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Iterated oscillation criteria for delay partial difference equations

ITERATED OSCILLATION CRITERIA FOR DELAY
PARTIAL DIFFERENCE EQUATIONS

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Abstract. In this paper, by using an iterative scheme, we advance the main oscillation result of Zhang and Liu (1997). We not only extend this important result but also drop a superfluous condition even in the noniterated case. Moreover, we present some illustrative examples for which the previous results cannot deliver answers for the oscillation of solutions but with our new efficient test, we can give affirmative answers for the oscillatory behaviour of solutions. For a visual explanation of the examples, we also provide 3D graphics, which are plotted by a mathematical programming language.

Keywords: partial difference equation; oscillation; variable coefficient

MSC 2010: 39A14, 39A21

1. Introduction

Partial difference equations are some kinds of difference equations that involve at least two (discrete) variables. Such equations come out from the mathematical modeling of random walk problems, molecular structure problems, and numerical difference approximation of solutions to partial delay differential equations (see [1], [6]). Because of its many application fields, the oscillation/nonoscillation problem for delay partial difference equations is now receiving much attention (see [2], [5]).

In this work, we consider the linear partial difference equation:

$$(1.1) \qquad x(m+1, n) + x(m, n+1) - x(m, n) + p(m, n)x(m-k, n-l) = 0$$
$$\text{for } (m, n) \in \mathbb{Z}_0^2,$$

where $\{p(m, n)\}_{(m,n)\in\mathbb{Z}_0^2}$ is a nonnegative double sequence of reals and $k, l \in \mathbb{N}$, and for simplicity of notation, we set $K := \max\{k, l\}$, $L := \min\{k, l\}$ and $\mathbb{Z}_{r_0} := \{r \in \mathbb{Z} \colon r \geqslant r_0\}$ for $r_0 \in \mathbb{Z}$.

We now define the initial problem for (1.1) (see [3]). To this end, we define the *primary* and the *secondary initial lattices* by $\Omega(-k,-l) := \mathbb{Z}_{-k} \times \mathbb{Z}_{-l} \setminus \mathbb{Z}_0 \times \mathbb{Z}_1$ and $\Upsilon(-k,-l) := \mathbb{Z}_{-k} \times \mathbb{Z}_{-l} \setminus \mathbb{Z}_1 \times \mathbb{Z}_0$, respectively. By a *solution* of (1.1) we mean a double sequence $\{x(m,n)\}_{(m,n)\in\mathbb{Z}_{-k}\times\mathbb{Z}_{-l}}$ of reals which satisfies the recursive equation (1.1) identically on $\mathbb{Z}_0^2$. Clearly, if an *initial double sequence* $\{\varphi(m,n)\}_{(m,n)\in\Omega(-k,-l)}$ is given, then one can easily iterate (1.1) and obtain all the values of the unique solution x, which satisfies $x = \varphi$ on the primary initial lattice $\Omega(-k,-l)$, by rewriting (1.1) in the following form:

$$(1.2) \quad x(m,n+1) = x(m,n) - x(m+1,n) - p(m,n)x(m-k,n-l) \quad \text{for } (m,n) \in \mathbb{Z}_0^2.$$

Alternatively, if $\{\psi(m,n)\}_{(m,n)\in\Upsilon(-k,-l)}$ is given, then one can also obtain the unique solution x of (1.1) satisfying $x = \psi$ on the secondary lattice $\Upsilon(-k,-l)$ by iterating

$$(1.3) \quad x(m+1,n) = x(m,n) - x(m,n+1) - p(m,n)x(m-k,n-l) \quad \text{for } (m,n) \in \mathbb{Z}_0^2.$$

