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NEW RESULTS FOR EP MATRICES IN INDEFINITE INNER PRODUCT SPACES

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Abstract. In this paper we study J-EP matrices, as a generalization of EP-matrices in indefinite inner product spaces, with respect to indefinite matrix product. We give some properties concerning EP and J-EP matrices and find connection between them. Also, we present some results for reverse order law for Moore-Penrose inverse in indefinite setting. Finally, we deal with the star partial ordering and improve some results given in the "EP matrices in indefinite inner product spaces" (2012), by relaxing some conditions.

Keywords: EP matrix; indefinite matrix product; reverse order law; partial order; indefinite inner product space

MSC 2010: 15A09, 06A06, 47B50

1. Introduction

An indefinite inner product in $\mathbb{C}^n$ is a sesquilinear form $[x,y]$, $x,y \in \mathbb{C}^n$, defined by the equation

$$[x,y] = \langle x, Jy \rangle.$$

Here, $\langle .,. \rangle$ is the standard Euclidean inner product, J is an invertible Hermitian matrix. We make an additional assumption that $J^2 = I$, motivated by the notion of Minkowski space which has been studied by physicists in optics. In some results herein this assumption is not restrictive at all. On the other hand, it lets us make a nice comparison with results in the Euclidean case. As in [8], we use a new matrix product, called the indefinite matrix multiplication. We give some basic notions.

Definition 1.1. Let $J_n \in \mathbb{C}^{n\times n}$ be such that $J_n = J_n^* = J_n^{-1}$. The indefinite matrix product of matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times l}$ is defined by $A \circ B = AJ_nB$.

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Definition 1.2. Let A be an $m \times n$ complex matrix. The adjoint $A^{[*]}$ of A is defined by $A^{[*]} = J_n A^* J_m$.

Definition 1.3. For $A \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{n\times m}$ is called the Moore-Penrose inverse of A if it satisfies the following equations: $A \circ X \circ A = A$, $X \circ A \circ X = X$, $(A \circ X)^{[*]} = A \circ X$ and $(X \circ A)^{[*]} = X \circ A$.

Definition 1.4. For $A \in \mathbb{C}^{n\times n}$, a matrix $X \in \mathbb{C}^{n\times n}$ is called the group inverse of A if it satisfies the following equations: $A\circ X\circ A = A$, $X\circ A\circ X = X$, $A\circ X = X\circ A$.

We are familiar with the fact that for $A \in \mathbb{C}^{m\times n}$ the Moore-Penrose inverse has the form $A^{[\dagger]} = J_n A^\dagger J_m$, and it always exists because the condition $\operatorname{rank}(A^{[*]} \circ A) = \operatorname{rank}(A \circ A^{[*]}) = \operatorname{rank}(A)$ is always satisfied. On the other hand, it is not the case that a similar formula for the group inverse holds. It may happen that the group inverse in the Euclidean space exists, but in the space with indefinite matrix product it does not, and vice versa, for example, for $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Anyway, $A^{[\#]} = (AJ)^\# J$, and it exists if and only if $\operatorname{rank}(A^{(2)}) = \operatorname{rank}(A)$, i.e. $\operatorname{rank}(AJA) = \operatorname{rank}(A)$, while $A^\#$ exists if and only if $\operatorname{rank}(A^2) = \operatorname{rank}(A)$. Clearly, if A and J commute, then both the group inverses exist at the same time and, in that case, $A^{[\#]} = A^\#$.

Definition 1.5. Let $A \in \mathbb{C}^{m\times n}$. Then the range space $\operatorname{Ra}(A)$ is defined by $\operatorname{Ra}(A) = \{y = A \circ x \in \mathbb{C}^m \colon x \in \mathbb{C}^n\}$ and the null space $\operatorname{Nu}(A)$ is defined by $\operatorname{Nu}(A) = \{x \in \mathbb{C}^n \colon A \circ x = 0\}$.

It is easy to see that $\operatorname{Ra}(A) = \mathrm{R}(AJ) = \mathrm{R}(A)$ and $\operatorname{Nu}(A) = \mathrm{N}(AJ)$. It is also clear that $\operatorname{Ra}(A^{[*]}) = \mathrm{R}((AJ)^*)$ and $\operatorname{Nu}(A^{[*]}) = \mathrm{N}(A^*)$, where $\mathrm{R}(A)$ and $\mathrm{N}(A)$ denote the standard range and null space of A, respectively.

Definition 1.6. A matrix $A \in \mathbb{C}^{n\times n}$ is called range-Hermitian if $\mathrm{R}(A^*) = \mathrm{R}(A)$, or, equivalently, if $\mathrm{N}(A^*) = \mathrm{N}(A)$.

