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PICONE'S IDENTITY FOR A FINSLER p-LAPLACIAN AND COMPARISON OF NONLINEAR ELLIPTIC EQUATIONS

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Abstract. In the paper we present an identity of the Picone type for a class of nonlinear differential operators of the second order involving an arbitrary norm H in $\mathbb{R}^n$ which is continuously differentiable for $x \neq 0$ and such that H^p is strictly convex for some $p > 1$. Two important special cases are the p-Laplacian and the so-called pseudo p-Laplacian. The identity is then used to establish a variety of comparison results concerning nonlinear degenerate elliptic equations which involve such operators. We also get criteria for the nonexistence of positive solutions in exterior domains for such equations by means of comparison with the equation exhibiting a kind of "anisotropic radial symmetry".

Keywords: Picone identity; Finsler p-Laplacian

MSC 2010: 35B05, 35J70

1. Introduction

Let $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with piecewise smooth boundary $\partial\Omega$. Fix $p \in (1, \infty)$ and consider an operator of the form

$$\Delta_{H,p} v := \operatorname{div}(A(x) H(\nabla v)^{p-1} \nabla_\xi H(\nabla v)) \tag{1.1}$$

where $A \in C^1(\overline{\Omega})$ with $A(x) > 0$ on $\overline{\Omega}$, $H\colon \mathbb{R}^n \to [0, \infty)$, $n \geqslant 2$, is a convex function of the class $C^1(\mathbb{R}^n \setminus \{0\})$ which is positively homogeneous of degree 1, and ∇ and ∇_ξ stand for the usual gradient operators with respect to the variables x and ξ, respectively. We refer to the operator $\Delta_{H,p}$ as the (weighted) Finsler p-Laplacian. A typical example of H satisfying the above conditions is the l_r-norm

$$H(\xi) = \|\xi\|_r = \left(\sum_{i=1}^{n} |\xi_i|^r \right)^{1/r}, \quad r > 1, \tag{1.2}$$

for which the operator defined by (1.1) has the form

$$\Delta_{r,p} v := \operatorname{div}(A(x)\|\nabla v\|_r^{p-r}\nabla^r v)$$

where

$$\nabla^r v := \Big(\Big|\frac{\partial v}{\partial x_1}\Big|^{r-2}\frac{\partial v}{\partial x_1}, \ldots, \Big|\frac{\partial v}{\partial x_n}\Big|^{r-2}\frac{\partial v}{\partial x_n}\Big). \tag{1.3}$$

Note that $\Delta_{r,p}$ is a nonlinear operator unless $p = r = 2$ when it reduces to the usual weighted Laplacian $\operatorname{div}(A\nabla v)$. Two important special cases are $r = 2$ and general $p \in (1,\infty)$ when $\Delta_{2,p}$ coincides with the usual p-Laplace operator and the case $r = p$ when $\Delta_{p,p}$ is the so-called pseudo p-Laplacian.

Various problems involving the general Finsler p-Laplacian $\Delta_{H,p}$ with $A \equiv 1$ have recently been studied by several authors including [2], [3], [4], [7], [8], [13] and [21], [22], [23].

In the linear case $p = 2$ and $H(\xi) = \|\xi\|_2$, $\xi \in \mathbb{R}^n$, the following simple formula is well known (see [17]):

Lemma 1.1 (Picone's identity). *If u, v and $A\nabla v$ are differentiable in a given domain $\Omega \subset \mathbb{R}^n$ and $v(x) \neq 0$ in Ω, then*

$$\operatorname{div}\Big(\frac{u^2}{v}A(x)\nabla v\Big) = \frac{u^2}{v}\operatorname{div}(A(x)\nabla v) + A(x)\|\nabla u\|_2^2 - A(x)\Big\|\nabla u - \frac{u}{v}\nabla v\Big\|_2^2. \tag{1.4}$$

The formula (1.4) is frequently used in the qualitative and comparison theory of linear differential equations and there are many extensions of (1.4) to more general linear elliptic operators (see for example [5], [19], [20] and [24]). A breakthrough in the "linear period" of the history of Picone's identity occured by the end of the 1990s, when several authors including Dunninger [12], Allegretto and Huang [1], Došlý and Mařík [10] and Kusano et al. [14] have independently extended (1.4) to the nonlinear p-Laplace operator $\Delta_p v := \operatorname{div}(\|\nabla v\|_2^{p-2}\nabla v)$. The generalized version of Lemma 1.1 reads as follows:

Lemma 1.2 (p-Laplace-Picone identity). *If u, v and $A\|\nabla v\|_2^{p-2}\nabla v$ are differentiable in a given domain $\Omega \subset \mathbb{R}^n$ and $v(x) \neq 0$ in Ω, then*

$$\begin{aligned}\operatorname{div}\Big(\frac{|u|^p}{|v|^{p-2}v}A(x)\|\nabla v\|_2^{p-2}\nabla v\Big) &= \frac{|u|^p}{|v|^{p-2}v}\operatorname{div}(A(x)\|\nabla v\|_2^{p-2}\nabla v) + A(x)\|\nabla u\|_2^p \\ &\quad - A(x)\Big\{\|\nabla u\|_2^p + (p-1)\frac{|u|^p}{|v|^p}\|\nabla v\|_2^p - p\frac{|u|^{p-2}u}{|v|^{p-2}v}\langle\|\nabla v\|_2^{p-2}\nabla v, \nabla u\rangle\Big\},\end{aligned} \tag{1.5}$$

where the bracketed expression, denoted by $\Phi(u,v)$, is a positive semidefinite form and $\Phi(u,v)=0$ if and only if u and v are proportional in Ω.

