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THE PRIMITIVE BOOLEAN MATRICES WITH THE SECOND LARGEST SCRAMBLING INDEX BY BOOLEAN RANK

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Abstract. The scrambling index of an $n \times n$ primitive Boolean matrix A is the smallest positive integer k such that $A^k(A^{\mathrm{T}})^k = J$, where A^{T} denotes the transpose of A and J denotes the $n \times n$ all ones matrix. For an $m \times n$ Boolean matrix M, its Boolean rank $b(M)$ is the smallest positive integer b such that $M = AB$ for some $m \times b$ Boolean matrix A and $b \times n$ Boolean matrix B. In 2009, M. Akelbek, S. Fital, and J. Shen gave an upper bound on the scrambling index of an $n \times n$ primitive matrix M in terms of its Boolean rank $b(M)$, and they also characterized all primitive matrices that achieve the upper bound. In this paper, we characterize primitive Boolean matrices that achieve the second largest scrambling index in terms of their Boolean rank.

Keywords: scrambling index; primitive matrix; Boolean rank

MSC 2010: 05C20, 05C50, 05C75

1. Introduction

A matrix over the binary Boolean algebra $\{0, 1\}$ is called a *Boolean matrix*. In this work, we sometimes use just the term matrix to mean Boolean matrix. For an $m \times n$ matrix A, we will denote its (i, j)-entry by A_{ij}, its ith row by $A_{i\cdot}$, and its jth column by $A_{\cdot j}$. For $m \times n$ matrices A and B, we say that B is dominated by A if $B_{ij} \leqslant A_{ij}$ for each i and j, and denote this by $B \leqslant A$. We denote the $m \times n$ all ones matrix by $J_{m,n}$ (and by J_n if $m = n$), the all ones n-vector by j_n, the $n \times n$ identity matrix by I_n and its ith column by $e_i(n)$. The subscripts m and n will be omitted whenever their values are clear from the context. Let A be an $m \times n$ matrix. For index sets $\alpha \subseteq \{1, 2, \ldots, m\}$, $\beta \subseteq \{1, 2, \ldots, n\}$, we denote the submatrix that lies in the rows of A indexed by α and the columns indexed by β as $A(\alpha, \beta)$.

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Let $D = (V, E)$ be a *digraph* on n vertices. Loops are permitted, but multiple arcs are not. A $u \to v$ *walk* in D is a sequence of vertices $u, u_1, \ldots, u_p = v$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_{p-1}, v)$, where the vertices and the arcs are not necessarily distinct. The *length* of a walk W is the number of arcs in W. The length of a shortest cycle in D is called the *girth* of D. The notation $u \xrightarrow{k} v$ is used to indicate that there is a $u \to v$ walk of length k.

For an $n \times n$ matrix $A = (a_{ij})$, the digraph $D(A)$ is the digraph with vertex set $V(D(A)) = \{1, 2, \ldots, n\}$, and (i, j) is an arc of $D(A)$ if and only if $a_{ij} \neq 0$. Then, for a positive integer $r \geqslant 1$, the (i, j)th entry of the matrix A^r is positive if and only if $i \xrightarrow{r} j$ in the digraph $D(A)$.

A digraph D is called *primitive* if for some positive integer k, there is a walk of length exactly k from each vertex u to each vertex v (possibly u again). Equivalently, a square matrix A of order n is called primitive if there exists a positive integer r such that $A^r > 0$.

The *scrambling index* of a primitive digraph D, denoted by $k(D)$, is the smallest positive integer k such that for every pair of vertices u and v, there exists some vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D. An analogous definition for scrambling index can be given for primitive matrices. The scrambling index of a primitive matrix A, denoted by $k(A)$, is the smallest positive integer k such that any two rows of A^k have at least one positive element in a coincident position. The scrambling index of a primitive matrix A can also be equivalently defined as the smallest positive integer k such that $A^k (A^{\mathrm{T}})^k = J$, where A^{T} denotes the transpose of A. If A is the adjacency matrix of a primitive digraph D, then $k(D) = k(A)$.

For an $m \times n$ Boolean matrix M, its *Boolean rank* $b(M)$ is defined to be the smallest positive integer b such that for some $m \times b$ Boolean matrix A and $b \times n$ Boolean matrix B, $M = AB$. The Boolean rank of the zero matrix is defined to be zero. $M = AB$ is called a *Boolean rank factorization* of M.

For additional terminology and notation we follow [3].

Let D_1 and D_2 be primitive digraphs of order n in Figure 1.1 and Figure 1.2, respectively.

Let

$$W_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad W_n = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ldots & \vdots \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 1 & 0 & \ldots & \ldots & 0 & 1 \\ 1 & 0 & \ldots & \ldots & 0 & 0 \end{bmatrix},$$

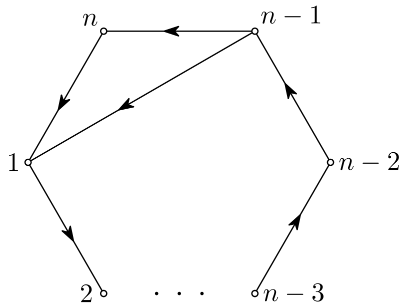

Figure 1.1 The digraph D_1 (Wielandt digraph, $n \geqslant 3$).

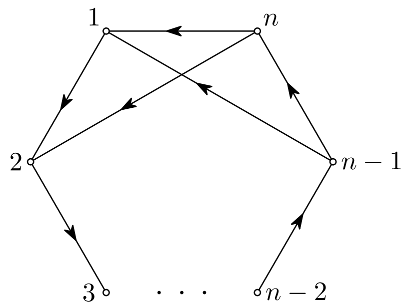

Figure 1.2 The digraph D_2 ($n \geqslant 4$).

where $n \geqslant 3$, and

$$H_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \end{bmatrix},$$

where $n \geqslant 4$. Then $D(W_n) = D_1$ for $n \geqslant 3$, and $D(H_n) = D_2$ for $n \geqslant 4$.

In [1], M. Akelbek, and S. Kirkland obtained an upper bound on the scrambling index of an $n \times n$ primitive matrix M in terms of its order n, and they also characterized all primitive matrices that achieve the upper bound.

Lemma 1.1 ([1]). *Let A be a primitive matrix of order $n \geqslant 2$. Then*

$$k(A) \leqslant \left\lceil \frac{(n-1)^2+1}{2} \right\rceil.$$

Equality holds if and only if there is a permutation matrix P such that PAP^{T} is equal to W_2 or J_2 when $n = 2$ and W_n when $n \geqslant 3$.

In [2], M. Akelbek, S. Fital, and J. Shen gave an upper bound on the scrambling index of an $n \times n$ primitive matrix M in terms of its Boolean rank $b(M)$, and they characterized all primitive matrices that achieve the upper bound, too.

