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COMMUTATORS OF THE FRACTIONAL MAXIMAL FUNCTION ON VARIABLE EXPONENT LEBESGUE SPACES

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Abstract. Let M_β be the fractional maximal function. The commutator generated by M_β and a suitable function b is defined by $[M_\beta, b]f = M_\beta(bf) - bM_\beta(f)$. Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot)\colon \mathbb{R}^n \to [1,\infty)$ such that

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty,$$

and by $\mathscr{B}(\mathbb{R}^n)$ the set of all $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal function M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. In this paper, the authors give some characterizations of b for which $[M_\beta, b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$, when $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < \beta < n/p_+$ and $1/q(\cdot) = 1/p(\cdot) - \beta/n$ with $q(\cdot)(n-\beta)/n \in \mathscr{B}(\mathbb{R}^n)$.

Keywords: commutator; BMO; fractional maximal function; variable exponent Lebesgue space

MSC 2010: 42B25, 46E30

1. Introduction and main result

Let T be the classical singular integral operator. The commutator $[T, b]$ generated by T and a suitable function b is defined by

$$[T, b]f = T(bf) - bT(f).$$

A classical result of Coifman, Rochberg and Weiss [3] states that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[T, b]$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. They also gave a characterization of

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BMO in virtue of the L^p-boundedness of the above commutator. In 1990, Milman and Schonbek [11] established a commutator result that applies to the Hardy-Littlewood maximal function as well as to a large class of nonlinear operators.

As usual, a cube $Q \subset \mathbb{R}^n$ always means its sides parallel to the coordinate axes. Denote by $|Q|$ the Lebesgue measure of Q and by χ_Q the characteristic function of Q. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we write $f_Q = |Q|^{-1} \int_Q f(x)\,\mathrm{d}x$.

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y,$$

the sharp function $M^\sharp f$ is defined by

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,\mathrm{d}y,$$

and the fractional maximal function M_β is defined by

$$M_\beta f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\beta/n}} \int_Q |f(y)|\,\mathrm{d}y, \quad 0 < \beta < n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

Let Q_0 be a fixed cube in $\mathbb{R}^n$. The Hardy-Littlewood maximal function and the fractional maximal function relative to Q_0 are given by

$$M_{Q_0}(f)(x) = \sup_{\substack{Q \ni x \\ Q \subset Q_0}} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y,$$

and

$$M_{\beta,Q_0}(f)(x) = \sup_{\substack{Q \ni x \\ Q \subset Q_0}} \frac{1}{|Q|^{1-\beta/n}} \int_Q |f(y)|\,\mathrm{d}y, \quad 0 < \beta < n,$$

where the supremum is taken over all cubes $Q \subset Q_0$ and $x \in Q$.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) = \begin{cases} 0, & \text{if } b(x) \geqslant 0, \\ |b(x)|, & \text{if } b(x) < 0, \end{cases}$$

and $b^+(x) = |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

The commutator generated by M_β and a suitable function b is formally defined by

$$[M_\beta, b]f = M_\beta(bf) - bM_\beta(f),$$

and the commutators of M and $M^\sharp$ are defined by

$$[M,b]f = M(bf) - bM(f) \quad \text{and} \quad [M^\sharp,b]f = M^\sharp(bf) - bM^\sharp(f).$$

In 2000, Bastero, Milman and Ruiz [1] studied the necessary and sufficient condition for the boundedness of $[M,b]$ and $[M^\sharp,b]$ on L^p spaces. In 2009, the authors [15] considered the same problem for $[M_\beta,b]$.

In this paper, we will extend the results of Zhang and Wu [15] to the variable exponent Lebesgue spaces. To state our result, we first recall some notation.

Definition 1.1. Let $p(\cdot)\colon\ \mathbb{R}^n \to [1,\infty)$ be a measurable function. The variable exponent Lebesgue space, $L^{p(\cdot)}(\mathbb{R}^n)$, is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \Big\{ f \text{ measurable: } \int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\eta}\Big)^{p(x)}\,\mathrm{d}x < \infty \text{ for some constant } \eta > 0 \Big\}.$$

It is well known that the set $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\Big\{ \eta > 0\colon\ \int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\eta}\Big)^{p(x)}\,\mathrm{d}x \leqslant 1 \Big\}.$$

Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot)\colon\ \mathbb{R}^n \to [1,\infty)$ such that

$$1 < p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p_+ : = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty,$$

and by $\mathscr{B}(\mathbb{R}^n)$ the set of all $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Remark 1.1. If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ and $\lambda > 1$, then by Jensen's inequality, $\lambda p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ (see Remark 2.13 in [4]).