Below, we revisit the definition of oscillation of a double sequence on the first discrete quadrant $\mathbb{Z}_0^2$. We call a double sequence $\{x(m,n)\}_{(m,n)\in\mathbb{Z}_0^2}$ *eventually positive* if there exists $(m_0,n_0) \in \mathbb{Z}_0^2$ such that $x > 0$ on $\mathbb{Z}_{m_0} \times \mathbb{Z}_0 \cup \mathbb{Z}_0 \times \mathbb{Z}_{n_0}$, and *eventually negative* if $-x$ is eventually positive. If a double sequence is neither eventually positive nor eventually negative, we call it oscillatory. To point out what is the difference between our definition and the oscillation definition given in [3], we give an example as follows: Let $\{x(m,n)\}_{(m,n)\in\mathbb{Z}_0^2}$ be defined by $x(r,0) = (-1)^r$, $x(0,r) = (-1)^r$ for all $r \in \mathbb{Z}_0$ and $x(m,n) = 1$ for all $(m,n) \in \mathbb{Z}_1^2$. Due to the oscillation definition given in [3], this sequence is eventually positive and hence is nonoscillatory since we have $x(m,n) = 1 > 0$ for all large m, n; however, it is oscillatory according to our definition. It is easy to infer that if it is nonoscillatory in our sense, then it is nonoscillatory according to the definition in [3], too. And conversely, if a double sequence is oscillatory according to the definition in [3], it is oscillatory in our sense. We call a solution x of (1.1) *oscillatory* if $\{x(m,n)\}_{(m,n)\in\mathbb{Z}_{-k}\times\mathbb{Z}_{-l}\setminus D(-k,-l)}$ is oscillatory, where $D(-k,-l)$ is the initial lattice, i.e., either $D(-k,-l) = \Omega(-k,-l)$ or $D(-k,-l) = \Upsilon(-k,-l)$ depending on the choice of the data of the initial value problem (see (1.2) and (1.3)). If every solution of (1.1) is oscillatory independently of the choice of the initial lattice and the initial sequence, then (1.1) is called oscillatory. Throughout the paper, we will focus our attention on those solutions of (1.1) which do not vanish on $\mathbb{Z}_{m_0} \times \mathbb{Z}_{-l} \cup \mathbb{Z}_{-k} \times \mathbb{Z}_{n_0} \setminus D(-k,-l)$ for any $(m_0,n_0) \in \mathbb{Z}_0^2$.

Below, we quote an important result due to Zhang and Liu.

Theorem A ([4], Theorem 2.1)**.** *Assume that*

(1.4)
$$\limsup_{\substack{m\to\infty\\n\to\infty}} p(m,n) > 0$$

and that

(1.5)
$$\liminf_{\substack{m\to\infty\\n\to\infty}} \Gamma(m,n) > 1,$$

where

(1.6)
$$\Gamma(m,n) := \inf_{\lambda\in\Lambda(m,n)}\left\{\frac{1}{\lambda}\left(\prod_{i=m-k}^{m-1}\prod_{j=n-l}^{n-1}\frac{1}{1-\lambda p(i,j)}\right)^{1/L}\right\}$$
$$\text{for } (m,n)\in\mathbb{Z}_k\times\mathbb{Z}_l$$

and
(1.7)
$$\Lambda(m,n) := \{\lambda > 0\colon 1-\lambda p(i,j) > 0 \text{ for all } (i,j)\in[m-k,m)\times[n-l,n)\cap\mathbb{Z}^2\}$$
$$\text{for } (m,n)\in\mathbb{Z}_k\times\mathbb{Z}_l.$$

Then (1.1) *is oscillatory.*

In this paper, we shall advance the conclusion of Theorem A. To this end, we give the following simple example, which illustrates the significance of the results of this paper.

E x a m p l e 1.1. Consider the delay partial difference equation for $(m,n)\in\mathbb{Z}_0^2$

(1.8)
$$x(m+1,n) + x(m,n+1) - x(m,n) + p(m,n)x(m-2,n-1) = 0,$$

where

$$p(m,n) := \begin{cases} \dfrac{2}{9}, & \mod(m,3) = 0, \\[2mm] \dfrac{1}{9}, & \text{otherwise.} \end{cases}$$

Due to Theorem A, we have for $(m,n)\in\mathbb{Z}_2\times\mathbb{Z}_1$

$$\Gamma(m,n) := \inf_{\lambda\in\Lambda(m,n)} \begin{cases} \dfrac{1}{\lambda(1-\lambda/9)^2}, & \mod(m,3) = 0, \\[3mm] \dfrac{1}{\lambda(1-\lambda/9)(1-2\lambda/9)}, & \text{otherwise} \end{cases}$$

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and

$$\Lambda(m,n) := \begin{cases} (0,9), & \mathrm{mod}(m,3) = 0, \\ (0,9/2), & \text{otherwise} \end{cases}$$

which yields

$$\Gamma(m,n) := \begin{cases} \dfrac{2\sqrt{3}}{3}, & \mathrm{mod}(m,3) = 0, \\ \dfrac{3}{4}, & \text{otherwise.} \end{cases}$$