Definition 1.7. Let M be a subset of $\mathbb{C}^n$. The orthogonal companion of M in $\mathbb{C}^n$ with respect to the indefinite inner product is defined by $M^{[\perp]} = \{x \in \mathbb{C}^n \colon [x, y] = 0$ for all $y \in M\}$.

In this paper we establish some properties of J-EP matrices and their connection with other classes of matrices. Besides the new results, the improvement of existing ones is also made.

This paper is organized as follows. In Section 2, we give some results concerning EP and J-EP matrices. We also investigate the relation between them. Some of the results in this section are nice generalizations of theorems which deal with EP matrices. It is important to mention that we also improve some of the results from [7]. In many theorems here we relaxed the conditions from [7] (Theorems 2.3, 2.4, 2.11).

In Section 3, the reverse order law with respect to the indefinite matrix product is studied. There are several theorems which give necessary and sufficient conditions for that. We also give a theorem and example (Theorem 3.3 and Example 3.1) showing that in Theorem 3.14, [7], the assumption that a matrix B is J-EP can be totally excluded. Moreover, it does not have to be an EP matrix, either.

Section 3 deals with the notion and properties of the star partial ordering with respect to an indefinite matrix product and gives the parallel with the original star ordering. Theorem 4.1 and Corollary 4.1 give a generalization of some results from [10]. They are also the improvements of Theorem 4.3 and Theorem 4.4 from [7].

2. EP and J-EP matrices

We start by introducing the notion of J-EP matrices and giving some of their properties.

Definition 2.1. A matrix $A \in \mathbb{C}^{n\times n}$ is J-EP if $A \circ A^{[\dagger]} = A^{[\dagger]} \circ A$.

The next well known lemma is often used to establish the relationship between J-EP and EP matrices.

Lemma 2.1. *A matrix $A \in \mathbb{C}^{n\times n}$ is a J-EP matrix if and only if AJ is an EP matrix.*

We have to emphasize that most of the properties of EP matrices can be generalized to J-EP matrices with respect to an indefinite matrix product. Also, their characterization can be given according to EP matrices. All of that can be done by using Lemma 2.1, i.e., by considering the AJ matrix instead of a matrix A and vice versa. Some of these results (without proofs) we give here as Theorem 2.1 (as a generalization of Theorem 7.5.1 in [3]).

Theorem 2.1. *Let $A \in \mathbb{C}^{n\times n}$. Then the following statements are equivalent:*

(1) *A is J-EP;*
(2) $A \circ A^{[\dagger]} = A^{(2)} \circ (A^{[\dagger]})^{(2)}$;
(3) $A^{[\dagger]} \circ A = (A^{[\dagger]})^{(2)} \circ A^{(2)}$;
(4) $A \circ A^{[\dagger]} \circ A^{[*]} \circ A = A^{[*]} \circ A \circ A \circ A^{[\dagger]}$;
(5) $A^{[\dagger]} \circ A \circ A \circ A^{[*]} = A \circ A^{[*]} \circ A^{[\dagger]} \circ A$;
(6) $A \circ A^{[\dagger]} \circ (A \circ A^{[*]} - A^{[*]} \circ A) = (A \circ A^{[*]} - A^{[*]} \circ A) \circ A \circ A^{[\dagger]}$;
(7) $A^{[\dagger]} \circ A \circ (A \circ A^{[*]} - A^{[*]} \circ A) = (A \circ A^{[*]} - A^{[*]} \circ A) \circ A \circ A^{[\dagger]}$;

(8) $A^{[*]} \circ A^{[\#]} \circ A + A \circ A^{[\#]} \circ A^{[*]} = 2A^{[*]}$;
(9) $A^{[\dagger]} \circ A^{[\#]} \circ A + A \circ A^{[\#]} \circ A^{[\dagger]} = 2A^{[\dagger]}$;
(10) $A \circ A \circ A^{[\dagger]} + A^{[\dagger]} \circ A \circ A = 2A$;
(11) $A \circ A \circ A^{[\dagger]} + (A \circ A \circ A^{[\dagger]})^{[*]} = A + A^{[*]}$;
(12) $A^{[\dagger]} \circ A \circ A + (A^{[\dagger]} \circ A \circ A)^{[*]} = A + A^{[*]}$.

Theorem 2.2. *Let $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times n}$ and $A^{[*]} = A$. Then $A \circ B$ is J-*EP *if and only if A^*B is* EP.