We say an ordered pair of variable exponents $(p(\cdot), q(\cdot))$ belongs to $\mathscr{P}^\beta_{p,q}(\mathbb{R}^n)$, if $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < \beta < n/p_+$ and $1/q(\cdot) = 1/p(\cdot) - \beta/n$ with $q(\cdot)(n-\beta)/n \in \mathscr{B}(\mathbb{R}^n)$.

Our main result can be stated as follows.

Theorem 1.1. *Let $b(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the following assertions are equivalent:*

(1) *$b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(2) *$[M_\beta,b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$ for all $(p(\cdot),q(\cdot)) \in \mathscr{P}^\beta_{p,q}(\mathbb{R}^n)$.*

(3) *$[M_\beta,b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$ for some $(p(\cdot),q(\cdot)) \in \mathscr{P}^\beta_{p,q}(\mathbb{R}^n)$.*

(4) *There exists $(p(\cdot),q(\cdot)) \in \mathscr{P}^\beta_{p,q}(\mathbb{R}^n)$, such that*

$$\sup_Q \frac{\|(b - M_Q(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}} < \infty.$$

(5) *For all* $(p(\cdot), q(\cdot)) \in \mathscr{P}^{\beta}_{p,q}(\mathbb{R}^n)$, *we have*

$$\sup_{Q} \frac{\|(b - M_Q(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}} < \infty.$$

Remark 1.2. By Remark 2.13 in [4], we know that $q(\cdot)(n-\beta)/n \in \mathscr{B}(\mathbb{R}^n)$ is equivalent to saying that there exists s with $n/(n-\beta) < s < \infty$ such that $q(\cdot)/s \in \mathscr{B}(\mathbb{R}^n)$. Moreover, it follows from Remark 1.1 that $q(\cdot)(n-\beta)/n \in \mathscr{B}(\mathbb{R}^n)$ implies $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

One of the most interesting problems on spaces with variable exponents is to give conditions guaranteing the boundedness of the Hardy-Littlewood maximal function. Important sufficient conditions called log-Hölder have been obtained by Cruz-Uribe, Fiorenza and Neugebauer [5].

Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, we say that $p(\cdot)$ satisfies the local log-Hölder condition if there exists a constant $C > 0$ such that for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \leqslant \frac{-C}{\log|x-y|}, \quad \text{if } |x-y| \leqslant 1/2. \tag{1.1}$$

We say that $p(\cdot)$ satisfies the log-Hölder decay condition if there exists a constant $C > 0$ such that for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \leqslant \frac{C}{\log(\mathrm{e} + |x|)}, \quad \text{if } \ |y| \geqslant |x|. \tag{1.2}$$

If both the conditions (1.1) and (1.2) are satisfied, we say that $p(\cdot)$ satisfies the log-Hölder condition, abbreviated to $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$.

Remark 1.3. By Theorem 1.5 in [5], if $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ then $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Furthermore, for $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, $0 < \beta < n/p_+$ and $1/q(\cdot) = 1/p(\cdot) - \beta/n$, it is easy to check that $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ and $q(\cdot)(n-\beta)/n \in \mathscr{P}^{\log}(\mathbb{R}^n)$, which implies $(p(\cdot), q(\cdot)) \in \mathscr{P}^{\beta}_{p,q}(\mathbb{R}^n)$.

This along with Theorem 1.1 gives the following result, a special case of Theorem 1.1.

Corollary 1.1. *Let* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, $0 < \beta < n/p_+$ *and* $1/q(\cdot) = 1/p(\cdot) - \beta/n$. *If* $b(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *then the following assertions are equivalent:*

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

(ii) $[M_\beta, b]$ *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to* $L^{q(\cdot)}(\mathbb{R}^n)$.

(iii)

$$\sup_Q \frac{\|(b - M_Q(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}} < \infty.$$

When $p(\cdot), q(\cdot)$ are constant exponents, Corollary 1.1 was proved by the authors in [15].

The remainder of this paper is organized as follows. In Section 2, we recall some known results in the context of variable Lebesgue spaces. In Section 3, we will give some auxiliary results which are of independent interest and will be used in the proof of the main result. In the last section, we will prove Theorem 1.1.

