm{M}_0$ induced by $\boldsymbol{\chi}$, then

$$\nabla^1_{\dot{\sigma}_1(t)} V_1(t) = \boldsymbol{\chi}(t)_*^{-1}(\nabla^0_{\dot{\sigma}_0(t)} V_0(t)) \in \mathrm{T}_{\sigma_1(t)}\mathrm{M}_1;$$

***normal*:** for any normal vector field Λ_1 along $\sigma_1 \colon I \to \mathrm{M}_1$ let Λ_0 be the vector field along the development curve $\sigma_0 \colon I \to \mathrm{M}_0$ induced by $\boldsymbol{\chi}$, then

$${\nabla^1}^{\perp}_{\dot{\sigma}_1(t)} \Lambda_1(t) = \boldsymbol{\chi}(t)_*^{-1}({\nabla^0}^{\perp}_{\dot{\sigma}_0(t)} \Lambda_0(t)) \in \big(\mathrm{T}_{\sigma_1(t)}\mathrm{M}_1\big)^{\perp},$$

where ${\nabla^0}^{\perp}$ and ${\nabla^1}^{\perp}$ are the *normal connections* on M_0 and M_1, respectively, compatible with the induced metric.

For a proof we refer the reader to [9].

3.1. The configuration space and the distribution

The *configuration space* Σ of the rolling map $\boldsymbol{\chi}$ is the space of all possible positions of M_0 tangent to M_1

$$\Sigma = \big\{ \big(p, g, q\big) \in \mathrm{M}_0 \times \mathsf{Isom}(\mathrm{M}) \times \mathrm{M}_1 \;:\; g(\mathrm{T}_q\mathrm{M}_1) = \mathrm{T}_p\mathrm{M}_0 \big\}.$$

Lemma 3.3. (Sharpe [22] generalised in Hüper et al. [9]) The mapping $\boldsymbol{\chi} \colon I \to \mathsf{Isom}(\mathrm{M})$ is a rolling map if and only if $\big(\sigma_0, \boldsymbol{\chi}_*, \sigma_1\big)$ is tangent to the n-dimensional distribution on Σ given by the following set of differential equations

(a) $\dot{\sigma}_0 = \boldsymbol{\chi}_* \dot{\sigma}_1$;

(b) $\big(\dot{\boldsymbol{\chi}}\, \boldsymbol{\chi}^{-1}\big)_* V = \mathrm{II}^0(\dot{\sigma}_0, V) - \boldsymbol{\chi}_*\, \mathrm{II}^1(\boldsymbol{\chi}_*^{-1}\dot{\sigma}_0, \boldsymbol{\chi}_*^{-1} V)$, for all $V \in \mathrm{T}_{\sigma_0(t)}\mathrm{M}_0$;

(c) $\big(\dot{\boldsymbol{\chi}}\, \boldsymbol{\chi}^{-1}\big)_* \Lambda = \Xi^0(\dot{\sigma}_0, \Lambda) - \boldsymbol{\chi}_*\, \Xi^1(\boldsymbol{\chi}_*^{-1}\dot{\sigma}_0, \boldsymbol{\chi}_*^{-1} \Lambda)$, for all $\Lambda \in \big(\mathrm{T}_{\sigma_0(t)}\mathrm{M}_0\big)^{\perp}$.

Here II is the second fundamental form and Ξ is the Weingarten map.

If M_0 is locally isometric to Euclidean space, i. e., if M_0 is *flat*, then $\mathrm{II}^0 \equiv 0 \equiv \Xi^0$ and the above conditions, (b) and (c), simplify to

$$\big(\dot{\boldsymbol{\chi}}\, \boldsymbol{\chi}^{-1}\big)_* V = -(\boldsymbol{\chi}_*\, \mathrm{II}^1)(\dot{\sigma}_0, V) \quad \text{and} \quad \big(\dot{\boldsymbol{\chi}}\, \boldsymbol{\chi}^{-1}\big)_* \Lambda = -(\boldsymbol{\chi}_*\, \Xi^1)(\dot{\sigma}_0, \Lambda), \tag{5}$$

where $(\boldsymbol{\chi}_*\, \mathrm{II})$ and $(\boldsymbol{\chi}_*\, \Xi)$ are push-forwards of the two tensors. We might as well write the above equations in terms of angular velocity in the *body coordinate system*. Then

$$\big(\boldsymbol{\chi}^{-1} \dot{\boldsymbol{\chi}}\big)_* V = -\mathrm{II}^1(\dot{\sigma}_1, V) \quad \text{and} \quad \big(\boldsymbol{\chi}^{-1} \dot{\boldsymbol{\chi}}\big)_* \Lambda = -\Xi^1(\dot{\sigma}_1, \Lambda), \tag{6}$$

for all $V \in \mathrm{T}_{\sigma_1(t)}\mathrm{M}_1$ and $\Lambda \in \big(\mathrm{T}_{\sigma_1(t)}\mathrm{M}_1\big)^{\perp}$.

Example 3.4. For the special case of $\mathrm{M}_1 = \mathbf{S}^n$, the unit sphere, rolling upon $\mathrm{M}_0 \simeq \mathbb{R}^n$, embedded in Euclidean space $\mathrm{M} = \mathbb{R}^{n+1}$ and $\mathsf{Isom}(\mathrm{M}) = \mathbb{SE}(n+1)$, the second fundamental form the Weingarten map are given by $\Pi(U,V) = \langle U, V\rangle N$ and $\Xi(U,N) = -U$, where N is the unit normal vector. Differential equations (5) and (6) have the rank two solutions, respectively

$$\big(\dot{\boldsymbol{\chi}}\boldsymbol{\chi}^{-1}\big)_* = \dot{\sigma}_0 \wedge N \quad \text{and} \quad \big(\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}}\big)_* = \dot{\sigma}_1 \wedge \sigma_1, \tag{7}$$

where the wedge product '$\wedge$' is thought to be $U \wedge V = U \otimes V - V \otimes U$. When $\sigma_1 \colon I \to \mathbf{S}^n$ is a geodesic then $\big(\dot{\boldsymbol{\chi}}\boldsymbol{\chi}^{-1}\big)_*$ and $\big(\boldsymbol{\chi}^{-1}\dot{\boldsymbol{\chi}}\big)_*$ are constant curves in the Lie algebra $\mathfrak{se}(n+1)$. This is easily seen by differentiating (7) with respect to t

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\sigma}_0 \wedge N\big) = \ddot{\sigma}_0 \wedge N = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\big(\dot{\sigma}_1 \wedge \sigma_1\big) = \ddot{\sigma}_1 \wedge \sigma_1 + \dot{\sigma}_1 \wedge \dot{\sigma}_1 = 0.$$

Here is a modification of the previous example, when the development curve is a circle. By symmetry, the rolling curve must be a small circle on the sphere. As a consequence of the *no-slip* condition the perimeters of the two circles are equal.