A prototype of results that can easily be obtained from (1.5) by integrating (1.5) over Ω and using the Gauss theorem asserts that the existence of a solution v of $\operatorname{div}(A\|\nabla v\|_2^{p-2}\nabla v)+C|v|^{p-2}v=0$ where $C\in C(\overline{\Omega})$ which satisfies $v(x)\neq 0$ in $\overline{\Omega}$ necessarily implies that

$$J[u;\Omega]:=\int_{\Omega}[A(x)\|\nabla u\|_2^p - C(x)|u|^p]\,\mathrm{d}x>0 \tag{1.6}$$

for all $u\in W_0^{1,p}(\Omega)\setminus\{0\}$.

Another standard result based on (1.5), or, more precisely, on its extended version

$$\operatorname{div}\Big(ua(x)\|\nabla u\|_2^{p-2}\nabla u-\frac{|u|^p}{|v|^{p-2}v}A(x)\|\nabla v\|_2^{p-2}\nabla v\Big)=u\operatorname{div}(a(x)\|\nabla u\|_2^{p-2}\nabla u)$$

$$-\frac{|u|^p}{|v|^{p-2}v}\operatorname{div}(A(x)\|\nabla v\|_2^{p-2}\nabla v)+(a(x)-A(x))\|\nabla u\|_2^p+A(x)\Phi(u,v) \tag{1.7}$$

where a satisfies the same conditions as A, is the comparison theorem of the Leighton type which says that if for some nontrivial solution of $\operatorname{div}(a\|\nabla u\|_2^{p-2}\nabla u)+c|u|^{p-2}u=0$ which satisfies $u=0$ on $\partial\Omega$, the condition

$$V[u;\Omega]:=\int_{\Omega}[(a(x)-A(x))\|\nabla u\|_2^p+(C(x)-c(x))|u|^p]\,\mathrm{d}x\geqslant 0, \tag{1.8}$$

is fulfilled, then any solution v of the majorant equation $\operatorname{div}(A\|\nabla v\|_2^{p-2}\nabla v)+C|v|^{p-2}v=0$ must have a zero in Ω unless v and u are linearly dependent (see [14]). The classical Sturm-Picone comparison theorem which assumes the validity of the pointwise inequalities $a(x)\geqslant A(x)$ and $C(x)\geqslant c(x)$ in Ω is clearly covered by the above result.

In [6] and [9], the classical Picone's identity (1.4) has been generalized to another nonlinear differential operator, namely, to the so called *pseudo p-Laplacian* defined by $\widetilde{\Delta}_p v:=\operatorname{div}(\nabla^p v)$ where $\nabla^p v$ is given by (1.3). The pseudo p-Laplace generalization of (1.4) reads as follows.

Lemma 1.3. *If u, v and $A\nabla^p v$ are differentiable in a given domain $\Omega \subset \mathbb{R}^n$ and $v(x) \neq 0$ in Ω, then*

$$\operatorname{div}\Big(\frac{|u|^p}{|v|^{p-2}v}A(x)\nabla^p v\Big) = \frac{|u|^p}{|v|^{p-2}v}\operatorname{div}(A(x)\nabla^p v) + A(x)\|\nabla u\|_p^p - A(x)\widetilde{\Phi}(u,v) \tag{1.9}$$

where

$$\widetilde{\Phi}(u,v) := \|\nabla u\|_p^p + (p-1)\frac{|u|^p}{|v|^p}\|\nabla^p v\|_q^q - p\frac{|u|^{p-2}u}{|v|^{p-2}v}\langle \nabla u, \nabla^p v\rangle$$

and $q = p/(p-1)$ is the Hölder conjugate exponent of p. Moreover, the form $\widetilde{\Phi}(u,v)$ is positive semidefinite and the equality $\widetilde{\Phi}(u,v) = 0$ occurs if and only if u and v are linearly dependent in Ω.

As before, integration of (1.9) over Ω and the use of the divergence theorem leads to a variety of results concerning nonlinear equations involving pseudo p-Laplacian analogous to the necessary condition for the existence of positive solutions (1.6) and the integral comparison criterion (1.8).

The purpose of this paper is to generalize the identities (1.5) and (1.9) to the case where the particular norms $\|\cdot\|_2$ and $\|\cdot\|_p$ are replaced by an arbitrary norm $H(\cdot)$ in $\mathbb{R}^n$ which is of class C^1 for $x \neq 0$ and such that H^p is a strictly convex function, and to obtain the Leighton-type comparison result concerning a pair of nonlinear degenerate elliptic equations of the form

$$\operatorname{div}(a(x)H(\nabla u)^{p-1}\nabla_\xi H(\nabla u)) + c(x)|u|^{p-2}u = 0 \tag{1.10}$$

and

$$\operatorname{div}(A(x)H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)) + C(x)|v|^{p-2}v = 0 \tag{1.11}$$

where a, c, A, C and H are as above.

The outline of the paper is the following. In Section 2 we recall some of the properties of general norms in $\mathbb{R}^n$. Section 3 contains an extension of Picone's identity to the Finsler p-Laplace operator and illustrates its use in the comparison theory of nonlinear elliptic equations involving such operators. In Section 4 we compare the equation (1.11) with another equation of the same form exhibiting a kind of "anisotropic radial symmetry" and get criteria for the nonexistence of positive solutions in exterior domains.

2. Preliminaries

In this section we survey some of the elementary properties of general norms in $\mathbb{R}^n$ which will be needed in the sequel. For the proofs see for instance [3] or [7].

