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COUNTING GRAPHS WITH DIFFERENT NUMBERS OF SPANNING TREES THROUGH THE COUNTING OF PRIME PARTITIONS

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Abstract. Let A_n $(n \geqslant 1)$ be the set of all integers x such that there exists a connected graph on n vertices with precisely x spanning trees. It was shown by Sedláček that $|A_n|$ grows faster than the linear function. In this paper, we show that $|A_n|$ grows faster than $\sqrt{n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}$ by making use of some asymptotic results for prime partitions. The result settles a question posed in J. Sedláček, On the number of spanning trees of finite graphs, Čas. Pěst. Mat., 94 (1969), 217–221.

Keywords: number of spanning trees; asymptotic; prime partition

MSC 2010: 05A16, 05C35

1. Introduction

J. Sedláček is regarded as one of the pioneers of Czech graph theory. He devoted much of his work to the study of subjects related to the number of spanning trees $\tau(G)$ of a graph G. In [7] he studied the function $\alpha(n)$ defined as the least number k for which there exists a graph on k vertices having precisely n spanning trees. He showed that for every $n > 6$, we have

$$\alpha(n) \leqslant \begin{cases} \dfrac{n+6}{3} & \text{if } n \equiv 0 \pmod 3, \\ \dfrac{n+4}{3} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Azarija and Škrekovski [1] later found out that if $n > 25$ then

$$\alpha(n) \leqslant \begin{cases} \dfrac{n+4}{3} & \text{if } n \equiv 2 \pmod 3, \\ \dfrac{n+9}{4} & \text{otherwise.} \end{cases}$$

Sedláček continued to study quantities related to the function τ. In [6] and [8] he considered the set B_n^t defined as the set of integers such that $x \in B_n^t$ whenever there is a t-regular graph on n vertices with precisely x spanning trees. He showed that for odd integers $t \geqslant 3$, $\lim_{a\to\infty} |B_{2a}^t| = \infty$ and whenever $t \geqslant 4$ is an even integer, $\lim_{a\to\infty} |B_a^t| = \infty$. In [6] he also studied a more general set A_n defined as a set of numbers such that $x \in A_n$ whenever there exists a connected graph on n vertices having precisely x spanning trees. One could think about $|A_n|$ as the maximal number of connected graphs on n vertices with mutually different numbers of spanning trees. Using a simple construction, he has shown that

$$\lim_{n\to\infty} \frac{|A_n|}{n} = \infty$$

and remarked: *it is not clear how the fraction $|A_n|/n^2$ behaves when n tends to infinity.* In modern terminology, we could write his result as $|A_n| = \omega(n)$ since $f(n) = \omega(g(n))$ whenever $|f(n)| \geqslant c|g(n)|$ for every $c > 0$ and $n > n_0$ for an appropriately chosen n_0.

In this paper, we extend his work and show that $|A_n| = \omega\big(\sqrt{n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}\big)$. In order to prove the result we define the graph $C_{x_1,\ldots,x_k}$ as follows. Let $3 \leqslant x_1 \leqslant \ldots \leqslant x_k$ be integers. By $C_{x_1,\ldots,x_k}$ we denote the graph that is obtained after identifying a vertex from the disjoint cycles $C_{x_1}, \ldots, C_{x_k}$. Since $C_{x_1}, \ldots, C_{x_k}$ are the blocks of $C_{x_1,\ldots,x_k}$, it follows that

$$\tau(C_{x_1,\ldots,x_k}) = \prod_{i=1}^{k} x_i \quad \text{and} \quad |V(C_{x_1,\ldots,x_k})| = \sum_{i=1}^{k} x_i - k + 1.$$

We also introduce some number theoretical concepts. We say that $\langle x_1, \ldots, x_k \rangle$ is a *partition* of n with integer *parts* $1 \leqslant x_1 \leqslant \ldots \leqslant x_k$ if $\sum_{i=1}^{k} x_i = n$. The study of partitions covers an extensive part of the research done in combinatorics and number theory. If we denote by $p(n)$ the number of partitions of n then the celebrated theorem of Hardy and Ramanujan [4] states that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \mathrm{e}^{\pi\sqrt{2n/3}}$$

or, equivalently,

$$\lim_{n\to\infty} \frac{p(n)}{\frac{1}{4n\sqrt{3}} \mathrm{e}^{\pi\sqrt{2n/3}}} = 1.$$

Since then, asymptotics for many types of partition functions have been studied [2]. Of interest in our paper is the function $p_p(n)$ which we define as the number of partitions of n into prime parts. In [5] Roth and Szekeres presented a theorem which can be used to derive the following asymptotic relation for p_p:

$$p_p(n) \sim \mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}.$$

Somehow surprising is the fact that if we disallow a constant number of primes as parts, the same asymptotic relation still holds. More specifically, if $p_{op}(n)$ is the number of partitions into odd primes then

$$p_{op}(n) \sim p_p(n).$$

2. Main result

In this section we prove that $|A_n| = \omega\big(\sqrt{n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}\big)$ by showing that

$$\lim_{n\to\infty} \frac{|A_n|}{\sqrt{n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}} = \infty.$$

We do so by estabilishing a lower bound for the number of partitions whose parts are only odd primes and whose sum is less than or equal to a given number n.