Example 3.5. Let $\sigma_0(t) = (r\cos(\omega t), r\sin(\omega t), -1)$, a circle of radius r, then (7) yields

$$\big(\dot{\boldsymbol{\chi}}\boldsymbol{\chi}^{-1}\big)_* = \omega r \begin{pmatrix} 0 & 0 & \sin(\omega t) \\ 0 & 0 & -\cos(\omega t) \\ -\sin(\omega t) & \cos(\omega t) & 0 \end{pmatrix}.$$

Solution can be found with a change of coordinates $\boldsymbol{\chi}_*(t) = e^{tQ}\widetilde{\boldsymbol{\chi}}_*(t)$, with $e^{tQ} = U(t)$. Take

$$Q = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{then} \quad U(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) & 0 \\ \sin(\omega t) & \cos(\omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$U^{\mathrm{T}}(t) \begin{pmatrix} 0 & 0 & \sin(\omega t) \\ 0 & 0 & -\cos(\omega t) \\ -\sin(\omega t) & \cos(\omega t) & 0 \end{pmatrix} U(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

is a constant matrix. The transformed equation becomes

$$\dot{\boldsymbol{\chi}}_*\boldsymbol{\chi}_*^{-1} = (\dot{U}\widetilde{\boldsymbol{\chi}}_* + U\dot{\widetilde{\boldsymbol{\chi}}}_*)\widetilde{\boldsymbol{\chi}}_*^{-1}U^{\mathrm{T}} = \dot{U}U^{\mathrm{T}} + U\dot{\widetilde{\boldsymbol{\chi}}}_*\widetilde{\boldsymbol{\chi}}_*^{-1}U^{\mathrm{T}} = Q + U\dot{\widetilde{\boldsymbol{\chi}}}_*\widetilde{\boldsymbol{\chi}}_*^{-1}U^{\mathrm{T}}.$$

Hence

$$\dot{\widetilde{\boldsymbol{\chi}}}_*\widetilde{\boldsymbol{\chi}}_*^{-1} = U^{\mathrm{T}}\dot{\boldsymbol{\chi}}_*\boldsymbol{\chi}_*^{-1}U - Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a constant matrix. This case now reduces to the simple case considered in Example 3.4. One can now easily solve the above equation for $\boldsymbol{\chi}_*$ and recover $\boldsymbol{\chi}$ from the *"no-slip"* condition (4).

4. THE GEOMETRY OF THE ELLIPSOID

Let the Euclidean metric in $\mathbb{R}^{n+1}$ be denoted by $\langle\cdot,\cdot\rangle$. The positive definite matrix $\mathbf{D} = \operatorname{diag}(d_1, d_2, \dots, d_{n+1}) \succ 0$ induces another metric on $\mathbb{R}^{n+1}$ defined by $(U, V) \mapsto \langle U, V\rangle_{\mathbf{D}^{-2}} := \langle U, \mathbf{D}^{-2}V\rangle$. This metric space will be denoted by $\mathrm{M} = (\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}})$. Since

$$\langle \mathbf{D}U, \mathbf{D}V\rangle_{\mathbf{D}^{-2}} = \langle \mathbf{D}U, \mathbf{D}^{-2}\mathbf{D}V\rangle = \langle U, V\rangle,$$

then the mapping

$$\varphi\colon (\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle) \to \mathrm{M}, \tag{8}$$

given in the standard coordinates by $\boldsymbol{x} \mapsto \mathbf{D}\boldsymbol{x}$, is an isometry. M is an example of a space equipped with a left-invariant metric. Unlike the Euclidean space, groups of isometries acting on objects from the left hand side is different than from the right. The main reason for introducing M is that the ellipsoid $\mathcal{E}^n$ behaves like the sphere in Euclidean space with its standard metric.

4.1. The Riemannian connection

Let $\overline{\nabla}$ be the standard Riemannian connection on $\mathbb{R}^{n+1}$. Then $\varphi_*\overline{\nabla}$ is the Riemannian connection on M, cf. [16, Proposition 5.6]. More precisely, $\forall X, Y \in \mathrm{TM}$, $\varphi_*(\overline{\nabla}_X Y) = \nabla_{\varphi_* X}(\varphi_* Y)$, hence

$$\nabla_X Y = \mathbf{D}\overline{\nabla}_{\mathbf{D}^{-1}X}(\mathbf{D}^{-1}Y) = \overline{\nabla}_{\mathbf{D}^{-1}X} Y \tag{9}$$

is the Riemannian connection on M. We leave it to the reader to check that ∇ defined by (9) is indeed the Riemannian connection (or Levi-Civita connection) on M, and it is compatible with respect to $\langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}$ and is torsion free.

Let $\gamma\colon I \to \mathrm{M}$ be a curve in M and let $V\colon I \to \mathrm{TM}$ be a vector field along γ, i. e., $V(t) \in \mathrm{T}_{\gamma(t)}\mathrm{M}$, for all $t \in I$. Let $\bar\gamma$ and $\overline{V}$ be their isometric images in $\mathbb{R}^{n+1}$. Then by (9) and because $\mathbf{D}$ is constant, there is

$$\nabla_{\dot\gamma} V = \mathbf{D}\nabla_{\dot{\bar\gamma}}\overline{V} = \mathbf{D}\dot{\overline{V}} = \dot V.$$

Hence the covariant derivative of the vector field V does not depend on γ!

4.2. The group of isometries of M

Let $\mathsf{Isom}(\mathrm{M})$ denote the (Lie) group of isometries of M. Suppose that $\varphi\colon \mathrm{M} \to \mathrm{M}$ is an isometry. Therefore, for any $p \in \mathrm{M}$ and $U, V \in \mathrm{T}_p\mathrm{M}$ the following equality holds

$$\langle U, V\rangle_{\mathbf{D}^{-2}} = \langle \varphi_* U, \varphi_* V\rangle_{\mathbf{D}^{-2}}$$

or, equivalently,

$$\langle U, \mathbf{D}^{-2}V\rangle = \langle \varphi_* U, \mathbf{D}^{-2}\varphi_* V\rangle = \langle U, \varphi_*^{\mathrm{T}}\mathbf{D}^{-2}\varphi_* V\rangle.$$

It follows now that $\mathbf{D}^{-2} = \varphi_*^{\mathrm{T}}\mathbf{D}^{-2}\varphi_*$. Then, $\varphi_* \in \mathcal{G}_{\mathbf{D}^{-2}}$, where $\mathcal{G}_{\mathbf{D}^{-2}}$ is the matrix quadratic Lie group defined as

$$\mathcal{G}_{\mathbf{D}^{-2}} := \{\, g \;:\; g^{\mathrm{T}}\mathbf{D}^{-2}g = \mathbf{D}^{-2} \,\}.$$