2. Preliminaries

In this section, we recall some known results in the context of variable Lebesgue spaces. In what follows, we denote by $p'(\cdot)$ the conjugate index of $p(\cdot)$, that is $1/p(\cdot) + 1/p'(\cdot) = 1$. It is easy to check that if $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ then $p'(\cdot) \in \mathscr{P}(\mathbb{R}^n)$.

The first lemma is known as the generalized Hölder's inequality on variable exponent Lebesgue spaces and the proof can be found in [10] and [8].

Lemma 2.1. (i) *Suppose that $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, then for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and any $g \in L^{p'(\cdot)}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)g(x)| \,\mathrm{d}x \leqslant C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $C_p = 1 + 1/p_- - 1/p_+$.

(ii) *Assume that $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $1/p(x) = 1/p_1(x) + 1/p_2(x)$, then for any $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and any $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$,*

$$\|fg\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C_{p,p_1} \|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p_2(\cdot)}(\mathbb{R}^n)},$$

where $C_{p,p_1} = (1 + 1/(p_1)_- - 1/(p_1)_+)^{1/p_-}$.

Lemma 2.2 ([6]). *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. Then the following conditions are equivalent:*

(a) $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$,
(b) $p'(\cdot) \in \mathscr{B}(\mathbb{R}^n)$,
(c) $p(\cdot)/r \in \mathscr{B}(\mathbb{R}^n)$ *for some* $1 < r < p_-$,
(d) $(p(\cdot)/r)' \in \mathscr{B}(\mathbb{R}^n)$ *for some* $1 < r < p_-$.

Lemma 2.3 ([9]). *Let $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that*

$$\frac{1}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leqslant C$$

for all cubes Q in $\mathbb{R}^n$.

The next result follows from Corollary 2.12 and Remark 2.13 of [4].

Lemma 2.4 ([4]). *Let $p(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < \beta < n/p_+$ and $1/q(x) = 1/p(x) - \beta/n$. If $q(\cdot)(n-\beta)/n \in \mathscr{B}(\mathbb{R}^n)$, then M_β is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$.*

It is easy to check that if $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 < \beta < n/p_+$ and $1/q(x) = 1/p(x) - \beta/n$, then $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. So, under the assumptions of Lemma 2.4, $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ automatically follows from $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$.

In 2007, Capone, Cruz-Uribe and Fiorenza [2] proved that if $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, $0 < \beta < n/p_+$ and $1/q(x) = 1/p(x) - \beta/n$, then M_β is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$. Remark 1.3 shows that Lemma 2.4 extends the corresponding result in [2].

To state the extrapolation theorems, we recall the Muckenhoupt weights.

A locally integrable function $\omega\colon \mathbb{R}^n \to (0,\infty)$ is called a weight. We say that $\omega \in A_p$, $1 < p < \infty$, if there is a constant $C > 0$ such that for any cube $Q \subset \mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,\mathrm{d}x\right)^{p-1} \leqslant C,$$

where $1/p + 1/p' = 1$. We say that $\omega \in A_1$ if there is a constant $C > 0$ such that $M\omega(x) \leqslant C\omega(x)$ almost everywhere.

The extrapolation theorems (Lemma 2.5 and Lemma 2.6 below) are originally due to Cruz-Uribe, Fiorenza, Martell and Pérez [4]. Here we use the form in [7], see Theorem 7.2.1 and Theorem 7.2.3 in [7].

Lemma 2.5 ([7]). *Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < \infty$, every $(f,g) \in \mathcal{F}$ and every $\omega \in A_1$,*

$$\int_{\mathbb{R}^n} |f(x)|^{p_0}\omega(x)\,\mathrm{d}x \leqslant C_0\int_{\mathbb{R}^n} |g(x)|^{p_0}\omega(x)\,\mathrm{d}x.$$

Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ with $p_0 \leqslant p_-$. If $(p(\cdot)/p_0)' \in \mathscr{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.6 ([7]). *Given a family $\mathcal{F}$ of ordered pairs of measurable functions, suppose that for some fixed $0 < p_0 < q_0 < \infty$, every $(f,g) \in \mathcal{F}$ and every $\omega \in A_1$,*

$$\left(\int_{\mathbb{R}^n} |f(x)|^{q_0}\omega(x)\,\mathrm{d}x\right)^{1/q_0} \leqslant C_0\left(\int_{\mathbb{R}^n} |g(x)|^{p_0}\omega(x)^{p_0/q_0}\,\mathrm{d}x\right)^{1/p_0}.$$

Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ with $p_0 \leqslant p_-$ and $1/p_0 - 1/q_0 < 1/p_+$, and define $q(x)$ by

$$\frac{1}{q(x)} - \frac{1}{p(x)} = \frac{1}{q_0} - \frac{1}{p_0}.$$

If $(q(\cdot)/q_0)' \in \mathscr{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

3. Some auxiliary results

In this section we will give some auxiliary results which are of independent interest and will be used in the proof of the main result.