Hence, we have

$$\liminf_{\substack{m \to \infty \\ n \to \infty}} \Gamma(m,n) = \min\left\{\frac{3}{4}, \frac{2\sqrt{3}}{3}\right\} = \frac{3}{4} < 1,$$

which shows that Theorem A fails to deliver any conclusion on the oscillatory be-haviour of solutions of (1.8). However, our result (Theorem 2.1) in the following section gives the affirmative answer. The following graphics belong to the solution with the initial condition $x \equiv 1$ on $\Omega(-2,-1)$ and of 60 iterates.
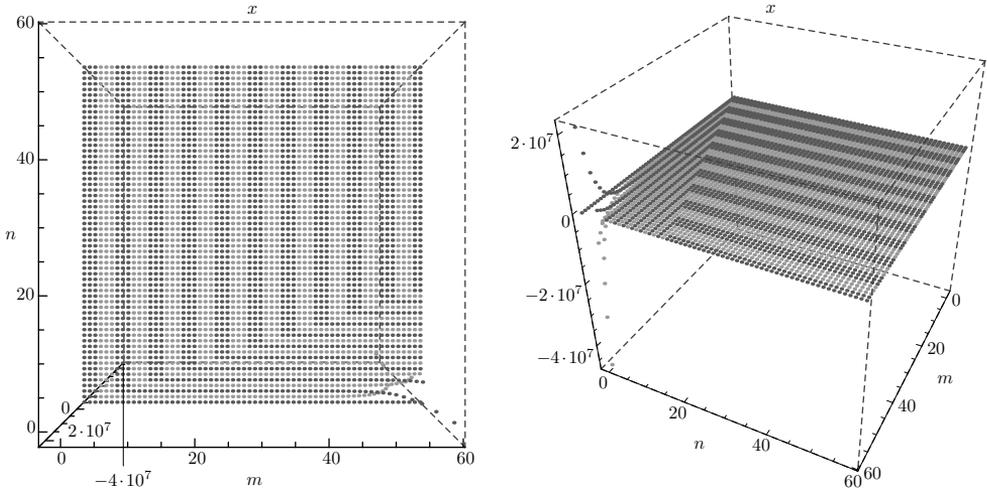
Here, the points painted with dark color represent the nonnegative terms while the points painted with light color represent the negative ones.

In the next section, we prove our main result and provide some simple examples to show its applicability.

2. Main results

We first prove a simple lemma, which will be needed in the sequel.

Lemma 2.1. *Let x be an eventually positive solution of* (1.1), *and suppose that*

$$
(2.1) \qquad \limsup_{\substack{m \to \infty \\ n \to \infty}} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} p(i,j) > 0.
$$

Then y_x defined by

$$
(2.2) \qquad y_x(m,n) := \frac{x(m-k, n-l)}{x(m,n)} \quad \text{for } (m,n) \in \mathbb{Z}_0^2
$$

satisfies

$$
(2.3) \qquad \liminf_{\substack{m \to \infty \\ n \to \infty}} y_x(m,n) < \infty.
$$

P r o o f.   Let x be an eventually positive solution of (1.1). Hence, we may suppose that $x(m,n) > 0$ for all $(m,n) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{n_1}$ for some fixed $(m_1, n_1) \in \mathbb{Z}_0^2$. Then from (1.1) it is obvious that x is decreasing on $\mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}$ where $(m_2, n_2) \in \mathbb{Z}_{m_1+k} \times \mathbb{Z}_{n_1+l}$, and thus y_x defined by (2.2) satisfies $y_x > 1$ on $\mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}$. By virtue of (2.1), there exist a constant $\varepsilon > 0$ and an increasing divergent double sequence $\{(\xi_r, \zeta_r)\}_{r \in \mathbb{N}} \subset \mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}$ such that $r \in \mathbb{N}$ implies

$$
(2.4) \qquad \sum_{i=\xi_r - k}^{\xi_r - 1} \sum_{j=\zeta_r - l}^{\zeta_r - 1} p(i,j) \geqslant \varepsilon.
$$