The Lie algebra of $\mathcal{G}_{\mathbf{D}^{-2}}$ is defined as:

$$\mathcal{L}_{\mathbf{D}^{-2}} := \{ A \ : \ A^{\mathrm{T}}\mathbf{D}^{-2} = -\mathbf{D}^{-2}A \}.$$

It can be easily seen that, for any $g \in \mathcal{G}_{\mathbf{D}^{-2}}$, there exists exactly one $R \in \mathbb{SO}(n+1)$ such that $g = \mathbf{D}R\mathbf{D}^{-1}$. Therefore $\mathcal{G}_{\mathbf{D}^{-2}} = \mathbf{D} \cdot \mathbb{SO}(n+1) \cdot \mathbf{D}^{-1}$ and the two groups are isomorphic, i. e., $\mathcal{G}_{\mathbf{D}^{-2}} \cong \mathbb{SO}(n+1)$. Also, for any $\Omega \in \mathcal{L}_{\mathbf{D}^{-2}}$, there exists exactly one $A \in \mathfrak{so}(n+1)$ such that $\Omega = \mathbf{D}A\mathbf{D}^{-1}$. At the same time, we established that $\mathsf{Isom}(\mathrm{M}) = \mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1} \cong \mathbb{SE}(m)$. In the reminder of this paper elements of $\mathsf{Isom}(\mathrm{M})$ will be denoted as pairs (g, s), with $g \in \mathcal{G}_{\mathbf{D}^{-2}}$ and $s \in \mathbb{R}^{n+1}$. The group operations in $\mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1}$ are defined as: $(g, s)^{-1} = \big(g^{-1}, -g^{-1}s\big)$ and $(g_1, s_1) \cdot (g_2, s_2) = (g_1g_2, g_1s_2 + s_1)$.

4.3. The ellipsoid as the unit sphere in M

The ellipsoid $\mathcal{E}^n$ is the unit sphere in M. It can be defined by

$$\mathcal{E}^n := \{ x \in \mathrm{M} \ : \ |x|_{\mathbf{D}^{-2}} = 1 \}. \tag{10}$$

For $\varepsilon > 0$, let $\gamma\colon (-\varepsilon, \varepsilon) \to \mathcal{E}^n$ be any differentiable curve with $\gamma(0) = p$ and $\dot\gamma(0) = V$. Differentiating the condition $|\gamma|_{\mathbf{D}^{-2}} = 1$ with respect to t yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\, |\gamma|^2_{\mathbf{D}^{-2}} = 2\big\langle \dot\gamma, \mathbf{D}^{-2}\gamma \big\rangle = 2\big\langle \dot\gamma, \gamma \big\rangle_{\mathbf{D}^{-2}}.$$

At $t = 0$, the above equality yields $\big\langle V, p\big\rangle_{\mathbf{D}^{-2}} = 0$. Henceforth the tangent space $\mathrm{T}_p\mathcal{E}^n$ is the subspace orthogonal to p in M with respect to its metric $\big\langle \cdot, \cdot \big\rangle_{\mathbf{D}^{-2}}$. The unit normal vector $\Lambda \in (\mathrm{T}_p\mathcal{E}^n)^{\perp}$ is given by $\Lambda = p/|p|_{\mathbf{D}^{-2}} = p$. Hence, the Weingarten map Ξ_Λ at $p \in \mathcal{E}^n$ is minus the identity, i. e., $\Xi_\Lambda = -\mathsf{id}$. The *scalar second fundamental form* h can be easily derived from the Weingarten equation $\big\langle \Xi_\Lambda(X), Y\big\rangle_{\mathbf{D}^{-2}} = -\big\langle X, Y\big\rangle_{\mathbf{D}^{-2}} = -h(X, Y)$. Hence the *second fundamental form* $\mathrm{II}(X, Y) = \big\langle X, Y\big\rangle_{\mathbf{D}^{-2}}p$.

The tangent space may be defined in terms of $\mathbf{D}$ as:

$$\mathrm{T}_p\mathcal{E}^n := \{ \mathbf{D}A\mathbf{D}^{-1}p \ : \ A \in \mathfrak{so}(n+1) \}. \tag{11}$$

4.4. Geodesics on the ellipsoid

Given a point $p_0 \in \mathcal{E}^n$ and a vector $V_0 \in T_{p_0}\mathcal{E}^n$, there exists unique geodesic $t \mapsto \gamma(t)$ satisfying $\gamma(0) = p_0$, $\dot\gamma(0) = V_0$. This geodesic is defined by

$$\gamma(t) = p_0\, \cos(t\,|V_0|) + V_0 \frac{\sin(t\,|V_0|)}{|V_0|}. \tag{12}$$

The algorithm to be presented in the last section depends on the implementation of geodesic arcs that join two points on the ellipsoid. So, at this stage we also present an explicit formula to compute the geodesic arc $t \mapsto \gamma(t)$ on $\big(\mathcal{E}^n, \big\langle \cdot, \cdot \big\rangle_{\mathbf{D}^{-2}}\big)$, joining the points p_i (at $t = t_i$) and p_{i+1} (at $t = t_{i+1}$) (with $p_i \neq \pm p_{i+1}$):

$$\gamma(t) = \frac{1}{\sin\theta_i} \left\{ \sin\Big(\frac{\theta_i}{t_{i+1} - t_i}\,(t_{i+1} - t)\Big)p_i + \sin\Big(\frac{\theta_i}{t_{i+1} - t_i}\,(t - t_i)\Big)p_{i+1} \right\}, \tag{13}$$

where $\theta_i = \arccos \big\langle p_i, p_{i+1} \big\rangle_{\mathbf{D}^{-2}}$.

This can be easily checked by computing $\ddot\gamma(t)$, to conclude that $\ddot\gamma(t) = -\theta_i^2\,\gamma(t)$, hence $\ddot\gamma(t)$ belongs to $\big(\mathrm{T}_{\gamma(t)}\mathcal{E}^n\big)^{\perp}$ in M.

5. ROLLING THE ELLIPSOID

We aim to write kinematic equations for the ellipsoid $\mathcal{E}^n$ rolling upon its affine tangent space, when both are embedded in $\mathrm{M} = \big(\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}\big)$. We derive the equations in a few steps, starting with the distribution of the rolling map.