For $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \beta < n$, define

$$M_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |b(x) - b(y)||f(y)|\,\mathrm{d}y$$

and

$$M_{\beta,b}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\beta/n}}\int_Q |b(x) - b(y)||f(y)|\,\mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The next result follows from Theorem 3 of Segovia and Torrea [12].

Lemma 3.1. *Let $1 < p < \infty$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then for any $\omega \in A_p$, we have*

$$\int_{\mathbb{R}^n} [M_b(f)(x)]^p\omega(x)\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^n} |f(x)|^p\omega(x)\,\mathrm{d}x.$$

Let v be a weight function. We say $v \in A(p,q)$ $(1 < p, q < \infty)$, if there exists a constant C such that for any cube $Q \subset \mathbb{R}^n$, we have

$$\left(\frac{1}{|Q|}\int_Q v(x)^{-p'}\,\mathrm{d}x\right)^{1/p'}\left(\frac{1}{|Q|}\int_Q v(x)^q\,\mathrm{d}x\right)^{1/q} \leqslant C.$$

By Theorem 3.2 of Segovia and Torrea [13], it is easy to get the following result.

Lemma 3.2. *Let $0<\beta<n$, $1<p<n/\beta$ and $1/q=1/p-\beta/n$. If $b\in\mathrm{BMO}(\mathbb{R}^n)$, then for any $v\in A(p,q)$,*

$$\Big(\int_{\mathbb{R}^n}[M_{\beta,b}(f)(x)v(x)]^q\,\mathrm{d}x\Big)^{1/q}\leqslant C\Big(\int_{\mathbb{R}^n}|f(x)v(x)|^p\,\mathrm{d}x\Big)^{1/p}.$$

Theorem 3.1. *Let $b(x)\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$. Then M_b is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself if and only if $b(x)\in\mathrm{BMO}(\mathbb{R}^n)$.*

Proof. (i) We prove the "if" part first. Since $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, it follows from Lemma 2.2 that there exists p_0 with $1<p_0<p_-$ such that $(p(\cdot)/p_0)'\in\mathscr{B}(\mathbb{R}^n)$. For this p_0 and every $\omega\in A_1\subset A_{p_0}$, by Lemma 3.1 we have

$$\int_{\mathbb{R}^n}[M_b(f)(x)]^{p_0}\omega(x)\,\mathrm{d}x\leqslant C_0\int_{\mathbb{R}^n}|f(x)|^{p_0}\omega(x)\,\mathrm{d}x.$$

Therefore, the "if" part follows from Lemma 2.5 applied to the pair $(M_b(f),f)$.

(ii) Now, let us prove the "only if" part. For any cube $Q\subset\mathbb{R}^n$ and any $y\in Q$, by the definition of M_b, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q|b(y)-b(x)|\chi_Q(x)\,\mathrm{d}x&\leqslant\sup_{Q'\ni y}\frac{1}{|Q'|}\int_{Q'}|b(y)-b(x)|\chi_Q(x)\,\mathrm{d}x\\&=M_b(\chi_Q)(y).\end{aligned}$$

Applying Lemma 2.1 (i), the boundedness of M_b on $L^{p(\cdot)}(\mathbb{R}^n)$ and Lemma 2.3, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q|b(y)-b_Q|\,\mathrm{d}y&=\frac{1}{|Q|}\int_Q\Big|\frac{1}{|Q|}\int_Q(b(y)-b(x))\,\mathrm{d}x\Big|\,\mathrm{d}y\\&\leqslant\frac{1}{|Q|}\int_Q\Big(\frac{1}{|Q|}\int_Q|b(y)-b(x)|\chi_Q(x)\,\mathrm{d}x\Big)\,\mathrm{d}y\\&\leqslant\frac{1}{|Q|}\int_Q M_b(\chi_Q)(y)\,\mathrm{d}y\\&=\frac{1}{|Q|}\int_{\mathbb{R}^n}M_b(\chi_Q)(y)\cdot\chi_Q(y)\,\mathrm{d}y\\&\leqslant\frac{C}{|Q|}\|M_b(\chi_Q)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\&\leqslant\frac{C}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\\&\leqslant C,\end{aligned}$$

which implies $b(x)\in\mathrm{BMO}(\mathbb{R}^n)$. $\square$

In 2007, Xu [14] proved that if $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ and $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$ then M_b is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself (see Theorem 1.3 in [14]). Obviously, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ is weaker than $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$. So, Theorem 3.1 extends Xu's result in [14].