Keeping in mind the Pigeonhole principle due to Dirichlet, we infer that (2.4) implies the existence of a double sequence $\{(\alpha_r, \beta_r)\}_{r \in \mathbb{N}} \subset \mathbb{Z}_0^2$ such that $\xi_r - k \leqslant \alpha_r < \xi_r$, $\zeta_r - l \leqslant \beta_r < \zeta_r$ and $p(\alpha_r, \beta_r) > \varepsilon/(kl)$ for all $r \in \mathbb{N}$. Substituting (α_r, β_r) for $r \in \mathbb{N}$ into (1.1) and considering the decreasing nature of x, we easily obtain

$$
\begin{aligned}
0 &= x(\alpha_r + 1, \beta_r) + x(\alpha_r, \beta_r + 1) - x(\alpha_r, \beta_r) + p(\alpha_r, \beta_r) x(\alpha_r - k, \beta_r - l) \\
&> -x(\alpha_r, \beta_r) + \frac{\varepsilon}{kl} x(\xi_r - k, \zeta_r - l),
\end{aligned}
$$

which yields

$$
(2.5) \qquad \frac{x(\alpha_r, \beta_r)}{x(\xi_r - k, \zeta_r - l)} > \frac{\varepsilon}{kl}
$$

for all $r \in \mathbb{N}$. Similarly, for all $r \in \mathbb{N}$, we have

$$
\begin{aligned}
0 &= x(\alpha_r + 1, \beta_r) + x(\alpha_r, \beta_r + 1) - x(\alpha_r, \beta_r) + p(\alpha_r, \beta_r) x(\alpha_r - k, \beta_r - l) \\
&> -x(\xi_r - k, \zeta_r - l) + \frac{\varepsilon}{kl} x(\alpha_r - k, \beta_r - l),
\end{aligned}
$$

which implies

$$
(2.6) \qquad \frac{x(\xi_r - k, \zeta_r - l)}{x(\alpha_r - k, \beta_r - l)} > \frac{\varepsilon}{kl}
$$

for all $r \in \mathbb{N}$. In view of (2.2), and taking the reciprocal after multiplying (2.5) and (2.6) yields

$$
(2.7) \qquad y_x(\alpha_r, \beta_r) < \left(\frac{kl}{\varepsilon} \right)^2
$$

for all $r \in \mathbb{N}$. It is clear that $\{(\alpha_r, \beta_r)\}_{r \in \mathbb{N}} \subset \mathbb{Z}_0^2$ is divergent and hence (2.7) implies (2.3), and this completes the proof. $\qquad \square$

Next, we have a new lemma, which will be applied in the proof of our main result. To state the lemma, we need to introduce

$$
(2.8) \quad \Gamma_r(m, n) := \begin{cases} 1, & r = 0, \\ \displaystyle\inf_{\lambda \in \Lambda_r(m,n)} \left\{ \frac{1}{\lambda} \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} \frac{1}{1 - \lambda \Gamma_{r-1}(i,j) p(i,j)} \right)^{1/L} \right\}, & r \in \mathbb{N} \end{cases}
$$

and

$$
\begin{aligned}
(2.9) \qquad \Lambda_r(m, n) := \{ \lambda > 0 \colon\ & 1 - \lambda \Gamma_{r-1}(i,j) p(i,j) > 0 \\
& \text{for all } (i,j) \in [m-k, m) \times [n-l, n) \cap \mathbb{Z}^2 \}
\end{aligned}
$$

for $(m, n) \in \mathbb{Z}_{rk} \times \mathbb{Z}_{rl}$ and $r \in \mathbb{N}$.

Lemma 2.2. *Let x be an eventually positive solution of* (1.1), *and suppose that*

$$
(2.10) \qquad \liminf_{\substack{m \to \infty \\ n \to \infty}} \Gamma_{r_0}(m, n) > 1
$$

for some $r_0 \in \mathbb{N}$. Then

$$
(2.11) \qquad \lim_{\substack{m \to \infty \\ n \to \infty}} y_x(m, n) = \infty.
$$

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P r o o f.   Let x be an eventually positive solution of (1.1). Hence, we may suppose that $x(m, n) > 0$ for all $(m, n) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{n_1}$ for some fixed $(m_1, n_1) \in \mathbb{Z}_0^2$. Then we may rewrite (1.1) as

(2.12)
$$[1 - y_x(m, n)p(m, n)]x(m, n) = x(m + 1, n) + x(m, n + 1) > 0$$

for all $(m, n) \in \mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}$ for some fixed $(m_2, n_2) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{n_1}$. Set

(2.13)
$$z_r(m, n) := \begin{cases} y_x(i, j), & r = 0, \\ \min_{(i,j)\in[m-k,m]\times[n-l,n]\cap\mathbb{Z}^2} z_{r-1}(i, j), & r \in \{1, 2, \ldots, r_0\} \end{cases}$$

for $(m, n) \in \mathbb{Z}_{m_2 + rk} \times \mathbb{Z}_{n_2 + rl}$. From (2.12), for all $(m, n) \in \mathbb{Z}_{m_2 + k} \times \mathbb{Z}_{n_2}$ we have