Proof. If $A^{[*]} = A$ then $JA^*J = A$, i.e. $A^*J = JA$ and $JA^* = AJ$. Thus, we have $(A \circ B)^{[\dagger]} \circ A \circ B = A \circ B \circ (A \circ B)^{[\dagger]}$ if and only if $J(AJB)^\dagger AJB = AJB(AJB)^\dagger J$ if and only if $J(JA^*B)^\dagger JA^*B = JA^*B(JA^*B)^\dagger J$, which is equivalent to $J(A^*B)^\dagger A^*B = JA^*B(A^*B)^\dagger$. Now, by premultiplying this by J we get $(A^*B)^\dagger A^*B = A^*B(A^*B)^\dagger$.

This theorem can be proved in a much easier way, so we give the alternative proof: Under the hypothesis that $A^{[*]} = A$, we have $A \circ B$ is J-EP if and only if AJB is J-EP, which is equivalent to JA^*B is J-EP. Now, by Lemma 2.1 we get the equivalence with A^*B is EP. □

Theorem 2.3. *Let J commute with $A^\dagger A$. Then A is J-*EP *if and only if A is* EP.

Proof. Let J commute with $A^\dagger A$ and let A be a J-EP matrix. Then $A^\dagger A = A^\dagger AJJ = JA^\dagger AJ = JA^\dagger JJAJ = A^{[\dagger]} \circ AJ = A \circ A^{[\dagger]}J = AA^\dagger$. Thus, A is an EP-matrix.

Conversely, let J commute with $A^\dagger A$ and let A be an EP matrix. Now, we have $A \circ A^{[\dagger]} = AJJA^\dagger J = AA^\dagger J = A^\dagger AJ = JA^\dagger A = A^{[\dagger]} \circ A$, proving that A is a J-EP matrix. □

Also, we have an analogous theorem.

Theorem 2.4. *Let J commute with $AA^\dagger$. Then A is J-*EP *if and only if A is* EP.

The conditions from the previous theorems are weaker than those in Theorem 3.7, [7]. We show that by the next theorem and example.

Theorem 2.5. *If $AJ = JA$ then $A^\dagger AJ = JA^\dagger A$ and $JAA^\dagger = AA^\dagger J$.*

Example 2.1. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. As A is an invertible matrix, $JA^\dagger A = A^\dagger AJ = J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. But neither A nor $A^\dagger$ commute with J.

We can find another condition that provides the equivalence between EP and J-EP matrices, given in the next theorem.

Theorem 2.6. *Let* $\mathrm{N}(AJ) = \mathrm{N}(A)$. *Then* A *is an* EP *matrix if and only if* A *is a* J-EP *matrix.*

P r o o f. From the condition $\mathrm{N}(AJ) = \mathrm{N}(A)$, taking the direct complements of both sides, we get $\mathrm{R}((AJ)^*) = \mathrm{R}(A^*)$.

Let A be a J-EP matrix. Then AJ is an EP matrix, so $\mathrm{R}((AJ)^*) = \mathrm{R}(AJ)$ and $\mathrm{N}((AJ)^*) = \mathrm{N}(AJ)$. Then $\mathrm{R}(A) = \mathrm{R}(AJ) = \mathrm{R}((AJ)^*) = \mathrm{R}(A^*)$ and $\mathrm{N}(A^*) = \mathrm{N}(JA^*) = \mathrm{N}((AJ)^*) = \mathrm{N}(AJ) = \mathrm{N}(A)$. Thus, A is an EP matrix.

The opposite direction can be shown similarly. $\square$

Of course, if $AJ = JA$ then $\mathrm{N}(AJ) = \mathrm{N}(A)$. But the opposite does not hold, so we relaxed the condition from [7], Theorem 3.7, (a).

The next two examples show that there are matrices that do not commute with a matrix J, which satisfy $\mathrm{N}(AJ) = \mathrm{N}(A)$ and are both EP-matrices and J-EP matrices or they are neither EP nor J-EP matrices, respectively.

Example 2.2. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $AJ = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $JA = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, so $AJ \neq JA$. As A and AJ are invertible matrices, we have $\mathrm{N}(AJ) = \mathrm{N}(A) = \{0\}$. Also $A^\dagger = A^{-1}$, so A is an EP and also a J-EP matrix.

Example 2.3. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $AJ = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = A$ and $JA = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, so $AJ \neq JA$. Also, $\mathrm{N}(AJ) = \mathrm{N}(A)$ and $A^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. A is neither an EP matrix, nor a J-EP matrix.

In Example 2.2 A is an invertible matrix. It is not surprising at all because every invertible matrix is both an EP and J-EP matrix and also $\mathrm{N}(AJ) = \mathrm{N}(A) = \{0\}$ holds. We are giving an example which shows that the matrix A does not have to be invertible.