Lemma 1.2 ([2]). *Let M be an $n \times n$ ($n \geqslant 2$) primitive matrix with Boolean rank $b(M) = b$. Then*

$$k(M) \leqslant \left\lceil \frac{(b-1)^2+1}{2} \right\rceil + 1.$$

Lemma 1.3 ([2]). *Suppose M is an $n \times n$ Boolean matrix with $3 \leqslant b = b(M) \leqslant n-1$. Then M is primitive and $k(M) = \lceil((b-1)^2+1)/2\rceil + 1$ if and only if M has a Boolean rank factorization $M = AB$, where A and B have the following properties:*
(i) $BA = W_b$,
(ii) *some row of A is $e^{\mathrm{T}}_{\lfloor b/2\rfloor}(b)$, some row of A is $e^{\mathrm{T}}_b(b)$, and*
(iii) *no column of B is $e_{b-1}(b) + e_b(b)$.*

In this paper, we characterize primitive Boolean matrices M with $5 \leqslant b = b(M) \leqslant n-1$ that achieve the second largest scrambling index in terms of its Boolean rank $b = b(M)$. The main result is the following theorem.

Theorem 1.4. *Suppose M is an $n \times n$ primitive Boolean matrix with $5 \leqslant b = b(M) \leqslant n-1$. Then $k(M) = h = \lceil\frac{1}{2}((b-1)^2+1)\rceil$ if and only if M has a Boolean rank factorization $M = AB$, where A and B satisfy one of the following conditions:*
(i) *$BA = W_b$, some row of A is $e^{\mathrm{T}}_{\lfloor b/2\rfloor}(b)$ and some row of A is $e^{\mathrm{T}}_b(b)$, some column of B is $e_{b-1}(b) + e_b(b)$.*
(ii) *$BA = W_b$, some row of A is $e^{\mathrm{T}}_1(b)$ and some row of A is $e^{\mathrm{T}}_{\lfloor b/2\rfloor+1}(b)$, either $e^{\mathrm{T}}_{\lfloor b/2\rfloor}(b)$ or $e^{\mathrm{T}}_b(b)$ is not a row of A, no column of B is $e_{b-1}(b) + e_b(b)$.*
(iii) *$BA = H_b$, some row of A is $e^{\mathrm{T}}_1(b)$ and some row of A is $e^{\mathrm{T}}_{\lfloor b/2\rfloor+1}(b)$, no column of B is $e_{b-1}(b) + e_b(b)$.*

2. Proof of the main result

Let $X \subseteq V(D)$. Denote by $R_t(X)$ the set of all vertices which can be reached by a walk of length t in digraph D starting from some vertex in X, and abbreviate $R_t(\{x\})$ as $R_t(x)$.

Lemma 2.1 ([5]). *Let A be a primitive matrix of order $n \geqslant 5$. Then*

$$k(A) = \left\lceil \frac{(n-1)^2+1}{2} \right\rceil - 1$$

if and only if there is a permutation matrix P such that PAP^{T} is equal to H_n.

Lemma 2.2 ([4]). *Let M be an $n \times n$ primitive Boolean matrix, and $M = AB$ be a Boolean rank factorization of M. Then neither A nor B has a zero line.*

Lemma 2.3 ([2]). *Suppose that A and B are $n\times m$ and $m\times n$ Boolean matrices respectively, and that neither A nor B has a zero line. Then*

(a) *AB is primitive if and only if BA is primitive.*

(b) *If AB and BA are primitive, then $|k(AB)-k(BA)|\leqslant 1$.*

For brevity, in the remainder of this paper, we let $h=\lceil\frac{1}{2}((b-1)^2+1)\rceil$.

Lemma 2.4. *Let $b\geqslant 4$ be odd. Then*

(1) $W_b^{h-2}(\{1,\frac{1}{2}(b-1),\frac{1}{2}(b+1),b\},\{b-1,b\})=\begin{bmatrix}1&0\\1&1\\0&1\\0&0\end{bmatrix}$.

(2) *The zero entries of $(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2}$ occur only in the $(b,\frac{1}{2}(b-1))$, $(\frac{1}{2}(b-1),b)$, $(1,\frac{1}{2}(b+1))$, and $(\frac{1}{2}(b+1),1)$ positions.*

(3) $H_b^{h-2}(\{1,\frac{1}{2}(b+1)\},\{b-1,b\})=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $H_b^{h-2}(\{1,\frac{1}{2}(b+1)\},\{1,b\})=\begin{bmatrix}0&0\\1&1\end{bmatrix}$.

(4) *The zero entries of $(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2}$ occur only in the $(1,\frac{1}{2}(b+1))$ and $(\frac{1}{2}(b+1),1)$ positions.*

Proof. Let $l=h-2=\lceil\frac{1}{2}((b-1)^2+1)\rceil-2=\frac{1}{2}((b-1)^2)-1$.

For the primitive digraph D_1 of order b, it is not difficult to verify that

▷ $R_l(b)=\{b-2,b-3,\ldots,\frac{1}{2}(b-1)\}$,

▷ $R_l(i)=\{i-1,i-2,\ldots,i-\frac{1}{2}(b-1)-1\}$ for $\frac{1}{2}(b+3)\leqslant i\leqslant b-1$,

▷ $R_l(\frac{1}{2}(b+1))=\{\frac{1}{2}(b-1),\frac{1}{2}(b-3),\ldots,1,b\}$,

▷ $R_l(\frac{1}{2}(b-1))=\{\frac{1}{2}(b-3),\frac{1}{2}(b-5),\ldots,1,b,b-1\}$,

▷ $R_l(i)=\{i-1,i-2,\ldots,1,b,b-1,\ldots,\frac{1}{2}(b-1)+i\}$ for $2\leqslant i\leqslant\frac{1}{2}(b-1)$, and

▷ $R_l(1)=\{b-1,b-2,\ldots,\frac{1}{2}(b+1)\}$.

(1) Note that $b-1\in R_l(1)$, $b\notin R_l(1)$. So $W_b^l(\{1\},\{b-1,b\})=[1\ 0]$. Similarly, we have $W_b^l(\{\frac{1}{2}(b-1)\},\{b-1,b\})=[1\ 1]$, $W_b^l(\{\frac{1}{2}(b+1)\},\{b-1,b\})=[0\ 1]$, and $W_b^l(\{b\},\{b-1,b\})=[0\ 0]$. Therefore, result (1) holds.

(2) Note that we have $R_l(i)\cap R_l(j)\neq\emptyset$ except $R_l(b)\cap R_l(\frac{1}{2}(b-1))=\emptyset$ and $R_l(1)\cap R_l(\frac{1}{2}(b+1))=\emptyset$. So in W_b^l every pair of rows intersect with each other except rows b and $\frac{1}{2}(b-1)$, 1 and $\frac{1}{2}(b+1)$. Thus the only zero entries of $(W_b)^l(W_b^{\mathrm{T}})^l$ are in the $(b,\frac{1}{2}(b-1))$, $(\frac{1}{2}(b-1),b)$, $(1,\frac{1}{2}(b+1))$, and $(\frac{1}{2}(b+1),1)$ positions.