(2.14)
$$\begin{aligned}
x(m, n) &< [1 - y_x(m - 1, n)p(m - 1, n)]x(m - 1, n) \\
&< [1 - y_x(m - 1, n)p(m - 1, n)] \\
&\quad \times [1 - y_x(m - 2, n)p(m - 2, n)]x(m - 2, n) \\
&\;\;\vdots \\
&< \prod_{i=m-k}^{m-1} [1 - y_x(i, n)p(i, n)]x(m - k, n),
\end{aligned}$$

which yields

(2.15)
$$\begin{aligned}
x(m, n) &< \left(\prod_{j=n-l}^{n-1} x(m, j) \right)^{1/l} \\
&< \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} [1 - y_x(i, j)p(i, j)]x(m - k, j) \right)^{1/l} \\
&< \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} [1 - y_x(i, j)p(i, j)] \right)^{1/l} x(m - k, n - l) \\
&< \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} [1 - z_1(m, n)p(i, j)] \right)^{1/l} x(m - k, n - l)
\end{aligned}$$

for all $(m, n) \in \mathbb{Z}_{m_2 + k} \times \mathbb{Z}_{n_2 + l}$. By following similar arguments, we can show easily that

(2.16)
$$x(m, n) < \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} [1 - z_1(m, n)p(i, j)] \right)^{1/k} x(m - k, n - l)$$

for all $(m, n) \in \mathbb{Z}_{m_2+k} \times \mathbb{Z}_{n_2+l}$. Combining (2.15) and (2.16) and using the definition in (2.2), we obtain

$$(2.17) \qquad y_x(m, n) > \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} \frac{1}{1 - z_1(m, n)p(i, j)} \right)^{1/L}$$

for all $(m, n) \in \mathbb{Z}_{m_2+k} \times \mathbb{Z}_{n_2+l}$; recall that L is defined to be the minimum of k and l. Using (2.12) and (2.13), we learn that $z_1(m, n) \in \Lambda_1(m, n)$ for all $(m, n) \in \mathbb{Z}_{m_2+k} \times \mathbb{Z}_{n_2+l}$. Hence, it follows from (2.8) and (2.17) that

$$(2.18) \qquad y_x(m, n) > \left(\frac{1}{z_1(m, n)} \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} \frac{1}{1 - z_1(m, n)p(i, j)} \right)^{1/L} \right) z_1(m, n)$$

$$\geqslant \Gamma_1(m, n) z_1(m, n)$$

holds for all $(m, n) \in \mathbb{Z}_{m_2+k} \times \mathbb{Z}_{n_2+l}$. Substituting (2.18) into (2.12), we obtain that

$$(2.19) \qquad [1 - z_1(m, n)\Gamma_1(m, n)p(m, n)]x(m, n) > x(m+1, n) + x(m, n+1) > 0$$

holds for all $(m, n) \in \mathbb{Z}_{m_2+k} \times \mathbb{Z}_{n_2+l}$. Following steps similar to those above, we can obtain

$$(2.20) \qquad y_x(m, n) > \left(\frac{1}{z_2(m, n)} \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} \frac{1}{1 - z_2(m, n)p(i, j)} \right)^{1/L} \right) z_2(m, n)$$

$$\geqslant \Gamma_2(m, n) z_2(m, n)$$

for all $(m, n) \in \mathbb{Z}_{m_2+2k} \times \mathbb{Z}_{n_2+2l}$. Repeating the emerging pattern for a total of $(r_0 - 2)$ times more, we finally obtain

$$(2.21) \qquad y_x(m, n) > \Gamma_{r_0}(m, n) z_{r_0}(m, n)$$

for all $(m, n) \in \mathbb{Z}_{m_2+r_0 k} \times \mathbb{Z}_{n_2+r_0 l}$. Suppose now that (2.11) does not hold, i.e.,

$$(2.22) \qquad \liminf_{\substack{m \to \infty \\ n \to \infty}} y_x(m, n) < \infty.$$

On the other hand, by the definition in (2.13), we know that

$$(2.23) \qquad \liminf_{\substack{m \to \infty \\ n \to \infty}} z_{r_0}(m, n) = \liminf_{\substack{m \to \infty \\ n \to \infty}} y_x(m, n) \geqslant 1.$$

Then, from (2.21)–(2.23) we obtain

$$\liminf_{\substack{m \to \infty \\ n \to \infty}} y_x(m, n) \geqslant \liminf_{\substack{m \to \infty \\ n \to \infty}} (\Gamma_{r_0}(m, n) z_{r_0}(m, n))$$

$$\geqslant \liminf_{\substack{m \to \infty \\ n \to \infty}} \Gamma_{r_0}(m, n) \liminf_{\substack{m \to \infty \\ n \to \infty}} z_{r_0}(m, n) > \liminf_{\substack{m \to \infty \\ n \to \infty}} y_x(m, n),$$

which is an obvious contradiction. Therefore, (2.11) must be true. $\qquad \square$

The following is the main result of the paper.