Let H be an arbitrary norm in $\mathbb{R}^n$, i.e., a convex function $H\colon \mathbb{R}^n \to [0,\infty)$ satisfying $H(\xi) > 0$ for all $\xi \neq 0$ which is positively homogeneous of degree 1, so that

$$H(t\xi) = |t|H(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } t \in \mathbb{R}. \tag{2.1}$$

Since all norms in $\mathbb{R}^n$ are equivalent, for H there exist positive constants α and β such that

$$\alpha\|\xi\|_2 \leqslant H(\xi) \leqslant \beta\|\xi\|_2$$

for all $\xi \in \mathbb{R}^n$. Let $\langle , \rangle$ denote the usual inner product in $\mathbb{R}^n$ and define the dual norm H_0 of H by

$$H_0(x) = \sup_{\xi\neq 0} \frac{\langle x, \xi\rangle}{H(\xi)} \quad \text{for } x \in \mathbb{R}^n. \tag{2.2}$$

For example, if H is the l_r-norm given by (1.2) and $s > 1$ is such that $1/r + 1/s = 1$, then $H_0(x) = \|x\|_s$. In particular, the Euclidean norm $\|\cdot\|_2$ is self-dual.

The unit H_0-ball, i.e., the set $K = \{x \in \mathbb{R}^n\colon\ H_0(x) \leqslant 1\}$ is sometimes called the *Wulff shape* (or *equilibrium crystal shape*) of H.

If we assume that $H \in C^1(\mathbb{R}^n \setminus \{0\})$, then (2.1) yields that

$$\nabla_\xi H(t\xi) = \operatorname{sgn} t\, \nabla_\xi H(\xi) \quad \text{for all } \xi \neq 0 \text{ and } t \neq 0 \tag{2.3}$$

and

$$\langle \xi, \nabla_\xi H(\xi)\rangle = H(\xi) \quad \text{for all } \xi \in \mathbb{R}^n \tag{2.4}$$

where the left-hand side is defined to be 0 if $\xi = 0$. Moreover,

$$H_0(\nabla_\xi H(\xi)) = 1 \quad \text{for all } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.5}$$

Similarly, if H_0 is continuously differentiable for $x \neq 0$, then

$$H(\nabla H_0(x)) = 1 \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{2.6}$$

Also, the identities

$$H[H_0(x)\nabla H_0(x)]\nabla_\xi H[H_0(x)\nabla H_0(x)] = x, \tag{2.7}$$

and

$$H_0[H(\xi)\nabla_\xi H(\xi)]\nabla H_0[H(\xi)\nabla_\xi H(\xi)] = \xi, \tag{2.8}$$

hold for all $x, \xi \in \mathbb{R}^n$, where $H(0)\nabla_\xi H(0)$ and $H_0(0)\nabla H_0(0)$ are defined to be 0.

From (2.2) we easily obtain the Hölder-type inequality

$$\langle x, \xi\rangle \leqslant H(\xi)H_0(x) \quad \text{for all } x, \xi \in \mathbb{R}^n \tag{2.9}$$

with equality holding if and only if

$$x = H(\xi)\nabla_\xi H(\xi) \quad (\text{or, equivalently, } H_0(x) = H(\xi)). \tag{2.10}$$

Remark 2.1. There is a close relationship between the dual H_0 of an arbitrary norm H and the so-called Fenchel transform (or conjugate) of H defined by

$$H^*(x) := \sup_{\xi\in\mathbb{R}^n} \{\langle x, \xi\rangle - H(\xi)\}.$$

More precisely, if we define the indicator function of a nonempty set $C \subset \mathbb{R}^n$ by $\mathbb{I}_C(x) = 0$ if $x \in C$ and $\mathbb{I}_C(x) = \infty$ if $x \notin C$, then $H^*(x) = \mathbb{I}_K(x)$, i.e., the Fenchel conjugate of H is nothing else than the indicator function of the Wulff ball K. For more details see [18].

In the proof of our main result we will also need the following lemma which is a consequence of the well-known result asserting that a continuously differentiable function F defined in an open convex subset of $\mathbb{R}^n$ is strictly convex if and only if

$$F(y) - F(x) - \langle\nabla F(x), y - x\rangle > 0 \tag{2.11}$$

for all $x \neq y$.

Lemma 2.1. *Let H be a norm in $\mathbb{R}^n$ such that $H \in C^1(\mathbb{R}^n \setminus \{0\})$ and H^p, $1 < p < \infty$, is strictly convex. If*

$$H(\xi)^p + (p-1)H(\eta)^p - p\langle\xi, H(\eta)^{p-1}\nabla H(\eta)\rangle = 0 \tag{2.12}$$

for some $\xi, \eta \in \mathbb{R}^n$, $\eta \neq 0$, and $H(\xi) = H(\eta)$, then $\xi = \eta$.

Proof. Given any $\xi, \eta \in \mathbb{R}^n$ with $\eta \neq 0$ satisfying $H(\xi) = H(\eta)$ and (2.12), we obtain

$$\begin{aligned} 0 &= pH(\eta)^p - p\langle\eta, H(\eta)^{p-1}\nabla H(\eta)\rangle + p\langle\eta - \xi, H(\eta)^{p-1}\nabla H(\eta)\rangle \\ &= p\langle\eta - \xi, H(\eta)^{p-1}\nabla H(\eta)\rangle. \end{aligned} \tag{2.13}$$

Notice that $pH(\eta)^{p-1}\nabla H(\eta) = \nabla(H(\eta))^p \neq 0$. Indeed, if $\nabla(H(\eta))^p$ were the zero vector for some $\eta \in \mathbb{R}^n$, i.e., the even strictly convex function $H(\eta)^p$ attained its global minimum at η, then η would necessarily be equal to 0, a contradiction. Therefore, by strict convexity of H^p, $\xi = \eta$, and the proof is complete. $\square$

The next lemma contains another simple norm inequality which will be used in establishing our results. It can easily be obtained from the multivariate mean value theorem combined with the generalized Hölder inequality (2.9).