Example 2.4. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. We have $AJ = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $JA = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, so $AJ \neq JA$. It is easy to see that $\mathrm{N}(AJ) = \mathrm{N}(A) = \mathrm{Span}\{(0,1,-1)^T\}$. Also, we have $A^\dagger = \frac{1}{4}\begin{pmatrix} 0 & 2 & 2 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{pmatrix}$. As $AA^\dagger = A^\dagger A = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$, the matrix A is EP.

By direct computation we have $A^{[\dagger]} \circ A = A \circ A^{[\dagger]} = \frac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{pmatrix}$, proving that A is also a J-EP matrix.

Theorem 2.7. *Let A be a J-idempotent J-EP matrix. Then A is* EP *if and only if A commutes with J.*

Proof. Let A be a J-idempotent J-EP matrix. Thus,

$$AJA = A \tag{1}$$

and

$$JA^{\dagger}A = AA^{\dagger}J. \tag{2}$$

By premultiplying and postmultiplying (2) by J we get $A^{\dagger}AJ = JAA^{\dagger}$. Now, postmultiplication by A gives $A^{\dagger}AJA = JAA^{\dagger}A$, which is equivalent to $A^{\dagger}A = JA$, as (1) holds.

On the other hand, if we premultiply (2) by AJ we get $AJJA^{\dagger}A = AJAA^{\dagger}J$ or $A = AA^{\dagger}J$, which is equivalent to $AJ = AA^{\dagger}$.

Thus, we have $AA^{\dagger} = A^{\dagger}A$ if and only if $AJ = JA$. □

To show that the condition $A^{[2]} = A$ cannot be dropped, we give the next example.

Example 2.5. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $AJA = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, so $A^{[2]} \neq A$. We can see that A is J-EP and A is an EP matrix, as $A^{\dagger} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $A^{[\dagger]} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

It is easy to see that $AJ = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $JA = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, so $AJ \neq JA$.

The following example shows that the condition that A is a J-EP matrix cannot be omitted, either.

Example 2.6. Let $A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then, $A \circ A = AJA = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} = A$ and $A^{\dagger} = \frac{1}{9}A$, so A is an EP-matrix. On the other hand $A^{[\dagger]} \circ A = \frac{1}{3}JA = \frac{1}{3}\begin{pmatrix} 2 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix}$ and $A \circ A^{[\dagger]} = \frac{1}{3}AJ = \frac{1}{3}\begin{pmatrix} 2 & -\sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$, which means that A is not a J-EP matrix.

Of course, $AJ \neq JA$, either.

We are familiar with the fact that $\mathrm{Ra}(A^{[*]}) = (\mathrm{Nu}(A))^{\perp}$, where the orthogonality is meant with respect to the standard inner product in $\mathbb{C}^n$. We also know that it is not true that $\mathrm{Ra}(A^{[*]}) = (\mathrm{Nu}(A))^{[\perp]}$. In [11], Theorem 2.5, it was shown that for any $n \times m$ real matrix $\mathrm{Ra}(I \circ A) = \mathrm{Nu}(A^{[*]})^{[\perp]}$ holds.

Theorem 2.8. *Let $A \in \mathbb{C}^{n\times n}$. Each two of the following statements imply the third one.*

(1) $\mathrm{Ra}(A^{[*]}) = (\mathrm{Nu}(A))^{[\perp]}$,

(2) *A is an* EP *matrix,*

(3) *A is a J-*EP *matrix.*

P r o o f. (1,2)$\Longrightarrow$(3) (and (1,3)$\Longrightarrow$(2)): Let $\mathrm{Ra}(A^{[*]}) = (\mathrm{Nu}(A))^{[\perp]}$. Then we have $\mathrm{R}(JA^*) = (\mathrm{N}(AJ))^{[\perp]}$. By [5], (2.2.3), we have $\mathrm{R}(JA^*) = J(\mathrm{N}(AJ))^{\perp}$. If we premultiply this equality by J, we get $\mathrm{R}(A^*) = (\mathrm{N}(AJ))^{\perp}$. We also know that $(\mathrm{N}(AJ))^{\perp} = \mathrm{R}((AJ)^*)$ so we finally get $\mathrm{R}(A^*) = \mathrm{R}((AJ)^*)$ and hence $\mathrm{N}(A) = \mathrm{N}(AJ)$. Now, by Theorem 2.6, we have that (3) (or (2)) holds.