Theorem 3.2. *Let $b(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(p(\cdot), q(\cdot)) \in \mathscr{P}^{\beta}_{p,q}(\mathbb{R}^n)$. Then $M_{\beta,b}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$ if and only if $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$.*

Proof. (i) We first prove the "if" part. Since $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\beta}{n} \leqslant \frac{1}{p_-} - \frac{\beta}{n},$$

we have

$$\frac{n}{n-\beta} < \frac{np_-}{n-\beta p_-} \leqslant q(x).$$

Noting that $q(\cdot)(n-\beta)/n \in \mathscr{B}(\mathbb{R}^n)$, by Remark 1.2 there exists s with $n/(n-\beta) < s < \infty$ such that $q(\cdot)/s \in \mathscr{B}(\mathbb{R}^n)$. If $s \leqslant np_-/(n-\beta p_-)$, then we take $r = 1$. If $s > np_-/(n-\beta p_-)$, then we take $r = s(n-\beta p_-)/np_- > 1$. Setting $q_0 = s/r$, we have

$$\frac{n}{n-\beta} < q_0 \leqslant \frac{np_-}{n-\beta p_-} \leqslant q(x).$$

Since $r \geqslant 1$ and $q(\cdot)/s \in \mathscr{B}(\mathbb{R}^n)$, we have $q(\cdot)/q_0 = rq(\cdot)/s \in \mathscr{B}(\mathbb{R}^n)$ from Remark 1.1. By Lemma 2.2, we have $(q(\cdot)/q_0)' \in \mathscr{B}(\mathbb{R}^n)$.

Define p_0 by $1/q_0 = 1/p_0 - \beta/n$, then $1 < p_0 < n/\beta$ and

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{\beta}{n}.$$

This together with $q_0 \leqslant q(x)$ gives $p_0 \leqslant p_-$.

For any $\omega \in A_1$ we have $\omega \in A_{1+q_0/p_0'}$. Set $v(x)^{q_0} = \omega(x)$, then $v^{q_0} \in A_{1+q_0/p_0'}$, which implies $v \in A(p_0, q_0)$. By Lemma 3.2,

$$\biggl(\int_{\mathbb{R}^n} [M_{\beta,b}(f)(x)]^{q_0} \omega(x)\,\mathrm{d}x\biggr)^{1/q_0} \leqslant C\biggl(\int_{\mathbb{R}^n} |f(x)|^{p_0} \omega(x)^{p_0/q_0}\,\mathrm{d}x\biggr)^{1/p_0}.$$

Applying Lemma 2.6 to the pair $(M_{\beta,b}(f), f)$, we have

$$\|M_{\beta,b}(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

So, the proof of the "if" part is completed.

(ii) Now, we prove the "only if" part. For any cube $Q \subset \mathbb{R}^n$, by the definition of $M_{\beta,b}$ and applying Lemma 2.1 (i) and the boundedness of $M_{\beta,b}$, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |b(y)-b_Q|\,\mathrm{d}y &= \frac{1}{|Q|}\int_Q \left|\frac{1}{|Q|}\int_Q (b(y)-b(x))\,\mathrm{d}x\right|\mathrm{d}y \\ &\leqslant \frac{1}{|Q|^{1+\beta/n}}\int_Q \left(\frac{1}{|Q|^{1-\beta/n}}\int_Q |b(y)-b(x)|\chi_Q(x)\,\mathrm{d}x\right)\mathrm{d}y \\ &\leqslant \frac{1}{|Q|^{1+\beta/n}}\int_Q M_{\beta,b}(\chi_Q)(y)\,\mathrm{d}y \\ &= \frac{1}{|Q|^{1+\beta/n}}\int_{\mathbb{R}^n} M_{\beta,b}(\chi_Q)(y)\cdot\chi_Q(y)\,\mathrm{d}y \\ &\leqslant \frac{C}{|Q|^{1+\beta/n}}\|M_{\beta,b}(\chi_Q)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\leqslant \frac{C}{|Q|^{1+\beta/n}}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Noting that $1/p(x) = 1/q(x) + \beta/n$ and $p(\cdot), q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, by Lemma 2.1 (ii) we get