Lemma 2.2. *If H is an arbitrary norm in $\mathbb{R}^n$ which is of class C^1 for $x \neq 0$, then*

$$|H(\eta)^p - H(\xi)^p| \leqslant p[H(\xi) + H(\eta)]^{p-1} H(\eta - \xi) \tag{2.14}$$

for any $\xi, \eta \in \mathbb{R}^n$.

Proof. From the mean value theorem applied to the function H^p it follows that for any $\xi, \eta \in \mathbb{R}^n$ there exists $c \in (0,1)$ such that

$$H(\eta)^p - H(\xi)^p = p\langle H((1-c)\xi + c\eta)^{p-1}\nabla_\xi H((1-c)\xi + c\eta), \eta - \xi\rangle. \tag{2.15}$$

Using (2.9) and the properties of the norm H, we get

$$\begin{aligned}
|H(\eta)^p - H(\xi)^p| &\leqslant pH_0[H((1-c)\xi + c\eta)^{p-1}\nabla_\xi H((1-c)\xi + c\eta)]H(\eta - \xi) \\
&= pH((1-c)\xi + c\eta)^{p-1} H(\eta - \xi) \\
&\leqslant p[H((1-c)\xi) + H(c\eta)]^{p-1} H(\eta - \xi) \\
&\leqslant p[H(\xi) + H(\eta)]^{p-1} H(\eta - \xi).
\end{aligned}$$

$\square$

3. Finsler-Picone identity and comparison theorems

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a piecewise smooth boundary. The following is an extension of Picone's identity (1.4) to the Finsler p-Laplace operator $\Delta_{H,p}$ given by (1.1).

Theorem 3.1. *Let H be an arbitrary norm in $\mathbb{R}^n$ which is of class C^1 for $x \neq 0$. Assume that u, v and $AH(\nabla v)^{p-1}\nabla_\xi H(\nabla v)$ are differentiable in a given domain Ω and $v(x) \neq 0$ in Ω. Denote*

$$\Phi(u,v) := H(\nabla u)^p + (p-1)\frac{|u|^p}{|v|^p} H(\nabla v)^p - p\frac{|u|^{p-2}u}{|v|^{p-2}v}\langle \nabla u, H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\rangle.$$

Then

$$\text{(3.1)}\quad \operatorname{div}\Big(\frac{|u|^p}{|v|^{p-2}v}A(x)H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)\Big)$$
$$= \frac{|u|^p}{|v|^{p-2}v}\operatorname{div}(A(x)H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)) + A(x)H(\nabla u)^p - A(x)\Phi(u,v).$$

Moreover, $\Phi(u,v) \geqslant 0$ in Ω and if, in addition, $H(\xi)^p$ is strictly convex in $\mathbb{R}^n$, then $\Phi(u,v) = 0$ in Ω if and only if u is a constant multiple of v in each component of Ω.

Proof. The relation (3.1) can be verified by a routine differentiation. To prove the positive semidefiniteness of the form $\Phi(u,v)$ notice that it can be rewritten as $\Phi(u,v) = \Phi_1(u,v) + \Phi_2(u,v)$, where

$$\Phi_1(u,v) := H(\nabla u)^p - pH(\nabla u)H\Big(\frac{u}{v}\nabla v\Big)^{p-1} + (p-1)H\Big(\frac{u}{v}\nabla v\Big)^p$$

and

$$\Phi_2(u,v) := p\Big[H(\nabla u)H\Big(\frac{u}{v}\nabla v\Big)^{p-1} - \Big\langle \nabla u, H\Big(\frac{u}{v}\nabla v\Big)^{p-1}\nabla_\xi H\Big(\frac{u}{v}\nabla v\Big)\Big\rangle\Big].$$

Now, the nonnegativity of $\Phi_1(u,v)$ is an immediate consequence of Young's inequality in the form $a^p - pab^{p-1} + (p-1)b^p \geqslant 0$, $a \geqslant 0$, $b \geqslant 0$, while $\Phi_2(u,v) \geqslant 0$ follows from the generalized Hölder inequality (2.9).

The equality case in $\Phi(u,v) \geqslant 0$ means that both $\Phi_1(u,v) = 0$ and $\Phi_2(u,v) = 0$ in Ω. The first equality is attained if and only if

$$\text{(3.2)}\qquad H(\nabla u) = H\Big(\frac{u}{v}\nabla v\Big) \quad \text{in } \Omega.$$

Denote

$$S := \{x \in \Omega\colon\ \Phi(u,v) = 0\}.$$

If $(u\nabla v/v)(x_0) \neq 0$ for some $x_0 \in S$, then by Lemma 2.1 we have $\nabla u = u\nabla v/v$ at x_0, or, equivalently, $\nabla(u/v)(x_0) = 0$. On the other hand, if $u\nabla v/v = 0$ on some subset S_0 of S, then $\nabla u = 0$ in S_0, which implies $\nabla(u/v) = 0$ in S_0. Summarizing the above facts we get $\nabla(u/v) = 0$ in Ω which forces u/v to be constant in each component of Ω. $\square$

In the particular case when $H(\xi)$ is an r-norm (1.2), the identity (3.1) specializes as follows.