(2,3)$\Longrightarrow$(1): Let A be both an EP and a J-EP matrix. Then we have

$$\mathrm{R}(A) = \mathrm{R}(A^*) \quad \text{and} \quad \mathrm{N}(A) = \mathrm{N}(A^*) \tag{3}$$

and also

$$\mathrm{R}(AJ) = \mathrm{R}((AJ)^*) \quad \text{and} \quad \mathrm{N}(AJ) = \mathrm{N}((AJ)^*). \tag{4}$$

Now, $\mathrm{Ra}(A^{[*]}) = \mathrm{R}(JA^*) = J\mathrm{R}(A^*) \overset{(3)}{=} J\mathrm{R}(A) = J\mathrm{N}(A^*)^{\perp} \overset{(4)}{=} J\mathrm{N}(AJ)^{\perp} = (\mathrm{N}(AJ))^{[\perp]} = \mathrm{Nu}(A)^{[\perp]}$. Thus (1) holds. $\square$

Theorem 2.9. *Let A, B be square matrices of the same size. If A commutes with JB or B commutes with AJ then $A \circ B$ is J-*EP *if and only if BA is* EP.

P r o o f. Let A commute with JB. Then, by Lemma 2.1, $A \circ B$ is J-EP if and only if $AJBJ$ is an EP matrix. This is equivalent to $JBAJ$ is EP. Now, by using Lemma 2.1 twice, we get that BA is EP. The rest of the proof is analogous. $\square$

In [7] the author gave some interesting properties concerning EP and J-EP matrices. One of them is a generalization of Theorem 1, [9].

Theorem 2.10. *Let $A_1, \ldots, A_m$ be J-*EP *matrices and let $A := A_1 + \ldots + A_m$. Suppose $\mathrm{Nu}(A) \subseteq \mathrm{Nu}(A_i)$ for each $i = 1, \ldots, m$ and $A_i \circ A_j = 0$ for $i \neq j$. Then A is J-*EP.

We show that the condition $A_i \circ A_j = 0$ for $i \neq j$ can be excluded, as well as that the equivalence holds true, not just the implication.

Theorem 2.11. *Let $A_1, \ldots, A_m$ be J-EP matrices. Then $A := A_1 + \ldots + A_m$ is J-EP if and only if $\mathrm{Nu}(A) \subseteq \mathrm{Nu}(A_i)$ for each $i = 1, \ldots, m$.*

Proof. Let $A_1, \ldots, A_m$ be J-EP matrices and let $A := A_1 + \ldots + A_m$ be J-EP, which is, by Lemma 2.1 equivalent to $A_1 J, \ldots, A_m J$ being EP matrices and $AJ := A_1 J + \ldots + A_m J$ being EP. According to Theorem 1, [9], this is equivalent to $N(AJ) \subseteq N(A_i J)$ for each $i = 1, \ldots, m$. It is clear that the last fact is equivalent to $\mathrm{Nu}(A) \subseteq \mathrm{Nu}(A_i)$. □

As the previous theorem, the next one gives necessary and sufficient conditions for sums of J-EP matrices being J-EP. Herein, we use the Theorem 1, [9] for EP matrices.

Theorem 2.12. *Let $A_1, \ldots, A_m$ be J-EP matrices. Then $A := A_1 + \ldots + A_m$ is J-EP if and only if $\mathrm{rank}(A_1\ A_2\ \ldots\ A_m)^{\mathrm{T}} = \mathrm{rank}(A)$.*

Proof. Let $A_1, \ldots, A_m$ be J-EP matrices, i.e., $A_i J$ is an EP matrix for every $i = 1, \ldots, m$. Then $AJ = A_1 J + \ldots + A_m J$ is EP if and only if $\mathrm{rank}(A_1 J\ A_2 J\ \ldots\ A_m J)^{\mathrm{T}} = \mathrm{rank}(AJ)$. It is clear that $\mathrm{rank}(A_1 J\ A_2 J\ \ldots\ A_m J)^{\mathrm{T}} = \mathrm{rank}((A_1\ A_2\ \ldots\ A_m)^{\mathrm{T}} J) = \mathrm{rank}(A_1\ A_2\ \ldots\ A_m)^{\mathrm{T}}$ and $\mathrm{rank}(AJ) = \mathrm{rank}(A)$, which completes the proof. □

3. The reverse order law

In the sequel, we give some new results for the reverse order law with respect to the Moore-Penrose inverse in the indefinite setting.

As is well known, and can be found in [2], for matrices P and Q such that PQ exists, $(PQ)^\dagger = Q^\dagger P^\dagger$ if and only if $R(P^* PQ) \subseteq R(Q)$ and $R(QQ^* P^*) \subseteq R(P^*)$.