5.1. The rolling map

For $\mathrm{M}_1 = \mathcal{E}^n$ choose an initial point of contact p_0 that, without loss of generality, to be the "south pole" of the ellipsoid. Then $p_0 := -\mathbf{D}\mathbf{e}_{n+1} = -d_{n+1}\mathbf{e}_{n+1} \in \mathbf{S}^n_{\mathbf{D}}$. The affine tangent space at p_0 is defined by

$$\mathrm{M}_0 = \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n := \big\{\, x \in \mathrm{M} \;:\; x = p_0 + (p_0)^{\perp} \,\big\},$$

where $(p_0)^{\perp}$ denotes the vector space normal to p_0 with respect to the metric $\mathbf{D}^{-2}$. The configuration space $\Sigma \subset \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n \times \mathcal{G}_{\mathbf{D}^{-2}} \times \mathcal{E}^n$ of the rolling map $\boldsymbol{\chi}$ is the space of all possible positions of the unit sphere $\mathcal{E}^n$ tangent to its affine tangent space. Namely

$$\Sigma = \big\{\, \big(p, g, q\big) \in \mathrm{M}_0 \times \mathcal{G}_{\mathbf{D}^{-2}} \times \mathcal{E}^n \;:\; g(\mathrm{T}_q\mathcal{E}^n) = \mathrm{T}_p\mathrm{M}_0 \,\big\}.$$

Lemma 3.3 yields an analog of (7) as follows

$$\dot g g^{-1} = \big(\dot\sigma_0 \wedge p_0\big)\mathbf{D}^{-2}.$$

Denote $\boldsymbol{A} = \dot R\, R^{-1}$. Then $\dot g g^{-1} = \mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}$ and the above equality becomes

$$\boldsymbol{A}(t) = u(t)\, p_0^{\mathrm{T}}\, \mathbf{D}^{-1} - \mathbf{D}^{-1}\, p_0\, u^{\mathrm{T}}(t), \tag{14}$$

where $u = (u_1, \ldots, u_n, 0)^{\mathrm{T}} = -\mathbf{D}^{-1}\dot\sigma_0$.

Let $\sigma_1(t) = g^{-1}(t)(p_0)$, where $g\colon I \to \mathcal{G}_{\mathbf{D}^{-2}}$ satisfies $g(0) = id$. Since the metric $\langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}$ is left-invariant with respect to $\mathcal{G}_{\mathbf{D}^{-2}}$ and $\mathcal{G}_{\mathbf{D}^{-2}}$ acts transitively on $\mathcal{E}^n$, any curve can be parameterised in this way. The following proposition establishes kinematic equations for a rolling map of the ellipsoid.

Proposition 5.1. Let $R\colon I \to \mathbb{SO}(n+1)$ and $s\colon I \to \mathbb{R}^{n+1}$ be solutions to the following set of equations

$$\begin{cases} \dot s(t) = -\mathbf{D}\boldsymbol{A}(t)\mathbf{D}^{-1}p_0 \\ \dot R(t) = \boldsymbol{A}(t)R(t) \end{cases}, \tag{15}$$

with $R(0) = \mathsf{id}$ and $s(0) = 0$. Then, $\boldsymbol{\chi}\colon I \to \mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1}$ given by

$$\boldsymbol{\chi}(t) = \big(g(t), s(t)\big) = \big(\mathbf{D}R(t)\mathbf{D}^{-1}, s(t)\big) \tag{16}$$

is a rolling map of the ellipsoid rolling on its affine tangent space in M, with rolling curve $\sigma_1(t) = \mathbf{D}R^{-1}\mathbf{D}^{-1}p_0$ and its development $\sigma_0(t) = s(t) + p_0$.

P r o o f . This is just a matter of checking that all the conditions of Definition 3.1 hold. (One can find an alternative proof of Proposition 5.1 in [9].)

(Rolling) It is easy to verify that since $\dot{s}(t)$ is normal to p_0 in M then equality (a) holds:

$$\boldsymbol{\chi}(t)(\sigma_1(t)) = \big(\mathbf{D}R(t)\mathbf{D}^{-1}\big)\big(\mathbf{D}R(t)^{\mathrm{T}}\mathbf{D}^{-1}\big)p_0 + s(t) = p_0 + s(t) \in \mathrm{M}_0.$$

To verify (b) it is enough to see that since the metric on M is left invariant with respect to $\mathcal{G}_{\mathbf{D}^{-2}}$, this group sends the unit sphere to itself. Also, since the normal spaces of $\mathcal{E}^n$ and $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$ coincide at the point of contact, so do the tangent spaces.

(No-slip) From the above calculations of the development curve it follows that

$$\dot{\sigma}_0 = \dot{s} = -gg^{-1}\;\dot{g}g^{-1}p_0 = -\dot{g}g^{-1}p_0 = -\mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}p_0.$$

(No-twist) It is enough to verify the tangential part because the normal one follows immediately from (15). For any vector $V \in \mathrm{T}_{\sigma_0(t)}\mathrm{M}_0$.

$$\begin{aligned}\big(\dot{g}g^{-1}\big)(V) &= \mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}V = \mathbf{D}\big(up_0^{\mathrm{T}}\mathbf{D}^{-1} - \mathbf{D}^{-1}p_0u^{\mathrm{T}}\big)\mathbf{D}^{-1}V \\ &= \mathbf{D}u\big\langle p_0, V\big\rangle_{\mathbf{D}^{-2}} - p_0\big\langle u, \mathbf{D}^{-1}V\big\rangle = -\big\langle u, \mathbf{D}^{-1}V\big\rangle p_0 \in \mathrm{T}^{\perp}_{p_0}\mathcal{E}^n.\end{aligned}$$

The proof is now complete. □

In general, the kinematic equations (15) may be hard to solve. However, when $\boldsymbol{A}(t) = \boldsymbol{A}$ is constant, explicit solutions can be found. This corresponds to rolling motions along geodesics.

Corollary 5.2. For the special situation when $\boldsymbol{A}(t) = \boldsymbol{A}$ is constant, the solution of the kinematic equations (15) is given by

$$R(t) = \exp(t\boldsymbol{A}) \quad \text{and} \quad s(t) = -t\,\mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}p_0,$$

the rolling curve and its development, given respectively by

$$\begin{aligned}\sigma_1(t) &= g^{-1}(t)p_0 = \mathbf{D}\exp(-t\boldsymbol{A})\,\mathbf{D}^{-1}p_0, \\ \sigma_0(t) &= p_0 + s(t) = p_0 - t\,\mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}p_0,\end{aligned} \tag{17}$$

are geodesics on the ellipsoid $\big(\mathcal{E}^n, \langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}\big)$ and on its affine tangent space $\big(\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n, \langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}\big)$.