For the primitive digraph D_2 of order b, it is not difficult to verify that

▷ $R_l(1)=\{b-1,b-2,\ldots,b-\frac{1}{2}(b-1)\}$,

▷ $R_l(i)=\{i-1,i-2,\ldots,i-\frac{1}{2}(b-1)-1\}$ for $\frac{1}{2}(b+3)\leqslant i\leqslant b$,

▷ $R_l(\frac{1}{2}(b+1))=\{\frac{1}{2}(b-1),\frac{1}{2}(b-3),\ldots,1,b\}$, and

▷ $R_l(i)=\{i-1,i-2,\ldots,1,b,b-1,\ldots,\frac{1}{2}(b-1)+i\}$ for $2\leqslant i\leqslant\frac{1}{2}(b-1)$.

(3) Note that $b-1\in R_l(1)$, $b\notin R_l(1)$. So $H_b^l(\{1\},\{b-1,b\})=[1\ 0]$. Similarly, we have $H_b^l(\{\frac{1}{2}(b+1)\},\{b-1,b\})=[0\ 1]$, $H_b^l(\{1\},\{1,b\})=[0\ 0]$, and $H_b^l(\{\frac{1}{2}(b+1)\},\{1,b\})=[1\ 1]$. Therefore, result (3) holds.

(4) Note that we have $R_l(i)\cap R_l(j)\neq\emptyset$ except $R_l(1)\cap R_l(\frac12(b+1))=\emptyset$. So in H_b^l every pair of rows intersect with each other except rows 1 and $\frac12(b+1)$. Thus the only zero entries of $(H_b)^l(H_b^{\mathrm T})^l$ are in the $(1,\frac12(b+1))$ and $(\frac12(b+1),1)$ positions. $\square$

Lemma 2.5. *Let $b\geqslant 4$ be even. Then*

(1) $W_b^{h-2}(\{1,\frac12 b,\frac12 b+1,b\},\{b-1,b\})=\begin{bmatrix}0&1\\0&0\\1&0\\1&1\end{bmatrix}$.

(2) *the zero entries of $(W_b)^{h-2}(W_b^{\mathrm T})^{h-2}$ occur only in the $(b,\frac12 b)$, $(\frac12 b,b)$, $(1,\frac12 b+1)$, and $(\frac12 b+1,1)$ positions.*

(3) $H_b^{h-2}(\{1,\frac12 b+1\},\{b-1,b\})=\begin{bmatrix}0&1\\1&0\end{bmatrix}$, $H_b^{h-2}(\{1,\frac12 b+1\},\{1,b\})=\begin{bmatrix}1&1\\0&0\end{bmatrix}$.

(4) *the zero entries of $(H_b)^{h-2}(H_b^{\mathrm T})^{h-2}$ occur only in the $(1,\frac12 b+1)$ and $(\frac12 b+1,1)$ positions.*

Proof. Let $l=h-2=\lceil\frac12((b-1)^2+1)\rceil-2=\frac12(b-2)b-1$.

For the primitive digraph D_1 of order b, it is not difficult to verify that

▷ $R_l(b)=\{b-1,b,1,2,\dots,\frac12 b-2\}$,

▷ $R_l(i)=\{i-1,i,\dots,b-1,b,1,2,\dots,i-\frac12 b-1\}$ for $\frac12 b+2\leqslant i\leqslant b-1$,

▷ $R_l(\frac12 b+1)=\{\frac12 b,\frac12 b+1,\dots,b-1\}$,

▷ $R_l(i)=\{i-1,i,\dots,i+\frac12 b-2\}$ for $2\leqslant i\leqslant\frac12 b$, and

▷ $R_l(1)=\{b,1,2,\dots,\frac12 b-1\}$.

(1) Note that $b-1\notin R_l(1)$, $b\in R_l(1)$. So $W_b^l(\{1\},\{b-1,b\})=[0\ 1]$. Similarly, we have $W_b^l(\{\frac12 b\},\{b-1,b\})=[0\ 0]$, $W_b^l(\{\frac12 b+1\},\{b-1,b\})=[1\ 0]$, and $W_b^l(\{b\},\{b-1,b\})=[1\ 1]$. Therefore, result (1) holds.

(2) Note that in W_b^l every pair of rows intersect with each other except rows b and $\frac12 b$, 1 and $\frac12 b+1$. Thus the only zero entries of $(W_b)^l(W_b^{\mathrm T})^l$ are in the $(b,\frac12 b)$, $(\frac12 b,b)$, $(1,\frac12 b+1)$, and $(\frac12 b+1,1)$ positions.

For the primitive digraph D_2 of order b, it is not difficult to verify that

▷ $R_l(1)=\{b,1,2,\dots,\frac12 b-1\}$,

▷ $R_l(i)=\{i-1,i,\dots,i+\frac12 b-2\}$ for $2\leqslant i\leqslant\frac12 b$,

▷ $R_l(\frac12 b+1)=\{\frac12 b,\frac12 b+1,\dots,b-1\}$,

▷ $R_l(i)=\{i-1,i,\dots,b-1,b,1,2,\dots,i-\frac12 b-1\}$ for $\frac12 b+2\leqslant i\leqslant b-1$, and

▷ $R_l(b)=\{b-1,b,1,2,\dots,\frac12 b-1\}$.

(3) Note that $b-1\notin R_l(1)$, $b\in R_l(1)$. So $H_b^l(\{1\},\{b-1,b\})=[0\ 1]$. Similarly, we have $H_b^l(\{\frac12 b+1\},\{b-1,b\})=[1\ 0]$, $H_b^l(\{1\},\{1,b\})=[1\ 1]$, and $H_b^l(\{\frac12 b+1\},\{1,b\})=[0\ 0]$. Therefore, result (3) holds.

(4) Note that in H_b^l every pair of rows intersect with each other except rows 1 and $\frac12 b+1$. Thus the only zero entries of $(H_b)^l(H_b^{\mathrm T})^l$ are in the $(1,\frac12 b+1)$, and $(\frac12 b+1,1)$ positions. $\square$

Lemma 2.6 ([2]). *For $b \geqslant 3$, $W_b^{h-1}(\{\lfloor\frac{1}{2}b\rfloor, b\}, \{b-1, b\})$ is either* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *or* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Lemma 2.7 ([2]). *If $b \geqslant 3$, then the zero entries of $(W_b)^{h-1}(W_b^{\mathrm{T}})^{h-1}$ occur only in the $(b, \lfloor\frac{1}{2}b\rfloor)$ and $(\lfloor\frac{1}{2}b\rfloor, b)$ positions.*

Suppose that M is an $n \times n$ Boolean matrix with $1 \leqslant b = b(M) \leqslant n$. If $b = n \geqslant 3$, then by Lemma 1.1, $k(M) = \lceil\frac{1}{2}((n-1)^2 + 1)\rceil$ if and only if there is an $n \times n$ permutation matrix P such that $PMP^{\mathrm{T}} = W_n$. If $b = 1$, since the only $n \times n$ primitive Boolean matrix with Boolean rank 1 is J_n, then $k(M) = \lceil\frac{1}{2}((b-1)^2+1)\rceil = 1$ if and only if $A = J_n$. Thus we may assume that $2 \leqslant b \leqslant n-1$. In this paper, we consider $5 \leqslant b \leqslant n-1$.