Theorem 2.1. *Assume that* (2.10) *holds for some* $r_0 \in \mathbb{N}$. *Then* (1.1) *is oscillatory.*

P r o o f.　To complete the proof, it suffices to prove that (2.10) implies (2.1). Assume the contrary that (2.10) holds but (2.1) does not, i.e.,

$$\text{(2.24)} \qquad \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} p(i,j) = 0,$$

which yields

$$\text{(2.25)} \qquad \lim_{\substack{m\to\infty \\ n\to\infty}} p(m,n) = 0.$$

Then there exists $(m_1, n_1) \in \mathbb{Z}_0^2$ such that

$$\text{(2.26)} \qquad \Gamma_{r_0}(m,n) > 1$$

for all $(m,n) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{n_1}$. Let

$$\text{(2.27)} \qquad \varepsilon := \frac{\delta^{r_0}}{2} \qquad \text{and} \qquad \delta := \frac{K^K}{(K+1)^{K+1}}.$$

Therefore, there exists $(m_2, n_2) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{n_1}$ such that

$$\text{(2.28)} \qquad p(m,n) < \varepsilon$$

for all $(m,n) \in \mathbb{Z}_{m_2} \times \mathbb{Z}_{n_2}$. Note that $(0, 1/\varepsilon) \subset \Lambda_1(m,n)$ for all $(m,n) \in \mathbb{Z}_{m_2+k} \times \mathbb{Z}_{n_2+l}$. Hence, for all $(m,n) \in \mathbb{Z}_{m_2+k} \times \mathbb{Z}_{n_2+l}$ we have

$$\text{(2.29)} \qquad \frac{\varepsilon}{\delta} = \inf_{\lambda \in (0,1/\varepsilon)} \left\{ \frac{1}{\lambda} \left(\frac{1}{1-\lambda\varepsilon} \right)^K \right\} \geqslant \Gamma_1(m,n).$$

Note that $(0, \delta/\varepsilon) \subset \Lambda_2(m,n)$ for all $(m,n) \in \mathbb{Z}_{m_2+2k} \times \mathbb{Z}_{n_2+2l}$. Using (2.29) and the definition in (2.8), we get

$$\text{(2.30)} \qquad \frac{\varepsilon}{\delta^2} = \inf_{\lambda \in (0,\delta/\varepsilon)} \left\{ \frac{1}{\lambda} \left(\frac{1}{1-\lambda\varepsilon/\delta} \right)^K \right\} \geqslant \Gamma_2(m,n)$$

for all $(m,n) \in \mathbb{Z}_{m_2+2k} \times \mathbb{Z}_{n_2+2l}$. Repeating this procedure, we finally obtain

$$\text{(2.31)} \qquad \frac{\varepsilon}{\delta^{r_0}} = \inf_{\lambda \in (0,\delta^{(r_0-1)}/\varepsilon)} \left\{ \frac{1}{\lambda} \left(\frac{1}{1-\lambda\varepsilon/\delta^{r_0-1}} \right)^K \right\} \geqslant \Gamma_{r_0}(m,n)$$

for all $(m,n) \in \mathbb{Z}_{m_2+r_0 k} \times \mathbb{Z}_{n_2+r_0 l}$ since $(0, \delta^{(r_0-1)}/\varepsilon) \subset \Lambda_{r_0}(m,n)$ for all $(m,n) \in \mathbb{Z}_{m_2+r_0 k} \times \mathbb{Z}_{n_2+r_0 l}$. Using (2.26), (2.27) and (2.31), we are led to an apparent contradiction. The proof is therefore completed. $\qquad \square$

R e m a r k 2.1.   One can see easily by letting $r_0 = 1$ that Theorem 2.1 reduces to Theorem A by dropping the superfluous restriction (1.4).

As an illustrative application to Theorem 2.1, we have the following.