Theorem 3.1. *If $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$ and $\mathrm{Ra}(A^{[*]}) = \mathrm{Ra}(B)$, then $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$.*

Proof. We have $\mathrm{Ra}(A^{[*]}) = \mathrm{Ra}(JA^*J) = \mathrm{R}(JA^*)$ and $\mathrm{Ra}(B) = \mathrm{R}(BJ)$. From the condition of the theorem we get $\mathrm{R}(JA^*) = \mathrm{R}(BJ)$. We have $\mathrm{R}((AJ)^*AJBJ) \subseteq \mathrm{R}((AJ)^*) = \mathrm{R}(JA^*) = \mathrm{R}(BJ)$ and $\mathrm{R}(BJ(BJ)^*(AJ)^*) \subseteq \mathrm{R}(BJ) = \mathrm{R}(JA^*) = \mathrm{R}((AJ)^*)$. Hence, $(AJBJ)^\dagger = (BJ)^\dagger (AJ)^\dagger$. Finally, we get $(A \circ B)^{[\dagger]} = (AJB)^{[\dagger]} = J(AJB)^\dagger J = J^\dagger (AJB)^\dagger J = (AJBJ)^\dagger J = (BJ)^\dagger (AJ)^\dagger J = JB^\dagger JA^\dagger J = B^{[\dagger]} \circ A^{[\dagger]}$. □

Theorem 3.2. *If $A \in \mathbb{C}^{n\times n}$ is a J-*EP *matrix and $B \in \mathbb{C}^{n\times n}$ is an* EP *matrix and if* $\mathrm{R}(A) = \mathrm{R}(B)$, *then* $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$.

Proof. Let A be J-EP. Then AJ is an EP matrix. Further on, we have $\mathrm{R}(AJ) = \mathrm{R}(A) = \mathrm{R}(B)$. Since B is an EP-matrix, by Theorem 7.2.4, Chapter 7, [3], we get $(AJB)^{\dagger} = B^{\dagger}(AJ)^{\dagger}$.

Now we have $(A \circ B)^{[\dagger]} = (AJB)^{[\dagger]} = J(AJB)^{\dagger}J = JB^{\dagger}(AJ)^{\dagger}J = JB^{\dagger}JA^{\dagger}J = B^{[\dagger]} \circ A^{[\dagger]}$. □

Remark 3.1. In [7], Theorem 3.14 the matrix B was J-EP. We recall that there are EP matrices that are not J-EP and vice versa.

We can show that the previous statement does not depend of the J-EP-ness or the EP-ness of the matrix $B \in \mathbb{C}^{n\times n}$. The next theorem proves it.

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$ be a J-*EP *matrix, and $B \in \mathbb{C}^{n\times n}$ a matrix such that* $\mathrm{R}(A) = \mathrm{R}(B)$. *Then* $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$.

Proof. Let $\mathrm{R}(A) = \mathrm{R}(B)$ and let A be a J-EP matrix. That means that $JA^{\dagger}A = AA^{\dagger}J$. Then we have

$$\begin{aligned}\mathrm{R}((AJ)^*AJB) &\subseteq \mathrm{R}(JA^*) = \mathrm{R}(JA^{\dagger}AA^*) \subseteq \mathrm{R}(JA^{\dagger}A)\\ &= \mathrm{R}(AA^{\dagger}J) = \mathrm{R}(AA^{\dagger}) = \mathrm{R}(A) = \mathrm{R}(B)\end{aligned}$$

and

$$\begin{aligned}\mathrm{R}(BB^*(AJ)^*) \subseteq \mathrm{R}(B) &= \mathrm{R}(A) = \mathrm{R}(AA^{\dagger}J) = \mathrm{R}(JA^{\dagger}A) = \mathrm{R}(J(A^{\dagger}A)^*)\\ &= \mathrm{R}(JA^*(A^{\dagger})^*) \subseteq \mathrm{R}(JA^*) = \mathrm{R}((AJ)^*),\end{aligned}$$

so the well-known condition for the reverse order law is satisfied. Thus we have $(AJB)^{\dagger} = B^{\dagger}(AJ)^{\dagger}$ and $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$. □

We give an illustration of that by the next example.

Example 3.1. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. As we can verify easily, B is not an EP matrix, and it is not a J-EP matrix, either. $A^{\dagger} = \frac{1}{4}A$, so A is J-EP. We have that $\mathrm{R}(A) = \mathrm{R}(B)$, as well. Further on, $A \circ B = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$, $(A \circ B)^{[\dagger]} = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $B^{[\dagger]} \circ A^{[\dagger]} = \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, i.e., $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$.

Remark 3.2. Actually, it can be shown that in Theorem 7.2.4 in [3] we do not need the condition that B is an EP operator.

Corollary 3.1. *Let A and B be J-EP matrices of the same size with $\mathrm{R}(A) = \mathrm{R}(B)$. Then $A \circ B$ is a J-EP matrix.*

Proof. The condition of Theorem 3.14, [7] are satisfied so we have that $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$. Obviously, this is equivalent to $(AJBJ)^{\dagger} = (BJ)^{\dagger}(AJ)^{\dagger}$. Now, by Corollary 2 in [6], we get that $AJBJ$ is an EP matrix, which is by Lemma 2.1 equivalent to $A \circ B$ being a J-EP matrix. □

From Theorem 3.2 it is clear that this implication holds true also for an EP matrix B. That can be proved by appropriate changes in the previous proof.