Lemma 2.8. *Let M be an $n \times n$ primitive Boolean matrix with $5 \leqslant b = b(M) \leqslant n-1$. Suppose M has a Boolean rank factorization $M = AB$, where A and B have the following properties:*
(1) *$BA = W_b$,*
(2) *some row of A is $e^{\mathrm{T}}_{\lfloor b/2\rfloor}(b)$ and some row of A is $e_b^{\mathrm{T}}(b)$, and*
(3) *some column of B is $e_{b-1}(b) + e_b(b)$.*
Then M is primitive and $k(M) = h$.

Proof. By Lemma 2.2 and Lemma 2.3 (a), neither A nor B has a zero line and the matrix M is primitive since W_b is primitive. By Lemma 1.3, $k(M) \leqslant h$.

Since $BA = W_b$ and A has no zero row, each column of B is dominated by a column of W_b. Thus each column of B is in the set $S_1 = \{e_1(b), e_2(b), \ldots, e_b(b), u\}$, where $u = e_{b-1}(b) + e_b(b)$. Therefore, $BB^{\mathrm{T}} \leqslant I_b + uu^{\mathrm{T}}$. Also, since some column of B is $e_{b-1}(b) + e_b(b)$, $BB^{\mathrm{T}} \geqslant I_b + uu^{\mathrm{T}}$. Hence $BB^{\mathrm{T}} = I_b + uu^{\mathrm{T}}$. Thus

$$\begin{aligned} M^{h-1}(M^{\mathrm{T}})^{h-1} &= (AB)^{h-1}((AB)^{\mathrm{T}})^{h-1} \\ &= A(BA)^{h-2}BB^{\mathrm{T}}((BA)^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= A(W_b)^{h-2}BB^{\mathrm{T}}(W_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= A(W_b)^{h-2}(I_b + uu^{\mathrm{T}})(W_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= A[(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}]A^{\mathrm{T}} \\ &= AZA^{\mathrm{T}}. \end{aligned}$$

If b is odd, by Lemma 2.4, the zero entries of $W_b^{h-2}(W_b^{\mathrm{T}})^{h-2}$ occur only in the $(b, \frac{1}{2}(b-1))$, $(\frac{1}{2}(b-1), b)$, $(1, \frac{1}{2}(b+1))$, and $(\frac{1}{2}(b+1), 1)$ positions. Note that

$$W_b^{h-2}\Big(\Big\{1, \frac{b-1}{2}, \frac{b+1}{2}, b\Big\}, \{b-1, b\}\Big) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

So

$$(W_b^{h-2}u)\Big(\Big\{1,\frac{b-1}{2},\frac{b+1}{2},b\Big\}\Big)=\begin{bmatrix}1\\1\\1\\0\end{bmatrix}.$$

Hence, the entries of $Z=(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2}+(W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}$ in the $(b,\frac{1}{2}(b-1))$ and $(\frac{1}{2}(b-1),b)$ positions are zero. Since some row of A is $e^{\mathrm{T}}_{(b-1)/2}(b)$ and some row of A is $e_b^{\mathrm{T}}(b)$, without loss of generality, suppose row p of A is $e^{\mathrm{T}}_{(b-1)/2}(b)$ and row q of A is $e_b^{\mathrm{T}}(b)$. Then $(M^{h-1}(M^{\mathrm{T}})^{h-1})_{pq}=(AZA^{\mathrm{T}})_{pq}=0$. Hence $k(M)>h-1$ and we get $k(M)=h$.

If b is even, by Lemma 2.5, the zero entries of $W_b^{h-2}(W_b^{\mathrm{T}})^{h-2}$ occur only in the $(b,\frac{1}{2}b)$, $(\frac{1}{2}b,b)$, $(1,\frac{1}{2}b+1)$, and $(\frac{1}{2}b+1,1)$ positions. Note that

$$W_b^{h-2}\Big(\Big\{1,\frac{b}{2},\frac{b}{2}+1,b\Big\},\{b-1,b\}\Big)=\begin{bmatrix}0&1\\0&0\\1&0\\1&1\end{bmatrix}.$$

So

$$W_b^{h-2}u\Big(\Big\{1,\frac{b}{2},\frac{b}{2}+1,b\Big\}\Big)=\begin{bmatrix}1\\0\\1\\1\end{bmatrix}.$$

Therefore, the entries of $Z=(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2}+(W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}$ in the $(\frac{1}{2}b,b)$ and $(b,\frac{1}{2}b)$ positions are zero. Since some row of A is $e^{\mathrm{T}}_{b/2}(b)$ and some row of A is $e_b^{\mathrm{T}}(b)$, without loss of generality, suppose row p of A is $e^{\mathrm{T}}_{b/2}(b)$ and row q of A is $e_b^{\mathrm{T}}(b)$. Then $(M^{h-1}(M^{\mathrm{T}})^{h-1})_{pq}=(AZA^{\mathrm{T}})_{pq}=0$. Hence $k(M)>h-1$ and we get $k(M)=h$. □

Lemma 2.9. *Let M be an $n\times n$ primitive Boolean matrix with $5\leqslant b=b(M)\leqslant n-1$. Suppose M has a Boolean rank factorization $M=AB$, where A and B have the following properties:*
(1) *$BA=W_b$,*
(2) *some row of A is $e_1^{\mathrm{T}}(b)$, some row of A is $e^{\mathrm{T}}_{\lfloor b/2\rfloor+1}(b)$, either $e^{\mathrm{T}}_{\lfloor b/2\rfloor}(b)$ or $e_b^{\mathrm{T}}(b)$ is not a row of A, and*
(3) *no column of B is $e_{b-1}(b)+e_b(b)$.*
Then M is primitive and $k(M)=h$.

Proof. By Lemma 2.2 and Lemma 2.3 (a), neither A nor B has a zero line and the matrix M is primitive since W_b is primitive. By Lemma 1.3, $k(M)\leqslant h$.