P r o o f . The only statement that requires a computation is that σ_1 is a geodesic on the ellipsoid for the metric induced by $\mathbf{D}^{-2}$. This is easily checked by computing its second derivative and comparing with (11) as follows:

$$\begin{aligned}\dot{\sigma}_1(t) &= -\mathbf{D}\boldsymbol{A}\exp(-t\boldsymbol{A})\,\mathbf{D}^{-1}p_0 = -\mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}\sigma_1(t), \\ \ddot{\sigma}_1(t) &= \mathbf{D}\boldsymbol{A}^2\mathbf{D}^{-1}\sigma_1(t) = -|u|^2\,\sigma_1(t) \in (\mathrm{T}_{\sigma_1(t)}\mathcal{E}^n)^{\perp}.\end{aligned}$$

The last equality can be verified by noting that $\big\langle u, \mathbf{D}^{-1}p_0\big\rangle = 0$ and, because $\boldsymbol{A}^2 \times \exp(-t\boldsymbol{A})\,\mathbf{D}^{-1}p_0 = \exp(-t\boldsymbol{A})\,\boldsymbol{A}^2\mathbf{D}^{-1}p_0$, then it follows from (14) that

$$\boldsymbol{A}^2\mathbf{D}^{-1}p_0 = \boldsymbol{A}\left(\boldsymbol{A}\mathbf{D}^{-1}p_0\right) = \boldsymbol{A}u = -|u|^2\,\mathbf{D}^{-1}p_0.$$

What was to show. □

Remark 5.3. The power series

$$\begin{aligned}\exp(-t\boldsymbol{A})\,\mathbf{D}^{-1}p_0 &= \sum_{i=0}^{\infty}\frac{(-t)^i}{i!}\boldsymbol{A}^i\,\mathbf{D}^{-1}p_0\\ &= \sum_{k=0}^{\infty}(-1)^k\,\frac{t^{2k}}{(2k)!}\,|u|^{2k}\;\mathbf{D}^{-1}p_0 - \sum_{k=0}^{\infty}(-1)^k\,\frac{t^{2k+1}}{(2k+1)!}\,|u|^{2k}\,u\end{aligned}$$

yields the expression for the geometric *exponential map*. Hence,

$$\sigma_1(t) = p_0\,\cos(t\,|u|) - \mathbf{D}u\frac{\sin(t\,|u|)}{|u|}, \tag{18}$$

and $-\mathbf{D}\,u \in \mathrm{T}_{p_0}\mathcal{E}^n$ is the initial velocity vector of the geodesic σ_1. This agrees with the formula (12) and gives a geometric interpretation of the control vector u in (14).

From the point of view of Control Theory, the ellipsoid rolling on its affine tangent space is controllable. This is a direct consequence of the positivity of the Gaussian curvature of the ellipsoid. In turn, one can steer the ellipsoid from an admissible configuration (any configuration in which the ellipsoid is tangent to the affine tangent space at a point) to any other admissible configuration, only by rolling without twist and without slip. Interested reader is referred to [15] for more details.

6. ALGORITHM TO GENERATE AN INTERPOLATING CURVE ON THE ELLIPSOID $\mathcal{E}^N$

This algorithm is based on a procedure to generate interpolating curves on some manifolds embedded in Euclidean space, first described in [12] for the 2-sphere, generalised in [10] for the n-sphere and in [11] for the rotation group and Grassmann manifolds. Here we show how this algorithm can be extended to the ellipsoid $\mathcal{E}^n$ to generate an interpolating curve, given in closed form, that solves the Problem 2.1 stated in Subsection 2.1. We also implement the algorithm for the 2-dimensional ellipsoid.

The basic idea behind the algorithm is to project the data from $\mathcal{E}^n$ to $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$, solve an interpolation problem in this affine space and, finally, projecting back to $\mathcal{E}^n$ the interpolating curve on the affine space. The projection uses a mixed technique of rolling/unrolling and unwrapping/wrapping, performed by an appropriate rolling map and a convenient diffeomorphism. These two maps must satisfy some conditions, as follows.

1. The rolling map (to perform the rolling/unrolling):

   Choose a rolling map $\boldsymbol{\chi} = (\mathbf{D}R\mathbf{D}^{-1}, s)\colon [0,\tau] \to \mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1}$ of $\mathcal{E}^n$ on $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$, along a smooth curve α_1 that joins p_0 (at $t=0$) to p_k (at $t=\tau$), with development α_0.

2. The local diffeomorphism (to perform the unwrapping/wrapping):

Choose a suitable local diffeomorphism, on an open neighbourhood U of p_0,

$$\Phi\colon U \subset \mathcal{E}^n \to \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n,$$

so that

$$\Phi(p_0) = p_0 \quad \text{and} \quad \partial\Phi^{-1}(p_0) = \mathsf{id}_{n+1}, \tag{19}$$

where $\partial\Phi$ denotes the Jacobian matrix of Φ.

6.1. The Algorithm

The algorithm consists essentially of five steps.

Step 1. Compute the rolling curve

$$\alpha_1\colon [0,\tau] \to \mathcal{E}^n,$$

connecting p_0 with p_k, i. e., such that

$$\alpha_1(0) = p_0 \quad \text{and} \quad \alpha_1(\tau) = p_k.$$

Step 2. Unwrap the boundary data by rolling $\mathcal{E}^n$ along α_1, so that:

$$\begin{aligned} p_0 &\mapsto \boldsymbol{\chi}(0)p_0 := q_0 = p_0 \in \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n,\\ p_k &\mapsto \boldsymbol{\chi}(\tau)p_k := q_k \in \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n, \end{aligned}$$

as well as

$$\begin{aligned} V_0 &\mapsto \boldsymbol{\chi}_*(0)V_0 := W_0 = V_0 \in \mathrm{T}_{q_0}(\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n),\\ V_k &\mapsto \boldsymbol{\chi}_*(\tau)V_k := W_k \in \mathrm{T}_{q_k}(\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n). \end{aligned}$$

Step 3. Unwrap the remaining interpolating points p_i at t_i, $i = 1,\dots,k-1$, from $\mathcal{E}^n$ to $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$, using the diffeomorphism Φ and the time dependent rolling map $\boldsymbol{\chi}$, so that

$$p_i \mapsto \Phi\big(\boldsymbol{\chi}(t_i)p_i - \alpha_0(t_i) + p_0\big) + \alpha_0(t_i) - p_0 =: q_i. \tag{20}$$

Step 4. Solve the interpolating problem on $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$ for the projected data $\{q_0,\dots,q_k; W_0, W_k\}$, to generate a C^2-smooth curve

$$\beta\colon [0,\tau] \to \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$$

satisfying

$$\begin{aligned} &\beta(0) = p_0 = q_0, \qquad \beta(t_i) = q_i, \qquad \beta(\tau) = q_k,\\ &\dot\beta(0) = V_0 = W_0, \qquad \dot\beta(\tau) = W_k. \end{aligned}$$

Step 5. Wrap $\beta([0,\tau])$ back onto the ellipsoid using Φ^{-1}, while unrolling along α_1, to produce a curve γ, defined by the following explicit formula.

$$\gamma(t) := \boldsymbol{\chi}^{-1}(t)\big(\Phi^{-1}\big(\beta(t) - \alpha_0(t) + p_0\big) + \alpha_0(t) - p_0\big). \tag{21}$$

Theorem 6.1. The curve $\gamma\colon [0,\tau] \mapsto \mathcal{E}^n$ defined by (21) solves Problem 2.1.