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}|Q|^{\beta/n}.$$

Applying Lemma 2.3, we obtain

$$\frac{1}{|Q|}\int_Q |b(y)-b_Q|\,\mathrm{d}y \leqslant \frac{C}{|Q|^{1+\beta/n}}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}|Q|^{\beta/n}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leqslant C,$$

which implies $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$. So, the proof of Theorem 3.2 is completed. □

Theorem 3.3. *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. If $0 \leqslant b(x) \in \mathrm{BMO}(\mathbb{R}^n)$, then $[M, b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.*

Proof. For a fixed $x \in \mathbb{R}^n$ such that $Mf(x) < \infty$, noting that $b \geqslant 0$, we have

$$\begin{aligned}|[M,b]f(x)| &= |M(bf)(x) - b(x)M(f)(x)| \\ &= \left|\sup_{Q\ni x}\frac{1}{|Q|}\int_Q b(y)|f(y)|\,\mathrm{d}y - \sup_{Q\ni x}\frac{1}{|Q|}\int_Q b(x)|f(y)|\,\mathrm{d}y\right| \\ &\leqslant \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |b(x)-b(y)||f(y)|\,\mathrm{d}y \\ &= M_b(f)(x).\end{aligned}\tag{3.1}$$

Since $Mf(x) < \infty$ for a.e. $x \in \mathbb{R}^n$ when $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, (3.1) is valid almost everywhere in $\mathbb{R}^n$. Noting that $b \in \mathrm{BMO}(\mathbb{R}^n)$, it follows from Theorem 3.1 that $[M, b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself. □

Theorem 3.4. *Let $(p(\cdot),q(\cdot))\in\mathscr{P}^{\beta}_{p,q}(\mathbb{R}^n)$. If $0\leqslant b(x)\in\mathrm{BMO}(\mathbb{R}^n)$, then $[M_\beta,b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$.*

Proof. For a fixed $x\in\mathbb{R}^n$, noting that $b\geqslant 0$, we have

$$\begin{aligned}|[M_\beta,b]f(x)|&=|M_\beta(bf)(x)-b(x)M_\beta(f)(x)|\\&\leqslant\left|\sup_{Q\ni x}\frac{1}{|Q|^{1-\beta/n}}\int_Q b(y)|f(y)|\,\mathrm{d}y-\sup_{Q\ni x}\frac{1}{|Q|^{1-\beta/n}}\int_Q b(x)|f(y)|\,\mathrm{d}y\right|\\&\leqslant\sup_{Q\ni x}\frac{1}{|Q|^{1-\beta/n}}\left|\int_Q[b(y)-b(x)]|f(y)|\,\mathrm{d}y\right|\\&\leqslant\sup_{Q\ni x}\frac{1}{|Q|^{1-\beta/n}}\int_Q|b(x)-b(y)||f(y)|\,\mathrm{d}y\\&=M_{\beta,b}(f)(x).\end{aligned}$$

By Theorem 3.2, we see that $[M_\beta,b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$. □

It is easy to see that if $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$, then the conclusions of Theorem 3.1 and Theorem 3.3 also hold. If $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$, $0<\beta<n/p_+$ and $1/q(\cdot)=1/p(\cdot)-\beta/n$, then the conclusions of Theorem 3.2 and Theorem 3.4 also hold.

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. To do this, we give two lemmas first.

Lemma 4.1. *Let $b(x)\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(p(\cdot),q(\cdot))\in\mathscr{P}^{\beta}_{p,q}(\mathbb{R}^n)$. If $[M_\beta,b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$, then $b\in\mathrm{BMO}(\mathbb{R}^n)$ and*

$$\sup_Q\frac{\|(b-|Q|^{-\beta/n}M_{\beta,Q}(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}}<\infty. \tag{4.1}$$

Proof. Since for any fixed cube Q and all $x\in Q$ we have (see (2.4) in [15])

$$M_\beta(\chi_Q)(x)=M_{\beta,Q}(\chi_Q)(x)=|Q|^{\beta/n}\quad\text{and}\quad M_\beta(b\chi_Q)(x)=M_{\beta,Q}(b)(x), \tag{4.2}$$

and since $[M_\beta,b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$, we have

$$\begin{aligned}\|(b-|Q|^{-\beta/n}M_{\beta,Q}(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}&=|Q|^{-\beta/n}\|(b|Q|^{\beta/n}-M_{\beta,Q}(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\leqslant|Q|^{-\beta/n}\|bM_\beta(\chi_Q)-M_\beta(b\chi_Q)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&=|Q|^{-\beta/n}\|[M_\beta,b]\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\&\leqslant C|Q|^{-\beta/n}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Noting that $1/p(\cdot) = 1/q(\cdot) + \beta/n$ and applying Lemma 2.1 (ii), we have

$$\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}|Q|^{\beta/n}. \tag{4.3}$$

Then

$$\|(b - |Q|^{-\beta/n}M_{\beta,Q}(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

which implies (4.1).