Corollary 3.1. *Assume that u, v and $A(x)\|\nabla v\|_r^{p-r}\nabla^r v$ are differentiable in a given domain Ω and $v(x)\neq 0$ in Ω. Then*

$$
\begin{aligned}
(3.3)\qquad \operatorname{div}\Bigl(\frac{|u|^p}{|v|^{p-2}v}A(x)\|\nabla v\|_r^{p-r}\nabla^r v\Bigr) &= \frac{|u|^p}{|v|^{p-2}v}\operatorname{div}(A(x)\|\nabla v\|_r^{p-r}\nabla^r v)\\
&\quad + A(x)\|\nabla u\|_r^p - A(x)\Bigl\{\|\nabla u\|_r^p + (p-1)\frac{|u|^p}{|v|^p}\|\nabla v\|_r^p\\
&\quad - p\frac{|u|^{p-2}u}{|v|^{p-2}v}\langle\nabla u, \|\nabla v\|_r^{p-r}\nabla^r v\rangle\Bigr\},
\end{aligned}
$$

where the bracketed expression is a positive semidefinite form and equals zero if and only if u and v are proportional in Ω.

As an immediate consequence of the Finsler-Picone identity (3.1) we obtain the following necessary condition for the existence of positive (or negative) solutions in $\overline{\Omega}$ for the equation (1.11).

Theorem 3.2. *If* (1.11) *possesses a solution v which satisfies $v(x)\neq 0$ in $\overline{\Omega}$, then*

$$
(3.4)\qquad J_H[u;\Omega] := \int_\Omega [A(x)H(\nabla u)^p - C(x)|u|^p]\,\mathrm{d}x > 0
$$

for all $0\not\equiv u\in D(\Omega) := \{\varphi\in C^1(\overline{\Omega})\colon\ \varphi = 0 \text{ on } \partial\Omega\}$.

Proof. For $u\in D(\Omega)$ which is not identically zero in Ω and any solution v of (1.11) satisfying $v(x)\neq 0$ on $\overline{\Omega}$, it is a consequence of the identity (3.1) integrated over Ω that

$$
(3.5)\qquad J_H[u;\Omega] = \int_\Omega A(x)\Phi(u,v)\,\mathrm{d}x \geqslant 0.
$$

Since $u=0$ on $\partial\Omega$ and $v\neq 0$ on $\partial\Omega$, u cannot be a constant multiple of v, i.e., equality $J_H[u;\Omega]=0$ cannot occur in (3.5) and the proof is complete. $\square$

The above theorem can be reformulated as a criterion for the validity of a “weaker” Sturmian conclusion concerning solutions of (1.11) in the sense that it establishes the existence of a zero of the solution v in $\Omega\cup\partial\Omega$ rather than in Ω.

Corollary 3.2. *If there exists a function $u\in D(\overline{\Omega})$ not identically zero such that*

$$
J_H[u;\Omega]\leqslant 0,
$$

then any solution v of (1.11) *has a zero in $\overline{\Omega}$.*

If the function u appearing in Corollary 3.2 is a nontrivial solution of the differential equation (1.10) which satisfies $u = 0$ on $\partial\Omega$, then multiplying (1.10) by u, integrating by parts and making use of the divergence theorem, we obtain that

$$F_H[u;\Omega] := \int_\Omega [a(x)H(\nabla u)^p - c(x)|u|^p]\,\mathrm{d}x = 0. \tag{3.6}$$

Define

$$V_H[u;\Omega] := \int_\Omega [(a(x) - A(x))H(\nabla u)^p + (C(x) - c(x))|u|^p]\,\mathrm{d}x. \tag{3.7}$$

Then the validity of the condition $J_H[u;\Omega] \leqslant 0$ in Corollary 3.2 is guaranteed by

$$V_H[u;\Omega] = F_H[u;\Omega] - J_H[u;\Omega] \geqslant 0 \tag{3.8}$$

and we have the following weaker version of the integral comparison theorem of the Leighton type.

Corollary 3.3. *Let* (1.10) *have a nontrivial solution u vanishing on $\partial\Omega$ and satisfying* (3.8). *Then every solution v of* (1.11) *must have a zero in $\overline{\Omega}$.*

Our next comparison result based on the Finsler-Picone identity (3.1) is the nonlinear analogue of Theorem 5.5 in [20].

Theorem 3.3. *Suppose that* (1.10) *has a nontrivial solution u vanishing on $\partial\Omega$ and satisfying*

$$\int_\Omega \Big[\Big(C(x) - \frac{A(x)}{a(x)}c(x)\Big)|u|^p + a(x)u\Big\langle H(\nabla u)^{p-1}\nabla_\xi H(\nabla u), \nabla\Big(\frac{A(x)}{a(x)}\Big)\Big\rangle\Big]\,\mathrm{d}x \geqslant 0. \tag{3.9}$$

Then every solution of (1.11) *must have a zero in $\overline{\Omega}$.*

P r o o f. From (1.10) and (1.11) it follows that

$$\begin{aligned}
L_A u := {} & \operatorname{div}(A(x)H(\nabla u)^{p-1}\nabla_\xi H(\nabla u)) + C(x)|u|^{p-2}u \\
= {} & \operatorname{div}\Big(\frac{A(x)}{a(x)}a(x)H(\nabla u)^{p-1}\nabla_\xi H(\nabla u)\Big) + C(x)|u|^{p-2}u \\
= {} & \frac{A(x)}{a(x)}\operatorname{div}(a(x)H(\nabla u)^{p-1}\nabla_\xi H(\nabla u)) \\
& + a(x)\Big\langle H(\nabla u)^{p-1}\nabla_\xi H(\nabla u), \nabla\Big(\frac{A(x)}{a(x)}\Big)\Big\rangle + C(x)|u|^{p-2}u \\
= {} & \Big[C(x) - \frac{A(x)}{a(x)}c(x)\Big]|u|^{p-2}u + a(x)\Big\langle H(\nabla u)^{p-1}\nabla_\xi H(\nabla u), \nabla\Big(\frac{A(x)}{a(x)}\Big)\Big\rangle.
\end{aligned}$$