E x a m p l e 2.1.   Consider the delay partial difference equation

$$(2.32) \qquad x(m+1, n) + x(m, n+1) - x(m, n) + p(m, n)x(m-3, n-1) = 0$$
$$\text{for } (m, n) \in \mathbb{Z}_0^2,$$

where for $(m, n) \in \mathbb{Z}_0^2$

$$p(m, n) := \begin{cases} \dfrac{1}{4}, & \mod(n, 2) = 0, \\[2mm] \dfrac{1}{16}, & \text{otherwise.} \end{cases}$$

Simple computations show us that for $(m, n) \in \mathbb{Z}_3 \times \mathbb{Z}_1$

$$\Gamma_1(m, n) = \begin{cases} \dfrac{16}{27}, & \mod(n, 2) = 0, \\[2mm] \dfrac{64}{27}, & \text{otherwise.} \end{cases}$$

and

$$\Gamma_2(m, n) \equiv \frac{1024}{729} \qquad \text{for } (m, n) \in \mathbb{Z}_6 \times \mathbb{Z}_2.$$

Hence, we have

$$\liminf_{\substack{m \to \infty \\ n \to \infty}} \Gamma_2(m, n) = \frac{1024}{729} \approx 1.4 > 1.$$

It follows from Theorem 2.1 (with $r_0 = 2$) that every solution of (2.32) is oscillatory. Also note that Theorem A cannot be applied to this equation since

$$\liminf_{\substack{m \to \infty \\ n \to \infty}} \Gamma_1(m, n) = \frac{16}{27} \approx 0.6 < 1.$$

The following graphics belong to the solution with the initial condition $x \equiv 1$ on $\Omega(-3, -1)$ and of 50 iterates. Here, the points painted with dark color represent the nonnegative terms while the points painted with light color represent the negative ones.

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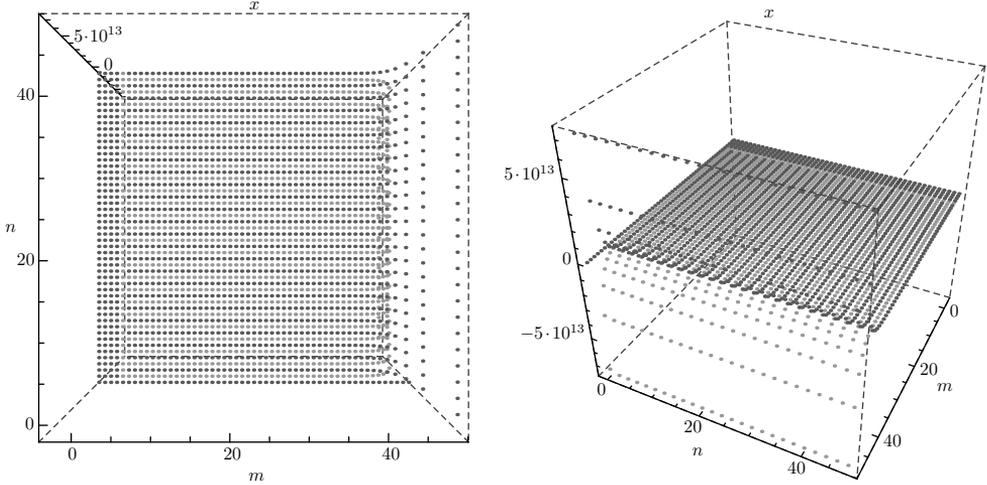
For the next corollary, we need to define

$$
P_r(m, n) := \begin{cases} 1, & r = 0, \\ \displaystyle\sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} P_{r-1}(i, j) p(i, j), & r \in \mathbb{N} \end{cases}
$$

for $(m, n) \in \mathbb{Z}_{rk} \times \mathbb{Z}_{rl}$ and $r \in \mathbb{N}$.

Corollary 2.1. *Assume that for some $r_0 \in \mathbb{N}$, we have*

$$
\tag{2.33} \liminf_{\substack{m \to \infty \\ n \to \infty}} P_{r_0}(m, n) > (\delta K L)^{r_0},
$$

where δ is defined by (2.27). Then (1.1) is oscillatory.