Now, let us prove $b \in \mathrm{BMO}(\mathbb{R}^n)$. For any cube Q, let $E = \{x \in Q\colon\ b(x) \leqslant b_Q\}$ and $F = \{x \in Q\colon\ b(x) > b_Q\}$. The following equality is trivially true (see [1], page 3331):

$$\int_E |b(x) - b_Q|\,\mathrm{d}x = \int_F |b(x) - b_Q|\,\mathrm{d}x.$$

Since $b(x) \leqslant b_Q \leqslant |b_Q| \leqslant |Q|^{-\beta/n}M_{\beta,Q}(b)(x)$ for any $x \in E$, we obtain

$$|b(x) - b_Q| \leqslant |b(x) - |Q|^{-\beta/n}M_{\beta,Q}(b)(x)|, \quad x \in E.$$

Therefore,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |b(x) - b_Q|\,\mathrm{d}x &= \frac{1}{|Q|}\int_{E\cup F} |b(x) - b_Q|\,\mathrm{d}x \\ &= \frac{2}{|Q|}\int_E |b(x) - b_Q|\,\mathrm{d}x \\ &\leqslant \frac{2}{|Q|}\int_E |b(x) - |Q|^{-\beta/n}M_{\beta,Q}(b)(x)|\,\mathrm{d}x \\ &\leqslant \frac{2}{|Q|}\int_Q |b(x) - |Q|^{-\beta/n}M_{\beta,Q}(b)(x)|\,\mathrm{d}x.\end{aligned} \tag{4.4}$$

On the other hand, by Lemma 2.1 (i), (4.1) and Lemma 2.3 we get

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |b(x) - |Q|^{-\beta/n}M_{\beta,Q}(b)(x)|\,\mathrm{d}x \\ &\quad\leqslant \frac{C}{|Q|}\|(b - |Q|^{-\beta/n}M_{\beta,Q}(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\quad\leqslant \frac{C}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\quad\leqslant C.\end{aligned}$$

This along with (4.4) gives

$$\frac{1}{|Q|}\int_Q |b(x) - b_Q|\,\mathrm{d}x \leqslant C.$$

So, by the definition of BMO, we obtain $b \in \mathrm{BMO}(\mathbb{R}^n)$. $\square$

The following result was proved by Bastero, Milman and Ruiz, see Proposition 4 in [1].

Lemma 4.2 ([1]). *Let* $b(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. *If*

$$\sup_Q \frac{1}{|Q|} \int_Q |b(x) - M_Q(b)(x)| \, \mathrm{d}x < \infty,$$

then $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $b^- \in L^\infty(\mathbb{R}^n)$.

Proof of Theorem 1.1. Since the implications (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) follow readily, we only have to prove (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) (the implication (2) $\Rightarrow$ (5) is similar to (3) $\Rightarrow$ (4)).

(1) $\Longrightarrow$ (2). Using the definition of $[M_\beta, b]$, the triangle's inequality, and noting that $|b(x)| - b(x) = 2b^-(x)$ and $M_\beta(bf)(x) = M_\beta(|b|f)(x)$, we obtain

$$\begin{aligned}
&|[M_\beta, b]f(x) - [M_\beta, |b|]f(x)| \\
&\quad = |M_\beta(bf)(x) - b(x)M_\beta(f)(x) - M_\beta(|b|f)(x) + |b(x)|M_\beta(f)(x)| \\
&\quad \leqslant |M_\beta(bf)(x) - M_\beta(|b|f)(x)| + |2b^-(x)M_\beta(f)(x)| \\
&\quad = 2b^-(x)M_\beta(f)(x).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
|[M_\beta, b]f(x)| &\leqslant |[M_\beta, b]f(x) - [M_\beta, |b|]f(x)| + |[M_\beta, |b|]f(x)| \\
&\leqslant 2b^-(x)M_\beta(f)(x) + |[M_\beta, |b|]f(x)|.
\end{aligned}$$