Thus, the integral on the left-hand side of (3.9) is equal to

$$\int_\Omega u L_A u \,\mathrm{d}x = -J_H[u;\Omega],$$

which implies that $J_H[u;\Omega] \leqslant 0$ and the assertion follows from Corollary 3.2. $\square$

Assuming that the boundary of a domain Ω is smooth and introducing a suitable Sobolev space, we can prove the following stronger analogues of the above results.

Theorem 3.4. *Let $\partial\Omega \in C^1$. Assume that there exists a nontrivial function $u \in C^1(\overline{\Omega})$ vanishing on $\partial\Omega$ and satisfying*

$$J_H[u;\Omega] := \int_\Omega [A(x)H(\nabla u)^p - C(x)|u|^p] \,\mathrm{d}x \leqslant 0. \tag{3.10}$$

Then every solution v of (1.11) *must have a zero in Ω unless v is a constant multiple of u.*

Proof. Suppose to the contrary that there exists a solution v of (1.11) such that $v(x) \neq 0$ in Ω. Since $\partial\Omega \in C^1$ and $u \in C^1(\overline{\Omega})$ with $u = 0$ on $\partial\Omega$, there exists a sequence $\{u_k\}$ of $C_0^\infty(\Omega)$ functions converging to u in the norm

$$\|w\| := \left(\int_\Omega [H(\nabla w)^p + |w|^p] \,\mathrm{d}x \right)^{1/p}.$$

Since v does not vanish in Ω, we can use the identity (3.1) with $u = u_k$ and integrate it over Ω to get

$$J_H[u_k;\Omega] = \int_\Omega A(x)\Phi(u_k, v) \,\mathrm{d}x \geqslant 0. \tag{3.11}$$

We will show that $\lim\limits_{k\to\infty} J_H[u_k;\Omega] = J_H[u;\Omega] = 0$. Indeed, since a, c, A and C are uniformly bounded, there is a constant $K_1 > 0$ such that

$$\begin{aligned} |J_H[u_k;\Omega] - J_H[u;\Omega]| \leqslant K_1 \int_\Omega |H(\nabla u_k)^p - H(\nabla u)^p| \,\mathrm{d}x \\ + K_1 \int_\Omega \big||u_k|^p - |u|^p\big| \,\mathrm{d}x. \end{aligned} \tag{3.12}$$

Observing that

$$|H(\nabla u_k)^p - H(\nabla u)^p| \leqslant p[H(\nabla u_k) + H(\nabla u)]^{p-1} H(\nabla(u_k - u)) \tag{3.13}$$

by (2.14) and using the Hölder inequality, we get

$$\tag{3.14} \begin{aligned} &\int_\Omega |H(\nabla u_k)^p - H(\nabla u)^p| \,\mathrm{d}x \\ &\quad \leqslant p\Big(\int_\Omega [H(\nabla u_k) + H(\nabla u)]^p \,\mathrm{d}x\Big)^{(p-1)/p} \Big(\int_\Omega H(\nabla(u_k - u))^p \,\mathrm{d}x\Big)^{1/p}. \end{aligned}$$

Similarly,

$$\tag{3.15} \int_\Omega \big||u_k|^p - |u|^p\big| \,\mathrm{d}x \leqslant p\Big(\int_\Omega (|u_k| + |u|)^p \,\mathrm{d}x\Big)^{(p-1)/p} \Big(\int_\Omega |u_k - u|^p \,\mathrm{d}x\Big)^{1/p}.$$

Collecting (3.12), (3.14) and (3.15), we have

$$|J_H[u_k;\Omega] - J_H[u;\Omega]| \leqslant K_2(\|u_k\| + \|u\|)^{p-1}\|u_k - u\|$$

for some positive constant K_2 which does not depend on k. It follows that $\lim_{k\to\infty} J_H[u_k;\Omega] = J_H[u;\Omega]$. From (3.6) we obtain $J_H[u;\Omega] \geqslant 0$, which contradicts the hypothesis (3.10) unless $J_H[u;\Omega] = 0$.

Now, let $J_H[u;\Omega] = 0$ and let S be an arbitrary domain with $\overline{S} \subset \Omega$. Then for sufficiently large k the support of u_k contains $\overline{S}$, so that

$$\tag{3.16} 0 \leqslant \int_S A(x)\Phi(u_k, v) \,\mathrm{d}x \leqslant \int_\Omega A(x)\Phi(u_k, v) \,\mathrm{d}x = J_H[u_k;\Omega]$$

for all such k. Using (3.13) and the Hölder inequality, we can show analogously to the first part of the proof that

$$\int_S A(x)\Phi(u_k, v) \,\mathrm{d}x \to \int_S A(x)\Phi(u, v) \,\mathrm{d}x \quad \text{as } \|u_k - u\| \to 0.$$

Passing to the limit as $k \to \infty$ in (3.16), we obtain that

$$\int_S A(x)\Phi(u, v) \,\mathrm{d}x = 0.$$

Since $A(x) > 0$ in Ω, it follows that $\Phi(u, v) \equiv 0$ identically in S. By Theorem 3.1, v must be a constant multiple of u in S and thus in Ω. This completes the proof. $\square$

Our next result is a "stronger" Leighton-type integral comparison theorem. The proof is similar to that of Corollary 3.3 and we omit it.