In [7] there is a theorem which gives a necessary and sufficient condition for the reverse order law in indefinite product. We give that theorem here.

Theorem 3.4 (Theorem 3.17, [7])**.** *Let A be such that $AJ = JA$. Then $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$ if and only if $A^*A \circ BB^*$ is J-EP.*

In the sequel we use the result from [1], saying that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if A^*ABB^* is range-Hermitian. That means that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if A^*ABB^* is an EP matrix.

We give an analogous result for the indefinite matrix product, proving that the assumption of commutativity of A and J in the previous theorem can be omitted.

Theorem 3.5. *Let $A, B \in \mathbb{C}^{n\times n}$. Then $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$ if and only if $A^*A \circ BB^*$ is a J-EP matrix.*

Proof. It is obvious that $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$ is equivalent to $J(AJB)^{\dagger} = JB^{\dagger}JA^{\dagger}$, i.e., $(AJBJ)^{\dagger} = (BJ)^{\dagger}(AJ)^{\dagger}$. From [2], Ex. 55 and [1], we have that it is equivalent to $(AJ)^*AJBJ(BJ)^*$ being range-Hermitian, i.e., JA^*AJBB^* is a range-Hermitian matrix. That means that JA^*AJBB^* is an EP matrix. Now, by Lemma 2.1 this is equivalent to $A^*A \circ BB^*$ being a J-EP matrix. □

The next example illustrates that.

Example 3.2. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. As we see, $AJ \neq JA$. Also, $A^*A \circ BB^* = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, which is a Hermitian and so a range-Hermitian matrix. We have that $A \circ B = A$, $(A \circ B)^{\dagger} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $(A \circ B)^{[\dagger]} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $B^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A^{\dagger} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. So, we get $B^{[\dagger]} \circ A^{[\dagger]} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. Thus, $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$.

Theorem 3.6. *Let A, B, C be square matrices of the same size such that $AJ = JA$, and let A^*ABB^* and $(ABJ)^*ABJCC^*$ be* EP*-matrices. Then $(A \circ B \circ C)^{[\dagger]} = C^{[\dagger]} \circ B^{[\dagger]} \circ A^{[\dagger]}$.*

P r o o f. Since A^*ABB^* is EP, so it is range-Hermitian and by [1] it follows that $(AB)^\dagger = B^\dagger A^\dagger$. Similarly, $(ABJ)^*ABJCC^*$ is an EP-matrix implies that $(ABJC)^\dagger = C^\dagger (ABJ)^\dagger$. Now, using $AJ = JA$, we have $(A \circ B \circ C)^{[\dagger]} = J(AJBJC)^\dagger J = J(JABJC)^\dagger J = J(ABJC)^\dagger = JC^\dagger (ABJ)^\dagger = JC^\dagger J(AB)^\dagger = JC^\dagger JB^\dagger A^\dagger JJ = C^{[\dagger]} \circ B^{[\dagger]} \circ A^{[\dagger]}$. $\square$

In the previous theorem, we did not have the condition $(A \circ B)J = J(A \circ B)$, as was the case in Theorem 3.20, [7], but we had the new condition $(ABJ)^*ABJCC^*$ is EP instead of $(AB)^*ABCC^*$ is EP.

4. The star partial ordering

There are several types of matrix partial orderings defined on $\mathbb{C}^{n\times m}$. One of them, the star ordering, was introduced by Drazin in [4] in the following way: If $A, B \in \mathbb{C}^{n\times m}$, then

$$A \overset{*}{\leqslant} B \Leftrightarrow A^*A = A^*B \text{ and } AA^* = BA^*.$$

S. Jayaraman in [7] defined a star ordering with respect to the indefinite matrix product as $A \overset{[*]}{\leqslant} B \Leftrightarrow A^{[*]} \circ A = A^{[*]} \circ B$ and $A \circ A^{[*]} = B \circ A^{[*]}$, and showed that this is equivalent to the original star ordering of the matrices A and B. Also, he gave a generalization of Theorem 5.4.3, [10], showing that under the assumption that $A \overset{*}{\leqslant} B$, we have that A is a J-EP matrix if and only if $A^{[\dagger]} \circ B = B \circ A^{[\dagger]}$. But he did not show the equivalence with $A \circ B^{[\dagger]} = B^{[\dagger]} \circ A$.

Now we give a full generalization of Theorem 5.4.3, [10].