Noting that $|b| \in \mathrm{BMO}(\mathbb{R}^n)$ when $b \in \mathrm{BMO}(\mathbb{R}^n)$, it follows from Lemma 2.4, Theorem 3.4 and $b^- \in L^\infty(\mathbb{R}^n)$ that for all $(p(\cdot), q(\cdot)) \in \mathscr{P}^\beta_{p,q}(\mathbb{R}^n)$,

$$\begin{aligned}
\|[M_\beta, b]f\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leqslant 2\|b^-\|_{L^\infty(\mathbb{R}^n)}\|M_\beta(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \|[M_\beta, |b|]f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

(3) $\Longrightarrow$ (4). For any fixed cubes $Q \subset \mathbb{R}^n$ and all $x \in Q$, we have (see the proof of Proposition 4 in [1])

$$M(\chi_Q)(x) = M_Q(\chi_Q)(x) = \chi_Q(x)$$

and

$$M(b\chi_Q)(x) = M_Q(b)(x),$$

which combined with (4.2) gives (for details see [15], page 1238)

$$M_{\beta,Q}(b)(x) - |Q|^{\beta/n}M_Q(b)(x) = [M_\beta, |b|](\chi_Q)(x) - |Q|^{\beta/n}[M, |b|](\chi_Q)(x). \tag{4.5}$$

Recall that assertion (3) says that for some $(p(\cdot), q(\cdot)) \in \mathscr{P}^{\beta}_{p,q}(\mathbb{R}^n)$, $[M_\beta, b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$. By Lemma 4.1 we have $b \in \mathrm{BMO}(\mathbb{R}^n)$ and there is a constant $C > 0$ such that for any cube $Q \subset \mathbb{R}^n$,

$$\|(b - |Q|^{-\beta/n} M_{\beta,Q}(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \tag{4.6}$$

Noting that $0 \leqslant |b| \in \mathrm{BMO}(\mathbb{R}^n)$ when $b \in \mathrm{BMO}(\mathbb{R}^n)$, it follows from (4.5), (4.6), Theorem 3.3 and Theorem 3.4 that

$$\begin{aligned}
&\|(b - M_Q(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad = \|(b - |Q|^{-\beta/n} M_{\beta,Q}(b) + |Q|^{-\beta/n} M_{\beta,Q}(b) - M_Q(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\|(b - |Q|^{-\beta/n} M_{\beta,Q}(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\qquad + C\|(|Q|^{-\beta/n} M_{\beta,Q}(b) - M_Q(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} + C|Q|^{-\beta/n}\|(M_{\beta,Q}(b) - |Q|^{\beta/n} M_Q(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} + C|Q|^{-\beta/n}\|[M_\beta, |b|](\chi_Q) - |Q|^{\beta/n}[M, |b|](\chi_Q)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} + C|Q|^{-\beta/n}\|[M_\beta, |b|](\chi_Q)\|_{L^{q(\cdot)}(\mathbb{R}^n)} + C\|[M, |b|](\chi_Q)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} + C|Q|^{-\beta/n}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} + C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\quad \leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

where in the last step we have used (4.3).

So, the proof of "(3) $\Longrightarrow$ (4)" is complete.

(4) $\Longrightarrow$ (1). Since $q(\cdot)(n-\beta)/n \in \mathscr{B}(\mathbb{R}^n)$ hence by Remark 1.1 we have $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. For any fixed cube Q, by Lemma 2.1 (i), assertion (4) and Lemma 2.3, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |b(x) - M_Q(b)(x)|\,\mathrm{d}x &\leqslant \frac{C}{|Q|}\|(b - M_Q(b))\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant \frac{C}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C.
\end{aligned}$$

This along with Lemma 4.2 gives that $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$. □

Remark 4.1. Lemma 4.1 says that if $b(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $(p(\cdot), q(\cdot)) \in \mathscr{P}^{\beta}_{p,q}(\mathbb{R}^n)$ and $[M_\beta, b]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$, then (4.1) holds. From the proof of Lemma 4.1 we see that (4.1) implies $b \in \mathrm{BMO}(\mathbb{R}^n)$, but we do not know whether (4.1) implies $b^- \in L^\infty(\mathbb{R}^n)$. So, it is natural to ask whether (4.1) is equivalent to any of the assertions in Theorem 1.1.

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