Theorem 3.5. *Let $\partial\Omega \in C^1$. Assume that there exists a nontrivial solution u of* (1.10) *vanishing on $\partial\Omega$ and satisfying*

$$V_H[u;\Omega] = \int_\Omega [(a(x) - A(x))H(\nabla u)^p + (C(x) - c(x))|u|^p]\,\mathrm{d}x \geqslant 0. \tag{3.17}$$

Then every solution v of (1.11) *must have a zero in Ω unless v is a constant multiple of u.*

The pointwise comparison principle of the Sturm-Picone type for the pair of nonlinear elliptic equations (1.10) and (1.11) is an immediate consequence of Theorem 3.5.

Corollary 3.4. *Assume that $a(x) \geqslant A(x)$ and $C(x) \geqslant c(x)$ in Ω and* (1.10) *has a nontrivial solution u such that $u = 0$ on $\partial\Omega$. Then any solution v of* (1.11) *is either zero at some point in Ω or else $v = ku$ for some nonzero constant k.*

4. Nonexistence of positive solutions in exterior domains

We apply comparison theorems from the preceding section to show that the equation

$$\operatorname{div}(A(x)H(\nabla v)^{p-1}\nabla_\xi H(\nabla v)) + C(x)|v|^{p-2}v = 0, \quad x \in \Omega_{r_0}, \tag{4.1}$$

may have no positive solutions in the exterior domain $\Omega_r := \{x \in \mathbb{R}^n\colon\ H_0(x) > r\}$ for any $r \geqslant r_0 > 0$, where H_0 is the dual norm of H. In particular, we compare (4.1) with an equation which is H_0-radially symmetric in the sense that its coefficients depend only on H_0, that is, it is of the form

$$\operatorname{div}(\tilde{a}(H_0(x))H(\nabla u)^{p-1}\nabla_\xi H(\nabla u)) + \tilde{c}(H_0(x))|u|^{p-2}u = 0, \quad x \in \Omega_{r_0}. \tag{4.2}$$

Clearly, if $u = y(H_0)$ is an H_0-radially symmetric solution of (4.2), then $y(r)$ satisfies the half-linear ordinary differential equation

$$(r^{n-1}\tilde{a}(r)|y'|^{p-2}y')' + r^{n-1}\tilde{c}(r)|y|^{p-2}y = 0, \quad r \geqslant r_0 > 0, \tag{4.3}$$

where "$'$" denotes the differentiation with respect to r.

Theorem 4.1. *Assume that there exist real-valued functions $\tilde{a}, \tilde{c} \in C([r_0, \infty))$ with $\tilde{a}(r) > 0$ for $r \geqslant r_0$ such that* (4.3) *is oscillatory in the sense that any of its solutions has a sequence of zeros clustering at infinity. Let*

$$\max_{H_0(x)=r} A(x) \leqslant \tilde{a}(r) \quad \text{and} \quad \min_{H_0(x)=r} C(x) \geqslant \tilde{c}(r), \quad r \geqslant r_0 > 0. \tag{4.4}$$

Then (4.1) *cannot have solutions v such that $v(x) \neq 0$ in Ω_r for any $r \geqslant r_0$.*

P r o o f. Let $y(r)$ be an oscillatory solution of (4.3) on $[r_0, \infty)$ and $\{r_i\}$ the sequence of its consecutive zeros satisfying $r_0 \leqslant r_1 < \ldots < r_i < \ldots$, $\lim\limits_{i\to\infty} r_i = \infty$. Then the function u defined by $u(x) := y(H_0(x))$ is an H_0-radially symmetric solution of (4.2) in Ω_{r_0} such that $u(x) = 0$ on $S_{r_i} := \{x \in \mathbb{R}^n \colon H_0(x) = r_i\}$, $i = 1, 2, \ldots$ Define

$$\Omega_{r_i, r_{i+1}} := \{x \in \mathbb{R}^n \colon r_i < H_0(x) < r_{i+1}\}, \quad i = 1, 2, \ldots$$

Let v be a solution of (4.1) in Ω_r for some $r \geqslant r_0$. Then $\Omega_{r_i, r_{i+1}} \subset \Omega_r$ for sufficiently large i and

$$\begin{aligned} V_H[u; \Omega_{r_i, r_{i+1}}] = \int_{\Omega_{r_i, r_{i+1}}} & [(\tilde{a}(H_0(x)) - A(x))H(\nabla u)^p \\ & + (C(x) - \tilde{c}(H_0(x)))|u|^p] \, \mathrm{d}x \geqslant 0 \end{aligned} \tag{4.5}$$

because of (4.4). Corollary 3.3 now implies that v must have a zero in $\overline{\Omega}_{r_i, r_{i+1}}$ for every i such that $r_i > r$, and the proof is complete. $\square$

An alternative way how to reduce the problem of the existence (nonexistence) of positive solutions of the PDE (4.1) in exterior domains to the one-dimensional oscillation problem is to replace $\tilde{a}(r)$ and $\tilde{c}(r)$ in (4.3) by the spherical means $\overline{a}(r)$ and $\overline{c}(r)$ of the coefficients $A(x)$ and $C(x)$ over the Wulff sphere $\{x \in \mathbb{R}^n \colon H_0(x) = r\}$, respectively, defined by

$$\bar{a}(r) := \frac{1}{\alpha_n r^{n-1}} \int_{H_0(x)=r} A(x) \, \mathrm{d}\sigma, \quad \bar{c}(r) := \frac{1}{\alpha_n r^{n-1}} \int_{H_0(x)=r} C(x) \, \mathrm{d}\sigma, \tag{4.6}$$

where α_n is the surface area of the unit H_0-sphere. In the special case of the usual p-Laplacian a similar averaging technique was used, for instance, in [10] and [14].