Proof. Recall that $s(t) = \alpha_0(t) - p_0$, $\boldsymbol{\chi}(t) = \big(\mathbf{D}R(t)\mathbf{D}^{-1}, s(t)\big) = \big(g(t), s(t)\big)$ and $\boldsymbol{\chi}^{-1}(t) = \big(g^{-1}(t), -g^{-1}s(t)\big)$. A simple calculation shows that

$$\gamma(t) = g^{-1}(t)\big(\Phi^{-1}\big(\beta(t) - s(t)\big)\big),$$

then

$$\begin{aligned}\dot\gamma(t) &= \dot g^{-1}(t)\big(\Phi^{-1}\big(\beta(t) - s(t)\big)\big) + g^{-1}(t)\big(\partial\Phi^{-1}\big(\beta(t) - s(t)\big)\big(\dot\beta(t) - \dot s(t)\big)\big)\\ &= -g^{-1}(t)\circ \dot g(t)\,\big(\gamma(t)\big) + g^{-1}(t)\big(\partial\Phi^{-1}\big(\beta(t) - s(t)\big)\big(\dot\beta(t) - \dot s(t)\big)\big).\end{aligned}$$

To compute the boundary conditions, note that $g(0) = \mathsf{id}$, $s(0) = 0$ and $\beta(0) = p_0$. So,

$$\gamma(0) = \Phi^{-1}\big(\beta(0) - s(0)\big) = \Phi^{-1}(p_0) = p_0.$$

Also, $\beta(\tau) = \alpha_0(\tau)$, which implies

$$\gamma(\tau) = g^{-1}(\tau)\big(\Phi^{-1}(p_0)\big) = g^{-1}(\tau)(p_0) = \alpha_1(\tau) = p_k.$$

Since $\dot\beta(\tau) = \dot\alpha_0(\tau) = \boldsymbol{\chi}_*(\tau)\dot\alpha_1(\tau)$, we have $\dot\beta(\tau) = g(\tau)V_k$. As a consequence,

$$\begin{aligned}\dot\gamma(0) &= -g^{-1}(0)\circ \dot g(0)\big(\gamma(0)\big) + g^{-1}(0)\big(\partial\Phi^{-1}(p_0)\big)\big(\dot\beta(0) - \dot s(0)\big)\\ &= \dot s(0) + \big(\dot\beta(0) - \dot s(0)\big) = V_0,\end{aligned}$$

$$\begin{aligned}\dot\gamma(\tau) &= -g^{-1}(\tau)\circ \dot g(\tau)\big(\gamma(\tau)\big) + g^{-1}(\tau)\big(\partial\Phi^{-1}(\beta(\tau) - s(\tau))(\dot\beta(\tau) - \dot s(\tau)\big)\\ &= -g^{-1}(\tau)\circ \dot g(\tau)\big(p_k\big) + g^{-1}(\tau)\big(\dot\beta(\tau) - \dot s(\tau)\big) = g^{-1}(\tau)\big(\dot\beta(\tau)\big) = V_k.\end{aligned}$$

Finally, looking at the expression of $\gamma(t_i)$ and using the expression

$$\beta(t_i) = \Phi\Big(\boldsymbol{\chi}(t_i)p_i - \alpha_0(t_i) + p_0\Big) + \alpha_0(t_i) - p_0,$$

that comes from (20), since $\beta(t_i) = q_i$, we obtain after simplifications $\gamma(t_i) = p_i$. The resulting curve is C^2-smooth by construction, since Φ and $\boldsymbol{\chi}$ are smooth and β is C^2-smooth. This concludes the proof. □

Remark 6.2. At this point it is important to point out that Step 4. can be easily implemented, although performed on a non-Euclidean submanifold. This is due to the fact that geodesics, and other polynomial curves are the same on $\big(\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n, \langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}\big)$ and $\big(\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n, \langle\cdot,\cdot\rangle\big)$. Indeed, the Euler Lagrange equation is the same for the two problems

$$\min_x \int_0^\tau \big\langle x^{(k)}(t), x^{(k)}(t)\big\rangle\,\mathrm{d}t$$

and

$$\min_x \int_0^\tau \big\langle x^{(k)}(t), x^{(k)}(t)\big\rangle_{\mathbf{D}^{-2}}\,\mathrm{d}t$$

and is given by $x^{(2k)} = 0$. In particular, for $k = 2$, cubic polynomials (cubic splines) in $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$ may be generated by the classical De Casteljau algorithm.

6.2. Rolling along piecewise smooth curves

In case we are only interested in an interpolating curve that is C^1-smooth, the condition that the rolling curve is smooth maybe replaced by piecewise smooth, as the following proposition shows. It remains an open question what are the necessary conditions on Φ to ensure that γ is C^2-smooth.

Proposition 6.3. If the rolling curve passes through the data points, i. e., $\alpha_1(t_i) = p_i$, and is smooth on each subinterval (t_i, t_{i+1}), $i = 0, \dots, k-1$, then the curve $\gamma\colon [0, \tau] \to \mathcal{E}^n$ is always C^1-smooth.

Proof. From *Step 4* of the algorithm it follows that

$$\beta(t_i) = \Phi\big(\boldsymbol{\chi}(t_i)(p_i) - s(t_i)\big) + s(t_i)$$

and consequently

$$\Phi^{-1}\left(\beta(t_i) - s(t_i)\right) = \Phi^{-1}\big(\Phi\big(\boldsymbol{\chi}(t_i)(p_i) - s(t_i)\big)\big) = \boldsymbol{\chi}(t_i)(p_i) - s(t_i).$$

Now, since $\alpha_1(t_i) = p_i$

$$\begin{aligned}\boldsymbol{\chi}(t_i)(p_i) - \alpha_0(t_i) + p_0 &= \boldsymbol{\chi}(t_i)(\alpha_1(t_i)) - \alpha_0(t_i) + p_0 \\ &= \boldsymbol{\chi}(t_i) \circ \boldsymbol{\chi}^{-1}(t_i)(\alpha_0(t_i)) - \alpha_0(t_i) + p_0 = p_0.\end{aligned}$$