Since $BA = W_b$ and A has no zero row, each column of B is dominated by a column of W_b. Note that no column of B is u. Hence each column of B is in the set $\{e_1(b), e_2(b), \ldots, e_b(b)\}$. Therefore, $BB^{\mathrm{T}} \leqslant I_b$. Also, since the matrix B has no zero row, $BB^{\mathrm{T}} \geqslant I_b$. Hence $BB^{\mathrm{T}} = I_b$. Thus

$$\begin{aligned} M^{h-1}(M^{\mathrm{T}})^{h-1} &= (AB)^{h-1}((AB)^{\mathrm{T}})^{h-1} \\ &= A(BA)^{h-2}BB^{\mathrm{T}}((BA)^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= A(W_b)^{h-2}I_b(W_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= A(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= AZA^{\mathrm{T}}, \end{aligned}$$

where, by Lemmas 2.4 and 2.5, $Z = (W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2}$ is the $b \times b$ matrix which has zero entries only in the $(b, \lfloor \frac{1}{2}b \rfloor)$, $(\lfloor \frac{1}{2}b \rfloor, b)$, $(1, \lfloor \frac{1}{2}b \rfloor + 1)$, and $(\lfloor \frac{1}{2}b \rfloor + 1, 1)$ positions. Since some row of A is $e_1^{\mathrm{T}}(b)$ and some row of A is $e_{\lfloor b/2 \rfloor + 1}^{\mathrm{T}}(b)$, without loss of generality, suppose row p of A is $e_1^{\mathrm{T}}(b)$ and row q of A is $e_{\lfloor b/2 \rfloor + 1}^{\mathrm{T}}(b)$. Then $(M^{h-1}(M^{\mathrm{T}})^{h-1})_{pq} = (AZA^{\mathrm{T}})_{pq} = 0$. Hence $k(M) > h - 1$ and we get $k(M) = h$.

□

Lemma 2.10. *Let M be an $n \times n$ primitive Boolean matrix with $5 \leqslant b = b(M) \leqslant n - 1$. Suppose M has a Boolean rank factorization $M = AB$, where A and B have the following properties:*

(1) *$BA = H_b$,*

(2) *some row of A is $e_1^{\mathrm{T}}(b)$ and some row of A is $e_{\lfloor b/2 \rfloor + 1}^{\mathrm{T}}(b)$, and*

(3) *no column of B is $e_{b-1}(b) + e_b(b)$.*

Then M is primitive and $k(M) = h$.

Proof. By Lemma 2.2 and Lemma 2.3 (a), neither A nor B has a zero line and the matrix M is primitive since W_b is primitive. By Lemma 1.3, $k(M) \leqslant h$.

Since $BA = H_b$ and A has no zero row, each column of B is dominated by a column of H_b. Note that no column of B is $e_{b-1}(b) + e_b(b)$. Hence each column of B is in the set $\{e_1(b), e_2(b), \ldots, e_b(b), v\}$, where $v = e_1(b) + e_b(b)$. Therefore, $BB^{\mathrm{T}} \leqslant I_b + vv^{\mathrm{T}}$. Thus

$$\begin{aligned} M^{h-1}(M^{\mathrm{T}})^{h-1} &= (AB)^{h-1}((AB)^{\mathrm{T}})^{h-1} \\ &= A(BA)^{h-2}BB^{\mathrm{T}}((BA)^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &\leqslant A(H_b)^{h-2}(I_b + vv^{\mathrm{T}})(H_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= A[(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2} + (H_b^{h-2}v)(H_b^{h-2}v)^{\mathrm{T}}]A^{\mathrm{T}} \\ &= AZA^{\mathrm{T}}. \end{aligned}$$

If b is odd, by Lemma 2.4, $(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2}$ is the $b\times b$ matrix which has zero entries only in the $(1,\frac12(b+1))$ and $(\frac12(b+1),1)$ positions. Note that

$$H_b^{h-2}\Big(\Big\{1,\frac{b+1}{2}\Big\},\{1,b\}\Big)=\begin{bmatrix}0&0\\1&1\end{bmatrix},\quad (H_b^{h-2}v)\Big(\Big\{1,\frac{b+1}{2}\Big\}\Big)=\begin{bmatrix}0\\1\end{bmatrix}.$$

So the entries of $Z=(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2}+(H_b^{h-2}v)(H_b^{h-2}v)^{\mathrm{T}}$ in the $(1,\frac12(b+1))$ and $(\frac12(b+1),1)$ positions are zero. Since some row of A is $e_1^{\mathrm{T}}(b)$ and some row of A is $e_{(b+1)/2}^{\mathrm{T}}(b)$, without loss of generality, suppose row p of A is $e_1^{\mathrm{T}}(b)$ and row q of A is $e_{(b+1)/2}^{\mathrm{T}}(b)$. Then $(M^{h-1}(M^{\mathrm{T}})^{h-1})_{pq}=(AZA^{\mathrm{T}})_{pq}=0$. Hence $k(M)>h-1$ and we get $k(M)=h$.

If b is even, by Lemma 2.5, $Z=(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2}$ is the $b\times b$ matrix which has zero entries only in the $(1,\frac12 b+1)$ and $(\frac12 b+1,1)$ positions. Note that

$$H_b^{h-2}\Big(\Big\{1,\frac{b}{2}+1\Big\},\{1,b\}\Big)=\begin{bmatrix}1&1\\0&0\end{bmatrix},\qquad (H_b^{h-2}v)\Big(\Big\{1,\frac{b}{2}+1\Big\}\Big)=\begin{bmatrix}1\\0\end{bmatrix}.$$

So the entries of $Z=(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2}+(H_b^{h-2}v)(H_b^{h-2}v)^{\mathrm{T}}$ in the $(\frac12 b+1,1)$ and $(1,\frac12 b+1)$ positions are zero. Since some row of A is $e_1^{\mathrm{T}}(b)$ and some row of A is $e_{b/2+1}^{\mathrm{T}}(b)$, without loss of generality, suppose row p of A is $e_1^{\mathrm{T}}(b)$ and row q of A is $e_{b/2+1}^{\mathrm{T}}(b)$. Then $(M^{h-1}(M^{\mathrm{T}})^{h-1})_{pq}=(AZA^{\mathrm{T}})_{pq}=0$. Hence $k(M)>h-1$ and we get $k(M)=h$. □

Lemma 2.11. *Let M be an $n\times n$ primitive Boolean matrix with $5\leqslant b=b(M)\leqslant n-1$. If $k(M)=h=\lceil\frac12((b-1)^2+1)\rceil$, then M has a Boolean rank factorization $M=AB$, such that A and B satisfy one of the following conditions:*

(i) *$BA=W_b$, some row of A is $e_{\lfloor b/2\rfloor}^{\mathrm{T}}(b)$ and some row of A is $e_b^{\mathrm{T}}(b)$, some column of B is $e_{b-1}(b)+e_b(b)$.*

(ii) *$BA=W_b$, some row of A is $e_1^{\mathrm{T}}(b)$ and some row of A is $e_{\lfloor b/2\rfloor+1}^{\mathrm{T}}(b)$, either $e_{\lfloor b/2\rfloor}^{\mathrm{T}}(b)$ or $e_b^{\mathrm{T}}(b)$ is not a row of A, no column of B is $e_{b-1}(b)+e_b(b)$.*