Theorem 4.2. *If the half-linear ODE*

$$(4.7) \qquad (r^{n-1}\bar{a}(r)|y'|^{p-2}y')' + r^{n-1}\bar{c}(r)|y|^{p-2}y = 0, \quad r \geqslant r_0 > 0,$$

with $\bar{a}$ and $\bar{c}$ given by (4.6) *is oscillatory, then the equation* (4.1) *cannot have positive (or negative) solutions in Ω_r for any $r > r_0$.*

P r o o f. Let $y(r)$ be an oscillatory solution of (4.7) and let $(r_0 \leqslant)\ r_1 < r_2 < \ldots r_i < \ldots$ be its consecutive zeros with $r_i \to \infty$ as $t \to \infty$. Integrating (4.7) from r_i to r_{i+1} by parts, we have

$$\int_{r_i}^{r_{i+1}} r^{n-1}[\bar{a}(r)|y'(r)|^p - \bar{c}(r)|y|^p]\,\mathrm{d}r = 0, \quad i = 1, 2, \ldots$$

Define the function u by $u(x) := y(H_0(x))$. Then

$$\begin{aligned} J_H[u; \Omega_{r_i, r_{i+1}}] &= \int_{\Omega_{r_i, r_{i+1}}} [A(x)H(\nabla u)^p - C(x)|u|^p]\,\mathrm{d}x \\ &= \int_{r_i}^{r_{i+1}} \Big[|y'(r)|^p \int_{H_0(x)=r} A(x)\,\mathrm{d}\sigma_r - |y(r)|^p \int_{H_0(x)=r} C(x)\,\mathrm{d}\sigma_r\Big]\,\mathrm{d}r \\ &= \alpha_n \int_{r_i}^{r_{i+1}} r^{n-1}[\bar{a}(r)|y'(r)|^p - \bar{c}(r)|y(r)|^p]\,\mathrm{d}r = 0. \end{aligned}$$

Thus, the conditions of Corollary 3.2 are satisfied in $\Omega_{r_i, r_{i+1}}$ and, consequently, any solution v of (4.1) must have a zero in $\overline{\Omega}_{r_i, r_{i+1}}$ which means that it cannot be positive (or negative) in Ω_r for any $r \geqslant r_0$. This completes the proof. □

There exists a voluminous literature (see, for example, [11] and the references therein) on oscillation of the one-dimensional half-linear differential equation

$$(4.8) \qquad (p(r)|y'|^{p-2}y')' + q(r)|y|^{p-2}y = 0$$

where p and q are continuous functions on $[r_0, \infty)$ with $p(r) > 0$ for $r \geqslant r_0 > 0$, and any of the available oscillation criteria for (4.8), when applied to (4.3) or (4.7), yield the corresponding nonexistence result for the partial differential equation (4.1). For example, the application of criteria from [15] and [16] gives the following result.

Corollary 4.1. *Suppose that continuous functions $\tilde{a}(r)$ and $\tilde{c}(r)$ defined on $[r_0, \infty)$ with $\tilde{a}(r) > 0$ in $[r_0, \infty)$ satisfy* (4.4).

(i) *Let*

$$\int_{r_0}^{\infty} (r^{n-1}\tilde{a}(r))^{-1/(p-1)}\,\mathrm{d}r = \infty \tag{4.9}$$

and denote

$$\widetilde{P}(r) := \int_{r_0}^{r} (s^{n-1}\tilde{a}(s))^{-1/(p-1)}\,\mathrm{d}s, \quad r \geqslant r_0 > 0. \tag{4.10}$$

If either

$$\int_{r_0}^{\infty} r^{n-1}\tilde{c}(r)\,\mathrm{d}r = \infty, \tag{4.11}$$

or

$$\liminf_{r\to\infty} (\widetilde{P}(r))^{p-1} \int_{r}^{\infty} s^{n-1}\tilde{c}(s)\,\mathrm{d}s > \frac{1}{p-1}\Big(\frac{p-1}{p}\Big)^{p}, \tag{4.12}$$

then (4.1) *has no positive solutions in the exterior domain Ω_r for any $r \geqslant r_0$.*

(ii) *Let*

$$\int_{r_0}^{\infty} (r^{n-1}\tilde{a}(r))^{-1/(p-1)}\,\mathrm{d}r < \infty \tag{4.13}$$

and denote

$$\widetilde{\pi}(r) := \int_{r}^{\infty} (s^{n-1}\tilde{a}(s))^{-1/(p-1)}\,\mathrm{d}s, \quad r \geqslant r_0 > 0. \tag{4.14}$$

If either

$$\int_{r_0}^{\infty} (\widetilde{\pi}(r))^{p} r^{n-1}\tilde{c}(r)\,\mathrm{d}r = \infty, \tag{4.15}$$

or

$$\liminf_{r\to\infty} \frac{1}{\widetilde{\pi}(r)} \int_{r}^{\infty} (\widetilde{\pi}(s))^{p} s^{n-1}\tilde{c}(s)\,\mathrm{d}s > \Big(\frac{p-1}{p}\Big)^{p}, \tag{4.16}$$

then (4.1) *has no positive solutions in the exterior domain Ω_r for any $r \geqslant r_0$.*

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