Lemma 2.1. *Let P_n be the set of all partitions $\langle x_1, \ldots, x_k\rangle$ with $\sum_{i=1}^{k} x_i \leqslant n$, all the numbers $x_1, \ldots, x_k$ being odd primes. Then there exists an n_0 such that for all $n \geqslant n_0$*

$$|P_n| \geqslant \frac{1}{4}\sqrt{n\log n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}.$$

Proof. Let n_1 be such a positive integer so that for all $n \geqslant n_1$

$$p_{op}(n) \geqslant \frac{1}{2}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}.$$

For $n \geqslant n_1$ we then have

$$|P_n| = \sum_{i=3}^{n} p_{op}(i) \geqslant \frac{1}{2}\sum_{i=n_1}^{n} \mathrm{e}^{(2\pi/\sqrt{3})\sqrt{i/\log i}} \geqslant \frac{1}{2}\int_{n_1-1}^{n} \mathrm{e}^{(2\pi/\sqrt{3})\sqrt{x/\log x}}\,\mathrm{d}x.$$

Showing that $\int_{n_1-1}^{n} \mathrm{e}^{(2\pi/\sqrt{3})\sqrt{x/\log x}}\,\mathrm{d}x \sim (\sqrt{3}/\pi)\sqrt{n\log n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}$ would imply the existence of a positive integer n_2 such that

$$\int_{n_1-1}^{n} \mathrm{e}^{(2\pi/\sqrt{3})\sqrt{x/\log x}}\,\mathrm{d}x \geqslant \frac{1}{2}\sqrt{n\log n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}$$

for all $n \geqslant n_2$. The statement of the theorem would then immediately follow for all $n \geqslant n_0$ where $n_0 = \max\{n_1, n_2\}$. To prove the last asymptotic identity we observe that it follows from L'Hospital's rule that

$$\lim_{n\to\infty} \frac{\int_{n_1-1}^{n} \mathrm{e}^{(2\pi/\sqrt{3})\sqrt{x/\log x}}\,\mathrm{d}x}{\frac{\sqrt{3}}{\pi}\sqrt{n\log n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}} = \lim_{n\to\infty} \frac{\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}}{\frac{\mathrm{d}}{\mathrm{d}n}\left(\frac{\sqrt{3}}{\pi}\sqrt{n\log n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}\right)} = 1.$$

□

Lemma 2.1 readily gives an asymptotic lower bound for $|A_n|$.

Theorem 2.1. $|A_n| = \omega\left(\sqrt{n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}\right)$.

Proof. Let P_n be defined in the same way as in the statement of Lemma 2.1. With every partition $\langle x_1, \ldots, x_k\rangle \in P_n$ with the sum $s = \sum_{i=1}^{k} x_i$ we associate the graph obtained after identifying a vertex from $C_{x_1,\ldots,x_k}$ with a vertex from the disjoint path P_{n-s+k}. Observe that the resulting graph has precisely n vertices and $\prod_{i=1}^{k} x_i$ spanning trees. Since all the parts in the partitions are primes it follows that any pair of graphs which were obtained from two different partitions in P_n have a different number of spanning trees. Thus

$$|A_n| \geqslant |P_n|$$

and therefore from Lemma 2.1 we know that for n large enough

$$|A_n| \geqslant \frac{1}{4}\sqrt{n\log n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}.$$

Since

$$\lim_{n\to\infty} \frac{\frac{1}{4}\sqrt{n\log n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}}{\sqrt{n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}} = \infty$$

it follows from the squeeze theorem that

$$\lim_{n\to\infty} \frac{|A_n|}{\sqrt{n}\,\mathrm{e}^{(2\pi/\sqrt{3})\sqrt{n/\log n}}} = \infty$$

from where the stated claim follows. □

Observe that all the graphs constructed in the proof of Theorem 2.1 contain a cut vertex. Since almost all graphs are 2-connected [3] it is reasonable to expect that there exists a construction of a class C_n of 2-connected graphs of order n with mutually different number of spanning trees such that $|C_n| = \omega(|P_n|)$. We believe the following statement to be true.

Conjecture 2.1. *For every number* $k > 0$

$$|A_n| = \omega(k^n).$$

In addition we leave as an open problem to evaluate the limit

$$\lim_{n\to\infty} \frac{|A_n|}{n^{n-2}}.$$

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