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REMARK ON INEQUALITIES FOR THE LAPLACIAN SPREAD OF GRAPHS

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Abstract. Two inequalities for the Laplacian spread of graphs are proved in this note. These inequalities are reverse to those obtained by Z. You, B. Liu: The Laplacian spread of graphs, Czech. Math. J. 62 (2012), 155–168.

Keywords: Laplacian eigenvalue; spread of a graph

MSC 2010: 15A18, 05C50

1. Introduction

Let $G=(V,E)$ be an undirected connected graph with m edges and n, $n\geqslant 3$ vertices, $V=\{x_1,x_2,\ldots,x_n\}$. Denote by $d_i=d(x_i)$, $i=1,2,\ldots,n$ the degree of each vertex, and by $M_1=\sum\limits_{i=1}^{n} d_i^2$ the first Zafreb index (see [1]). The Laplacian spectrum of G are the eigenvalues $\mu_1\geqslant\mu_2\geqslant\ldots\geqslant\mu_{n-1}>\mu_n=0$, whereas $P(\mu)=\mu(\mu^{n-1}+c_1\mu^{n-2}+\ldots+c_{n-1})$ is the Laplacian characteristic polynomial. The Laplacian spread of a graph is defined as

$$\mathrm{LS}(G)=\mu_1-\mu_{n-1}.$$

In [4] Z. You and B. Liu proved several inequalities for $\mathrm{LS}(G)$. Here, we are interested in the following two of them:

$$\mathrm{LS}(G)\geqslant\frac{2}{n-1}\sqrt{(n-1)(M_1+2m)-4m^2} \tag{1.1}$$

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and

$$\mathrm{LS}(G) \geqslant \mu_1 - \sqrt{\frac{M_1 + 2m - \mu_1^2}{n-2}}. \tag{1.2}$$

In this paper we are going to prove two inequalities which are reverse to (1.1) and (1.2).

2. Main result

Theorem 2.1. *For* LS(G) *of a connected undirected graph G with n, $n \geqslant 3$, vertices and m edges, the inequality*

$$\mathrm{LS}(G) \leqslant \sqrt{\frac{2}{n-1}}\sqrt{(n-1)(M_1+2m) - 4m^2} \tag{2.1}$$

is valid. Equality in (2.1) *holds if and only if $G \cong K_n$.*

Proof. Since the Laplacian eigenvalues of G, $\mu_1, \mu_2, \ldots, \mu_{n-1}$ are positive and form a decreasing sequence, according to the identity

$$T = (n-1)(M_1 + 2m) - 4m^2 = (n-1)\sum_{i=1}^{n-1}\mu_i^2 - \left(\sum_{i=1}^{n-1}\mu_i\right)^2 = \sum_{1 \leqslant i < j \leqslant n-1} (\mu_i - \mu_j)^2$$

we have that

$$T \geqslant \sum_{k=2}^{n-2}((\mu_1 - \mu_k)^2 + (\mu_k - \mu_{n-1})^2) + (\mu_1 - \mu_{n-1})^2. \tag{2.2}$$

Now, if in (2.2) we apply Jensen's discrete inequality for convex functions (see for example [2], [3]), we obtain

$$T \geqslant \frac{n-3}{2}(\mu_1 - \mu_{n-1})^2 + (\mu_1 - \mu_{n-1})^2 = \frac{n-1}{2}(\mu_1 - \mu_{n-1})^2.$$

Since $T \geqslant 0$ and $\mu_1 - \mu_{n-1} \geqslant 0$, the above inequality directly yields the inequality (2.1).

Equality in (2.2) and in Jensen's inequality holds if an only if $\mu_1 = \mu_2 = \ldots = \mu_{n-1}$. Accordingly, we conclude that equality in (2.1) holds if and only if $G \cong K_n$. □

Corollary 2.1. *If the observed graph $G = (V, E)$ is k-regular, then the inequality*

$$\mathrm{LS}(G) \leqslant \sqrt{\frac{2nk(n-k-1)}{n-1}}$$

holds.

Remark 2.1. Since

$$(n-1)\sum_{i=1}^{n-1}\mu_i^2 - \left(\sum_{i=1}^{n-1}\mu_i\right)^2 = (n-2)C_1^2 - 2(n-1)C_2,$$

where C_1 and C_2 are the coefficients of the Laplacian characteristic polynomial, the inequalities (1.1) and (2.1) can be represented as

$$\frac{2}{n-1}\sqrt{(n-2)C_1^2 - 2(n-1)C_2} \leqslant \mathrm{LS}(G) \leqslant \sqrt{\frac{2}{n-1}}\sqrt{(n-2)C_1^2 - 2(n-1)C_2}.$$

Theorem 2.2. *For* $\mathrm{LS}(G)$ *of a connected undirected graph* G *with* n *vertices and* m *edges, the inequality*

$$\mathrm{LS}(G) \leqslant \sqrt{M_1 + 2m - (n-2)\mu_{n-1}^2} - \mu_{n-1} \tag{2.3}$$

is valid. Equality in (2.3) *holds if and only if* $G \cong K_n$, *or* $G \cong K_{1,n}$, *or* $G \cong K_{n/2,n/2}$.

Proof. Inequality (2.3) can be easily obtained by using the inequality

$$M_1 + 2m = \mu_1^2 + \ldots + \mu_{n-1}^2 \geqslant \mu_1^2 + (n-2)\mu_{n-1}^2.$$

□

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