Theorem 4.1. *Let A and B be square matrices of the same order such that $A \overset{*}{\leqslant} B$. Then the following statements are equivalent:*

(1) *A is J-*EP*;*
(2) *$A^{[\dagger]} \circ B = B \circ A^{[\dagger]}$;*
(3) *$A \circ B^{[\dagger]} = B^{[\dagger]} \circ A$.*

P r o o f. First we show the equivalence of $A \overset{*}{\leqslant} B$ and $JA \overset{*}{\leqslant} JB$.

$$(JA)^*JA = (JA)^*JB \text{ and } JA(JA)^* = JB(JA)^*$$

is equivalent to $A^*A = A^*B$ and $JAA^*J = JBA^*J$, i.e.,

$$A^*A = A^*B \quad \text{and} \quad AA^* = BA^*.$$

Now, we have $A^{[\dagger]} \circ B = B \circ A^{[\dagger]}$ if and only if $JA^\dagger B = BA^\dagger J$. Premultiplication by J gives us the equivalence with $(JA)^\dagger JB = JB(JA)^\dagger$. This fact, $JA \overset{*}{\leqslant} JB$ and Theorem 5.4.3, [10] give the equivalence with $(JB)^\dagger JA = JA(JB)^\dagger$ and with AJ is EP. After some calculation we get $B^{[\dagger]} \circ A = A \circ B^{[\dagger]}$ and A is J-EP. □

This theorem also relaxed the conditions of Theorem 4.4 in [7], where the author showed that $A^{[\dagger]} \circ B = B \circ A^{[\dagger]}$ implies $B^{[\dagger]} \circ A = A \circ B^{[\dagger]}$, under the following assumptions: $AJ = JA$, $BJ = JB$, $AB^* = B^*A$ and $A \overset{*}{\leqslant} B$. We proved that we do not need the first three conditions and that even the equivalence holds true.

Corollary 4.1. *If A and B are square matrices of the same order such that $A \overset{*}{\leqslant} B$ and A commutes with J, then the following conditions are equivalent:*

(i) *A is J-EP;*
(ii) *A is EP;*
(iii) $A^{[\dagger]} \circ B = B \circ A^{[\dagger]}$;
(iv) $A^\dagger B = BA^\dagger$;
(v) $A \circ B^{[\dagger]} = B^{[\dagger]} \circ A$;
(vi) $AB^\dagger = B^\dagger A$.

P r o o f. The proof follows directly from the equivalence of EP and J-EP matrices when they commute with the matrix J, and from Theorem 4.1. □

The equivalence from the previous corollary does not hold without the assumption of A and B being in the star order. The next example illustrates it.

Example 4.1. Let $A = J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then we have $AJ = JA = I$, so A and J commute. $A^{[\dagger]} \circ B = B$ and $B \circ A^{[\dagger]} = B$. Hence, $A^{[\dagger]} \circ B = B \circ A^{[\dagger]}$, but $A^\dagger B = B$ and $BA^\dagger = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, so $A^\dagger B \neq BA^\dagger$. Notice that $A \overset{*}{\nleqslant} B$.

On the other hand, it can happen that $A^\dagger B = BA^\dagger$ but $A^{[\dagger]} \circ B \neq B \circ A^{[\dagger]}$. Take $A = I$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Corollary 4.2. *If A and B are square matrices of the same order, then the following statements are equivalent:*

(1) $A \overset{*}{\leqslant} B$;

(2) $A^{\dagger} \overset{*}{\leqslant} B^{\dagger}$;
(3) $AA^{\dagger}B = BA^{\dagger}A = BA^{\dagger}B = A$;
(4) $A^{\dagger}AB^{\dagger} = B^{\dagger}AA^{\dagger} = B^{\dagger}AB^{\dagger} = A^{\dagger}$;
(5) $A \overset{[*]}{\leqslant} B$;
(6) $A^{[\dagger]} \overset{[*]}{\leqslant} B^{[\dagger]}$;
(7) $A \circ A^{[\dagger]} \circ B = B \circ A^{[\dagger]} \circ A = B \circ A^{[\dagger]} \circ B = A$;
(8) $A^{[\dagger]} \circ A \circ B^{[\dagger]} = B^{[\dagger]} \circ A \circ A^{[\dagger]} = B^{[\dagger]} \circ A \circ B^{[\dagger]} = A^{[\dagger]}$.

Proof. The equivalence from (1) to (4) was shown in Corollary 5.2.9 in [10]. The rest of the proof follows by the fact that $A \overset{*}{\leqslant} B$, $A \overset{[*]}{\leqslant} B$ and $AJ \overset{*}{\leqslant} BJ$ are equivalent. $\square$

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