The left derivative of γ at t_i, $i = 0, \dots, k-1$, becomes now

$$\begin{aligned}\dot\gamma(t_i^-) &= \dot{\boldsymbol{\chi}}^{-1}(t_i^-) \circ \boldsymbol{\chi}(t_i)(p_i) + \boldsymbol{\chi}_*^{-1}(t_i)\left(\partial\Phi^{-1}(p_0)(\dot\beta(t_i) - \dot s(t_i^-)) + \dot s(t_i^-)\right) \\ &= \dot{\boldsymbol{\chi}}^{-1}(t_i^-) \circ \boldsymbol{\chi}(t_i)(p_i) + \boldsymbol{\chi}_*^{-1}(t_i)\big(\dot\beta(t_i)\big) = -\boldsymbol{\chi}_*^{-1}(t_i) \circ \dot{\boldsymbol{\chi}}(t_i^-)(p_i) + \boldsymbol{\chi}_*^{-1}(t_i)\big(\dot\beta(t_i)\big).\end{aligned}$$

By the *no-slip* condition

$$\boldsymbol{\chi}_*^{-1}(t_i) \circ \dot{\boldsymbol{\chi}}(t_i^-)(p_i) = \boldsymbol{\chi}_*^{-1}(t_i) \circ \dot{\boldsymbol{\chi}}(t_i^-)(\alpha_1(t_i)) = \boldsymbol{\chi}_*^{-1}(t_i)\,(\mathbf{0}) = \mathbf{0}$$

and therefore

$$\dot\gamma(t_i^-) = \boldsymbol{\chi}_*^{-1}(t_i)\big(\dot\beta(t_i)\big).$$

Right derivative of γ at t_i, $i = 1, \dots, k$, is derived in the same way. □

In the situation referred in the last statement, one particular choice for the rolling curve is the broken geodesic going through the given data, in which case the corresponding kinematic equations are easily solved.

Given a series $p_0, p_1, \dots, p_k$ of $k+1$ points in $\mathcal{E}^n$, choose each segment of the rolling curve $\sigma_1|_{[t_i, t_{i+1}]}$ from p_i to p_{i+1} to be a geodesic (13). To derive the rolling map $\boldsymbol{\chi}$, write each segment in terms of the exponent of some $(n+1)\times(n+1)$-matrix $\boldsymbol{A}_i$

$$\sigma_1|_{[t_i, t_{i+1}]}(t) = D\exp\big(-(t - t_i)\,\boldsymbol{A}_i\big)D^{-1}p_i. \tag{22}$$

It can be easily shown that the matrix $\boldsymbol{A}_i$ that satisfies (22) is of the following form

$$\boldsymbol{A}_i = \frac{\theta_i}{(t_{i+1} - t_i)\sin\theta_i} D^{-1}\left(p_i p_{i+1}^{\mathrm{T}} - p_{i+1} p_i^{\mathrm{T}}\right) D^{-1}.$$

The piecewise smooth rolling map $\boldsymbol{\chi}$ assumes the following form

$$\boldsymbol{\chi}(t) := \boldsymbol{\chi}_0(t_1) \circ \boldsymbol{\chi}_1(t_2) \circ \cdots \circ \boldsymbol{\chi}_{i-1}(t_i) \circ \boldsymbol{\chi}_i(t), \quad \text{for} \quad t \in [t_i, t_{t+1}],$$

where each segment $\boldsymbol{\chi}_i \colon [t_i, t_{i+1}] \to \mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1}$ is given by $\boldsymbol{\chi}_i(t) = \big(DR_i(t)D^{-1}, s_i(t)\big)$, with

$$R_i(t) = \exp\big((t - t_i)\,\boldsymbol{A}_i\big) \quad \text{and} \quad s_i(t) = -\,(t - t_i)\,D\boldsymbol{A}_i D^{-1} p_i.$$

For any $t_{i-1} < t < t_i$, the development curve σ_0 is given by

$$\sigma_0(t) = \boldsymbol{\chi}_0(t_1) \circ \boldsymbol{\chi}_1(t_2) \circ \cdots \circ \boldsymbol{\chi}_{i-1}(t)\big(\sigma_1|_{[t_{i-1}, t_i]}(t)\big),$$

where each segment of the rolling curve is mapped to

$$\begin{aligned}\boldsymbol{\chi}_i(t)\big(\sigma_1|_{[t_i, t_{i+1}]}(t)\big) = D\exp\big((t - t_i)\,\boldsymbol{A}_i\big)D^{-1}D\exp\big(-\,(t - t_i)\,\boldsymbol{A}_i\big)D^{-1}p_i \\ -\,(t - t_i)\,D\boldsymbol{A}_i D^{-1} p_i = p_i - (t - t_i)\,D\boldsymbol{A}_i D^{-1} p_i\end{aligned}$$

and each point p_i is mapped to

$$q_i = \sigma_0(t_i) = \boldsymbol{\chi}(t_i)\big(\sigma_1(t_i)\big) = \boldsymbol{\chi}_0(t_1) \circ \boldsymbol{\chi}_1(t_2) \circ \cdots \circ \boldsymbol{\chi}_{i-1}(t_i)\big(p_i\big).$$

7. IMPLEMENTATION OF THE ALGORITHM ON $\mathcal{E}^2$

In order to implement the algorithm on $\mathcal{E}^2$, we have to choose the rolling map so that the corresponding kinematic equations can be solved explicitly. For that reason, we choose $\boldsymbol{\chi} \colon [0, \tau] \to \mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1}$ to be the the rolling map of $\mathcal{E}^2$ on $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^2$, along the geodesic α_1 that joins p_0 (at $t = 0$) to p_k (at $t = \tau$), with development α_0. Our choice for the local diffeomorphism Φ is the stereographic projection from the "north pole". Before we proceed with the implementation of the algorithm, we give details about this projection.

7.1. Stereographic projection of $\mathcal{E}^2$

The stereographic projection from the "north pole" of the ellipsoid to the tangent space at the "south pole" $p_0 = \big(0, 0, -d_3\big)^\top \in \mathcal{E}^2$ is given by:

$$\begin{aligned}\Phi \colon \mathcal{E}^2 \setminus \{\big(0,0,d_3\big)^\top\} &\to \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^2 \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\mapsto \begin{pmatrix} \dfrac{2d_3 x_1}{d_3 - x_3} \\ \dfrac{2d_3 x_2}{d_3 - x_3} \\ -d_3 \end{pmatrix},\end{aligned}$$