(iii) *$BA=H_b$, some row of A is $e_1^{\mathrm{T}}(b)$ and some row of A is $e_{\lfloor b/2\rfloor+1}^{\mathrm{T}}(b)$, no column of B is $e_{b-1}(b)+e_b(b)$.*

P r o o f. Let M be primitive with $k(M)=h$, and $M=\widetilde{A}\widetilde{B}$ be a Boolean rank factorization of M. By Lemma 2.3, $\widetilde{B}\widetilde{A}$ is primitive and $h-1\leqslant k(\widetilde{B}\widetilde{A})\leqslant h+1$. Since $\widetilde{B}\widetilde{A}$ is a $b\times b$ matrix, by Lemma 1.1, $k(\widetilde{B}\widetilde{A})\leqslant h$. So there are two cases: $k(\widetilde{B}\widetilde{A})=h$ or $k(\widetilde{B}\widetilde{A})=h-1$.

Case 1. $k(\widetilde{B}\widetilde{A})=h$.

By Lemma 1.1, there is a permutation matrix P such that $P\widetilde{B}\widetilde{A}P^{\mathrm{T}}=W_b$. Let $B=P\widetilde{B}$ and $A=\widetilde{A}P^{\mathrm{T}}$. Then $BA=W_b$ and $AB=\widetilde{A}P^{\mathrm{T}}P\widetilde{B}=\widetilde{A}\widetilde{B}=M$.

Note that M is primitive, we have $\sum_{i=1}^{b} A_{.i} = j_n = \sum_{i=1}^{b} B_{i.}^{\mathrm{T}}$. Since $k(M) = h$, the matrix M^{h-1} must have two rows that do not intersect. Without loss of generality, suppose rows p and q of M^{h-1} do not intersect, that is, the inner product of $M_{p.}^{h-1}$ and $M_{q.}^{h-1}$ is zero. So entries in the (p,q) and (q,p) positions of $M^{h-1}(M^{\mathrm{T}})^{h-1}$ are zero. Since the matrix B has no zero row, we have $BB^{\mathrm{T}} \geqslant I_b$. Thus

$$\begin{aligned}
M^{h-1}(M^{\mathrm{T}})^{h-1} &= (AB)^{h-1}((AB)^{\mathrm{T}})^{h-1} \\
&= A(BA)^{h-2}BB^{\mathrm{T}}((BA)^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\
&= A(W_b)^{h-2}BB^{\mathrm{T}}(W_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\
&\geqslant A(W_b)^{h-2}I_b(W_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\
&= A(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\
&= AZA^{\mathrm{T}},
\end{aligned}$$

where $Z = (W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2}$ is the $b \times b$ matrix which has zero entries only in the $(\lfloor \frac{1}{2}b \rfloor, b)$, $(b, \lfloor \frac{1}{2}b \rfloor)$ $(1, \lfloor \frac{1}{2}b \rfloor + 1)$, and $(\lfloor \frac{1}{2}b \rfloor + 1, 1)$ positions. So

$$\begin{aligned}
AZA^{\mathrm{T}} &= (A_{.1}, A_{.2}, \ldots, A_{.b})ZA^{\mathrm{T}} \\
&= \left[\sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor + 1}}^{b} A_{.i} \middle| J_{n,\lfloor b/2 \rfloor - 2} \middle| \sum_{i=1}^{b-1} A_{.i} \middle| \sum_{i=2}^{b} A_{.i} \middle| J_{n,n-\lfloor b/2 \rfloor - 2} \middle| \sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor}}^{b} A_{.i} \right] A^{\mathrm{T}} \\
&= \left(\sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor + 1}}^{b} A_{.i} \right) (A_{.1})^{\mathrm{T}} + j_n \left(\sum_{i=2}^{\lfloor b/2 \rfloor - 1} A_{.i} \right)^{\mathrm{T}} + \left(\sum_{i=1}^{b-1} A_{.i} \right) (A_{.\lfloor b/2 \rfloor})^{\mathrm{T}} \\
&\quad + \left(\sum_{i=2}^{b} A_{.i} \right) (A_{.\lfloor b/2 \rfloor + 1})^{\mathrm{T}} + j_n \left(\sum_{i=\lfloor b/2 \rfloor + 2}^{b-1} A_{.i} \right)^{\mathrm{T}} + \left(\sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor}}^{b} A_{.i} \right) (A_{.b})^{\mathrm{T}}.
\end{aligned}$$

Since AZA^{T} is dominated by $M^{h-1}(M^{\mathrm{T}})^{h-1}$ and M^{h-1} $(M^{\mathrm{T}})^{h-1}$ has zero entries in the (p,q) and (q,p) positions, the entries $(AZA^{\mathrm{T}})_{pq}$ and $(AZA^{\mathrm{T}})_{qp}$ are also zero. Thus

$$\begin{aligned}
&\left(\sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor + 1}}^{b} A_{pi} \right) A_{q1} + \sum_{i=2}^{\lfloor b/2 \rfloor - 1} A_{qi} + \left(\sum_{i=1}^{b-1} A_{pi} \right) A_{q(\lfloor b/2 \rfloor)} + \left(\sum_{i=2}^{b} A_{pi} \right) A_{q(\lfloor b/2 \rfloor + 1)} \\
&\qquad + \sum_{i=\lfloor b/2 \rfloor + 2}^{b-1} A_{qi} + \left(\sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor}}^{b} A_{pi} \right) A_{qb} = 0,
\end{aligned}$$

and

$$\Bigg(\sum_{\substack{i=1\\ i\neq \lfloor b/2\rfloor+1}}^{b} A_{qi}\Bigg)A_{p1}+\sum_{i=2}^{\lfloor b/2\rfloor-1}A_{pi}+\Bigg(\sum_{i=1}^{b-1}A_{qi}\Bigg)A_{p(\lfloor b/2\rfloor)}+\Bigg(\sum_{i=2}^{b}A_{qi}\Bigg)A_{p(\lfloor b/2\rfloor+1)} + \sum_{i=\lfloor b/2\rfloor+2}^{b-1}A_{pi}+\Bigg(\sum_{\substack{i=1\\ i\neq \lfloor b/2\rfloor}}^{b} A_{qi}\Bigg)A_{pb}=0.$$

Then $A_{qi}=0$ and $A_{pi}=0$ for $i=2,\dots,\lfloor \frac{1}{2}b\rfloor-1,\lfloor \frac{1}{2}b\rfloor+2,\dots,b-1$. Substituting these back, we have

$$(A_{p1}+A_{p(\lfloor b/2\rfloor)}+A_{pb})A_{q1}+(A_{p1}+A_{p(\lfloor b/2\rfloor)}+A_{p(\lfloor b/2\rfloor+1)})A_{q(\lfloor b/2\rfloor)} + (A_{p(\lfloor b/2\rfloor)}+A_{p(\lfloor b/2\rfloor+1)}+A_{pb})A_{q(\lfloor b/2\rfloor+1)}+(A_{p1}+A_{p(\lfloor b/2\rfloor+1)}+A_{pb})A_{qb}=0.$$