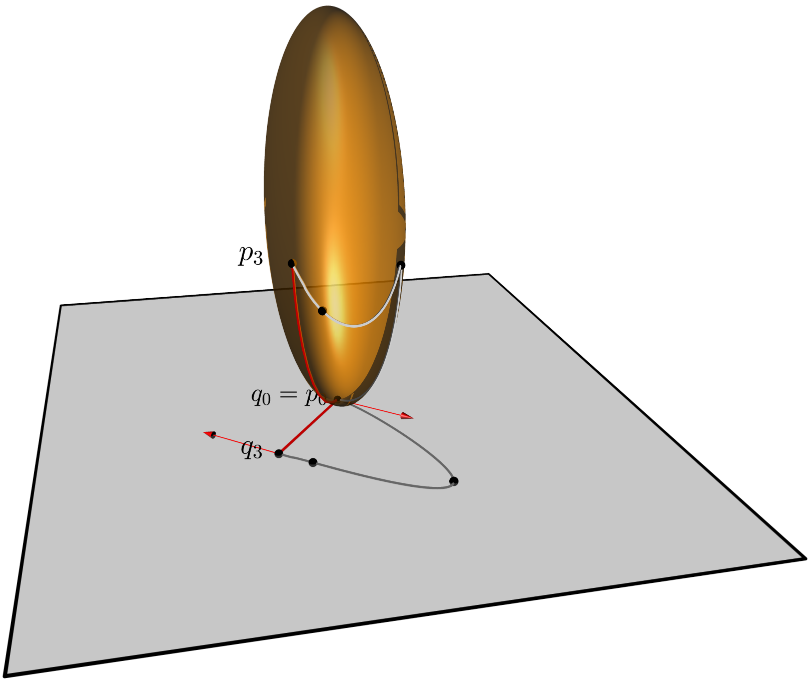

Fig. 2. Smooth interpolation on the ellipsoid.

with the inverse

$$\Phi^{-1}\colon \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^2 \to \qquad \mathcal{E}^2 \setminus \{(0,0,d_3)^\top\}$$

$$\begin{pmatrix}\xi_1\\ \xi_2\\ -d_3\end{pmatrix} \mapsto \begin{pmatrix} \dfrac{4d_1^2d_2^2\xi_1}{d_2^2\xi_1^2 + d_1^2\xi_2^2 + 4d_1^2d_2^2} \\[2ex] \dfrac{4d_1^2d_2^2\xi_2}{d_2^2\xi_1^2 + d_1^2\xi_2^2 + 4d_1^2d_2^2} \\[2ex] \dfrac{(d_2^2\xi_1^2 + d_1^2\xi_2^2 - 4d_1^2d_2^2)d_3}{d_2^2\xi_1^2 + d_1^2\xi_2^2 + 4d_1^2d_2^2} \end{pmatrix}.$$

Remark 7.1. It can easily be shown that Φ satisfies the following:

$$\Phi(p_0) = p_0, \qquad \partial\Phi^{-1}(p_0) = \mathsf{id}_3,$$

where $\partial\Phi^{-1}$ denotes the Jacobian matrix of the differentiable map Φ^{-1}.

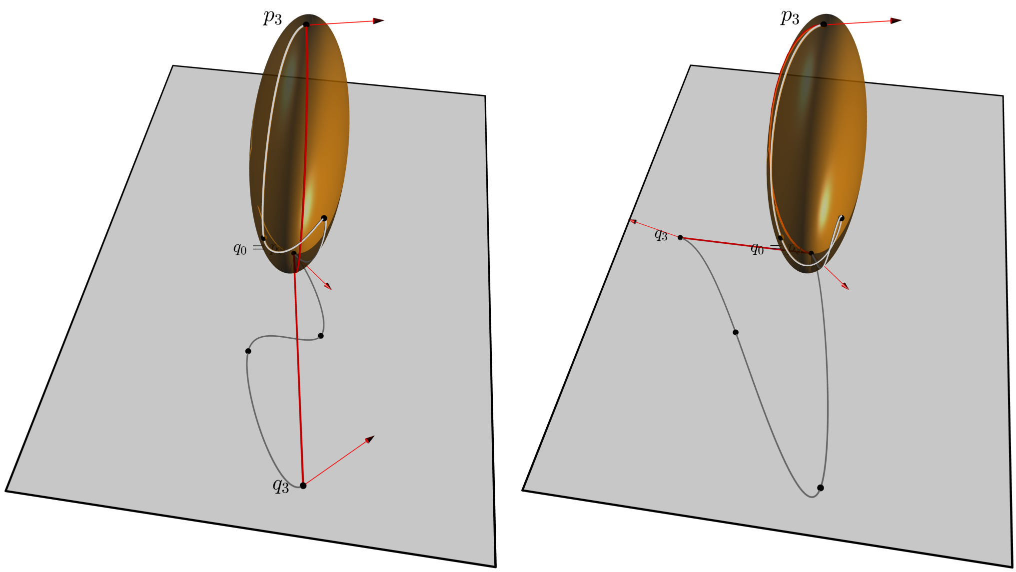

Fig. 3. Comparison of interpolation on the ellipsoid through two geodesic segments between conjugate (antipodal) points.

We now have all the necessary ingredients to generate interpolating curves on $\mathcal{E}^2$. Figure 2 shows the main steps of the algorithm and the resulting interpolating curve.

Although Theorem 6.1 guarantees that a solution to the interpolation problem exists, it says nothing about its uniqueness. It is clear from (21) that the interpolating curve γ depends on the choice of a rolling curve α_1 and a diffeomorphism Φ. But even when the later is fixed and the rolling curve is chosen to be a geodesic arc joining the initial and the final points, there might be many solution curves for the interpolating problem. This occurs, and was already expected, when those points are antipodal since there are infinitely many geodesics joining them. Figure 3 illustrates what happens when two different geodesic segments joining antipodal points are used as rolling curves. It is worth noting different directions of the transformed ending vectors W_k in each case. This is a result of the curvature of the ellipsoid.

8. CONCLUSIONS

We have presented an algorithm to generate a $\mathcal{C}^2$-smooth interpolating curve on the n-dimensional ellipsoid, based on a rolling and wrapping technique which produces a solution given in closed form. In order to accomplish this, the ellipsoid was embedded in $\mathbb{R}^{n+1}$ equipped with an appropriate non-Euclidean metric, and the kinematic equations for rolling the ellipsoid over the affine tangent space at a point were derived. As far as we know, this approach to rolling motions of the ellipsoid is new, having appeared only in our reference [14], a shorter version of the present paper.

This algorithm extends to any smooth manifold as long as the action of the first component of the rolling map keeps the manifold invariant, and the kinematic equations for rolling are known and can be solved explicitly. This algorithm can also be easily extended to generate $\mathcal{C}^k$-smooth interpolating curves, for $k > 2$, as long as the boundary conditions are adjusted to incorporate appropriate constraints on higher derivatives.

In which concerns the interpolation algorithm, the present work generalises ideas and results contained in [10, 12] and [11], where only certain manifolds embedded in Euclidean spaces have been considered. The novelty of our results is the approach to solve interpolation problems on the ellipsoid through rolling motions in a non-Euclidean space. The link between smooth curves in different spaces leads to a simple solution.

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