If $A_{q(\lfloor b/2\rfloor)}\neq 0$, then $A_{p1}=A_{p(\lfloor b/2\rfloor)}=A_{p(\lfloor b/2\rfloor+1)}=0$. Since A has no zero rows, $A_{pb}\neq 0$ and then $A_{q1}=A_{q(\lfloor b/2\rfloor+1)}=A_{qb}=0$. Therefore, some row of A is $e_b^{\mathrm{T}}(b)$ and some row of A is $e_{\lfloor b/2\rfloor}^{\mathrm{T}}(b)$. In this case, by Lemma 1.3, we know that if no column of B is $e_{b-1}(b)+e_b(b)$, then $k(M)=h+1$. So some column of B is $e_{b-1}(b)+e_b(b)$. This concludes (i).

If $A_{q1}\neq 0$, then $A_{p1}=A_{p(\lfloor b/2\rfloor)}=A_{pb}=0$. Since A has no zero rows, $A_{p(\lfloor b/2\rfloor+1)}\neq 0$ and then $A_{q(\lfloor b/2\rfloor)}=A_{q(\lfloor b/2\rfloor+1)}=A_{qb}=0$. Therefore, some row of A is $e_1^{\mathrm{T}}(b)$ and some row of A is $e_{\lfloor b/2\rfloor+1}^{\mathrm{T}}(b)$. If both $e_{\lfloor b/2\rfloor}^{\mathrm{T}}(b)$ and $e_b^{\mathrm{T}}(b)$ are rows of A, then we go to (i). If either $e_{\lfloor b/2\rfloor}^{\mathrm{T}}(b)$ or $e_b^{\mathrm{T}}(b)$ is not a row of A, we claim that B cannot have a column $u=e_{b-1}(b)+e_b(b)$. To the contrary, suppose that some column of B is u. Since B has no zero row, $BB^{\mathrm{T}}\geqslant I_b+uu^{\mathrm{T}}$. Thus

$$\begin{aligned}M^{h-1}(M^{\mathrm{T}})^{h-1}&=(AB)^{h-1}((AB)^{\mathrm{T}})^{h-1}\\ &=A(BA)^{h-2}BB^{\mathrm{T}}((BA)^{\mathrm{T}})^{h-2}A^{\mathrm{T}}\\ &=A(W_b)^{h-2}BB^{\mathrm{T}}(W_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}}\\ &\geqslant A(W_b)^{h-2}(I_b+uu^{\mathrm{T}})(W_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}}\\ &=A[(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2}+(W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}]A^{\mathrm{T}}.\end{aligned}$$

If b is odd, by Lemma 2.4, the zero entries of $W_b^{h-2}(W_b^{\mathrm{T}})^{h-2}$ occur only in the $(b,\frac{1}{2}(b-1))$, $(\frac{1}{2}(b-1),b)$, $(1,\frac{1}{2}(b+1))$, and $(\frac{1}{2}(b+1),1)$ positions. Note that

$$W_b^{h-2}\Big(\Big\{1,\frac{b-1}{2},\frac{b+1}{2},b\Big\},\{b-1,b\}\Big)=\begin{bmatrix}1&0\\1&1\\0&1\\0&0\end{bmatrix}.$$

So

$$(W_b^{h-2}u)\Big(\Big\{1, \frac{b-1}{2}, \frac{b+1}{2}, b\Big\}\Big) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, the zero entries of $(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}$ occur only in the $(b, \frac{1}{2}(b-1))$ and $(\frac{1}{2}(b-1), b)$ positions. If neither $e_b^{\mathrm{T}}(b)$ nor $e_{(b-1)/2}^{\mathrm{T}}(b)$ is a row of A, then $A[(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}] = J_{n\times b}$ and $A[(W_b)^{h-2} \times (W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}]A^{\mathrm{T}} = J_n$. If $e_b^{\mathrm{T}}(b)$ is a row of A, without loss of generality, suppose row p of A is $e_b^{\mathrm{T}}(b)$, then the zero entry of $A[(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}]$ occurs only in the $(p, \frac{1}{2}(b-1))$ position. Since $e_{(b-1)/2}^{\mathrm{T}}(b)$ is not a row of A, then $A[(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}]A^{\mathrm{T}} = J_n$. Similarly, if $e_{(b-1)/2}^{\mathrm{T}}(b)$ is a row of A, and $e_b^{\mathrm{T}}(b)$ is not a row of A, we can show that $A[(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}]A^{\mathrm{T}} = J_n$. Therefore,

$$M^{h-1}(M^{\mathrm{T}})^{h-1} \geqslant A[(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}]A^{\mathrm{T}} = J_n,$$

which contradicts $k(M) = h$.

If b is even, by Lemma 2.5, the zero entries of $W_b^{h-2}(W_b^{\mathrm{T}})^{h-2}$ occur only in the $(b, \frac{1}{2}b)$, $(\frac{1}{2}b, b)$, $(1, \frac{1}{2}b+1)$, and $(\frac{1}{2}b+1, 1)$ positions. Note that

$$W_b^{h-2}\Big(\Big\{1, \frac{b}{2}, \frac{b}{2}+1, b\Big\}, \{b-1, b\}\Big) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

So

$$W_b^{h-2}u\Big(\Big\{1, \frac{b}{2}, \frac{b}{2}+1, b\Big\}\Big) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the zero entries of $(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^t$ occur only in the $(\frac{1}{2}b, b)$ and $(b, \frac{1}{2}b)$ positions. Note that either $e_{b/2}^{\mathrm{T}}(b)$ or $e_b^{\mathrm{T}}(b)$ is not a row of A. We can show that $A[(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}]A^{\mathrm{T}} = J_n$. Therefore,

$$M^{h-1}(M^{\mathrm{T}})^{h-1} \geqslant A[(W_b)^{h-2}(W_b^{\mathrm{T}})^{h-2} + (W_b^{h-2}u)(W_b^{h-2}u)^{\mathrm{T}}]A^{\mathrm{T}} = J_n,$$

which contradicts $k(M) = h$. This proves (ii).

Case 2. $k(\widetilde{B}\widetilde{A}) = h - 1$.

By Lemma 2.1, there is a permutation matrix P such that $P\widetilde{B}\widetilde{A}P^{\mathrm{T}} = H_b$. Let $B = P\widetilde{B}$ and $A = \widetilde{A}P^{\mathrm{T}}$. Then $BA = H_b$ and $AB = \widetilde{A}P^{\mathrm{T}}P\widetilde{B} = \widetilde{A}\widetilde{B} = M$.

Since M is primitive, we have $\sum_{i=1}^{b} A_{.i} = j_n = \sum_{i=1}^{b} B_{i.}^{\mathrm{T}}$. Since $k(M) = h$, the matrix M^{h-1} must have two rows that do not intersect. Without loss of generality, suppose rows p and q of M^{h-1} do not intersect. Then entries in the (p,q) and (q,p) positions of $M^{h-1}(M^{\mathrm{T}})^{h-1}$ are zero. Since matrix B has no zero row, we have $BB^{\mathrm{T}} \geqslant I_b$. Thus

$$\begin{aligned} M^{h-1}(M^{\mathrm{T}})^{h-1} &= (AB)^{h-1}((AB)^{\mathrm{T}})^{h-1} \\ &= A(BA)^{h-2}BB^{\mathrm{T}}((BA)^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= A(H_b)^{h-2}BB^{\mathrm{T}}(H_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &\geqslant A(H_b)^{h-2}I_b(H_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= A(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\ &= AZA^{\mathrm{T}}, \end{aligned}$$

where $Z = (H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2}$ is the $b \times b$ matrix which has zero entries only in the $(1, \lfloor \frac{1}{2}b \rfloor + 1)$ and $(\lfloor \frac{1}{2}b \rfloor + 1, 1)$ positions. So

$$\begin{aligned} AZA^{\mathrm{T}} &= \left[\left| \sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor + 1}}^{b} A_{.i} \right| J_{n,\lfloor b/2 \rfloor - 1} \left| \sum_{i=2}^{b} A_{.i} \right| J_{n,n-\lfloor b/2 \rfloor - 1} \right] A^{\mathrm{T}} \\ &= \left(\sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor + 1}}^{b} A_{.i} \right) (A_{.1})^{\mathrm{T}} + j_n \left(\sum_{i=2}^{\lfloor b/2 \rfloor} A_{.i} \right)^{\mathrm{T}} \\ &\quad + \left(\sum_{i=2}^{b} A_{.i} \right) (A_{.\lfloor b/2 \rfloor + 1})^{\mathrm{T}} + j_n \left(\sum_{i=\lfloor b/2 \rfloor + 2}^{b} A_{.i} \right)^{\mathrm{T}}. \end{aligned}$$

Since AZA^{T} is dominated by $M^{h-1}(M^{\mathrm{T}})^{h-1}$ and $M^{h-1}(M^{\mathrm{T}})^{h-1}$ has zero entries in the (p,q) and (q,p) positions, the entries in the (p,q) and (q,p) positions of AZA^{T} are also zero. Thus

$$\left(\sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor + 1}}^{b} A_{pi} \right) A_{q1} + \sum_{i=2}^{\lfloor b/2 \rfloor} A_{qi} + \left(\sum_{i=2}^{b} A_{pi} \right) A_{q(\lfloor b/2 \rfloor + 1)} + \sum_{i=\lfloor b/2 \rfloor + 2}^{b} A_{qi} = 0,$$

and

$$\left(\sum_{\substack{i=1 \\ i \neq \lfloor b/2 \rfloor + 1}}^{b} A_{qi} \right) A_{p1} + \sum_{i=2}^{\lfloor b/2 \rfloor} A_{pi} + \left(\sum_{i=2}^{b} A_{qi} \right) A_{p(\lfloor b/2 \rfloor + 1)} + \sum_{i=\lfloor b/2 \rfloor + 2}^{b} A_{pi} = 0.$$

Then $A_{qi} = 0$ and $A_{pi} = 0$ for $i = 2, 3, \ldots, b$ and $i \neq \lfloor \frac{1}{2}b \rfloor + 1$. Substituting these back, we have

$$A_{p1}A_{q1} + A_{p(\lfloor b/2 \rfloor + 1)}A_{q(\lfloor b/2 \rfloor + 1)} = 0.$$

Thus rows $A_{p\cdot}$ and $A_{q\cdot}$ are disjoint. Since A has no zero rows, each of these rows has precisely one nonzero entry. Therefore, some row of A is $e_1^{\mathrm{T}}(b)$ and some row of A is $e^{\mathrm{T}}_{\lfloor b/2\rfloor+1}(b)$.

We claim B cannot have a column $u = e_{b-1}(b) + e_b(b)$. To the contrary, suppose that some column of B is u. Since B has no zero row, $BB^{\mathrm{T}} \geqslant I_b + uu^{\mathrm{T}}$. Thus

$$\begin{aligned}
M^{h-1}(M^{\mathrm{T}})^{h-1} &= (AB)^{h-1}((AB)^{\mathrm{T}})^{h-1} \\
&= A(BA)^{h-2}BB^{\mathrm{T}}((BA)^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\
&= A(H_b)^{h-2}BB^{\mathrm{T}}(H_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\
&\geqslant A(H_b)^{h-2}(I_b + uu^{\mathrm{T}})(H_b^{\mathrm{T}})^{h-2}A^{\mathrm{T}} \\
&= A[(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2} + (H_b^{h-2}u)(H_b^{h-2}u)^{\mathrm{T}}]A^{\mathrm{T}}.
\end{aligned}$$

If b is odd, by Lemma 2.4, the zero entries of $H_b^{h-2}(H_b^{\mathrm{T}})^{h-2}$ occur only in the $(1, \frac{1}{2}(b+1))$ and $(\frac{1}{2}(b+1), 1)$ positions. Note that $H_b^{h-2}(\{1, \frac{1}{2}(b+1)\}, \{b-1, b\}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $H_b^{h-2}u \geqslant e_1(b) + e_{(b+1)/2}(b)$. Therefore, $(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2} + (H_b^{h-2}u)(H_b^{h-2}u)^{\mathrm{T}} = J_b$. Since A has no zero lines, we have $M^{h-1}(M^{\mathrm{T}})^{h-1} = AJ_bA^{\mathrm{T}} = J_n$, which contradicts $k(M) = h$.

If b is even, by Lemma 2.5, the zero entries of $H_b^{h-2}(H_b^{\mathrm{T}})^{h-2}$ occur only in the $(1, \frac{1}{2}b+1)$ and $(\frac{1}{2}b+1, 1)$ positions. Note that $H_b^{h-2}(\{1, \frac{1}{2}b+1\}, \{b-1, b\}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $H_b^{h-2}u \geqslant e_1(b) + e_{b/2+1}(b)$. Therefore, $(H_b)^{h-2}(H_b^{\mathrm{T}})^{h-2} + (H_b^{h-2}u)(H_b^{h-2}u)^{\mathrm{T}} = J_b$. Since A has no zero lines, we have $M^{h-1}(M^{\mathrm{T}})^{h-1} = AJ_bA^{\mathrm{T}} = J_n$, which contradicts $k(M) = h$. This concludes (iii). □

Combining Lemmas 2.8–2.11, we get the main result.

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