P r o o f.   From (2.8) and (2.9) we have

$$
\begin{aligned}
\Gamma_1(m, n) &= \inf_{\lambda \in \Lambda_1(m,n)} \left\{ \frac{1}{\lambda} \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} [1 - \lambda p(i, j)] \right)^{-1/L} \right\} \\
&\geqslant \inf_{\lambda \in \Lambda_1(m,n)} \left\{ \frac{1}{\lambda} \left(\frac{1}{KL} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} [1 - \lambda p(i, j)] \right)^{-K} \right\} \\
&\geqslant \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \left(1 - \frac{\lambda}{KL} P_1(m, n) \right)^{-K} \right\} \\
&= \frac{1}{\delta KL} P_1(m, n)
\end{aligned}
$$

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for all $(m, n) \in \mathbb{Z}_k \times \mathbb{Z}_l$. In the second line above, the well-known inequality between the arithmetic and the geometric mean is used. Define

$$\Lambda'_r(m, n) := \{\lambda > 0 : 1 - \lambda P_{r-1}(i, j)p(i, j)/(\delta KL)^{(r_0-1)} > 0$$
$$\text{for all } (i, j) \in [m-k, m) \times [n-l, n) \cap \mathbb{Z}^2\}$$

for $(m, n) \in \mathbb{Z}_{rk} \times \mathbb{Z}_{rl}$ and $r \in \mathbb{N}$. In the next step, we see that

$$\Gamma_2(m, n) = \inf_{\lambda \in \Lambda_2(m,n)} \left\{ \frac{1}{\lambda} \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} [1 - \lambda \Gamma_1(i, j)p(i, j)] \right)^{-1/L} \right\}$$

$$\geqslant \inf_{\lambda \in \Lambda'_2(m,n)} \left\{ \frac{1}{\lambda} \left(\prod_{i=m-k}^{m-1} \prod_{j=n-l}^{n-1} \left[1 - \lambda \frac{1}{\delta KL} P_1(i, j)p(i, j)\right] \right)^{-1/L} \right\}$$

$$\geqslant \inf_{\lambda \in \Lambda'_2(m,n)} \left\{ \frac{1}{\lambda} \left(\frac{1}{KL} \sum_{i=m-k}^{m-1} \sum_{j=n-l}^{n-1} \left[1 - \lambda \frac{1}{\delta KL} P_1(i, j)p(i, j)\right] \right)^{-K} \right\}$$

$$\geqslant \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \left(1 - \frac{\lambda}{\delta (KL)^2} P_2(m, n) \right)^{-K} \right\} = \frac{1}{(\delta KL)^2} P_2(m, n)$$

for all $(m, n) \in \mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$. Note that in the second line above, we employed the fact that $\Lambda_2(m, n) \subset \Lambda'_2(m, n)$ for all $(m, n) \in \mathbb{Z}_{2k} \times \mathbb{Z}_{2l}$. By induction, we obtain

$$(2.34) \qquad \Gamma_{r_0}(m, n) \geqslant \frac{1}{(\delta KL)^{r_0}} P_{r_0}(m, n)$$

for all $(m, n) \in \mathbb{Z}_{r_0 k} \times \mathbb{Z}_{r_0 l}$. So (2.33) and (2.34) imply that (2.10) holds for $r_0 \in \mathbb{N}$, and thus the claim follows from Theorem 2.1. $\qquad \square$

The following example is an application of Corollary 2.6.

E x a m p l e 2.2. Consider the delay partial difference equation for $(m, n) \in \mathbb{Z}_0^2$

$$(2.35) \qquad x(m+1, n) + x(m, n+1) - x(m, n) + p(m, n)x(m-2, n-2) = 0,$$

where for $(m, n) \in \mathbb{Z}_0^2$

$$p(m, n) := \begin{cases} \dfrac{95}{1024}, & \mod(n, 3) = 0, \\ \dfrac{3}{16}, & \text{otherwise.} \end{cases}$$

In this case, for $(m, n) \in \mathbb{Z}_4^2$ we have

$$P_1(m, n) = \begin{cases} \dfrac{3}{4}, & \mod(n, 3) = 0, \\ \dfrac{287}{512}, & \text{otherwise} \end{cases} \quad \text{and} \quad P_2(m, n) = \begin{cases} \dfrac{861}{2048}, & \mod(n, 3) = 0, \\ \dfrac{1431}{4096}, & \text{otherwise} \end{cases}$$

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which implies

$$\liminf_{\substack{m \to \infty \\ n \to \infty}} P_1(m,n) = \frac{287}{512} \approx 0.561 < \frac{2^2}{3^3}2^2 \approx 0.593$$

and

$$\liminf_{\substack{m \to \infty \\ n \to \infty}} P_2(m,n) = \frac{1431}{4096} \approx 0.349 < \left(\frac{2^2}{3^3}2^2\right)^2 \approx 0.351 \,.$$

However, for $(m,n) \in \mathbb{Z}_6^2$ we have

$$P_3(m,n) = \begin{cases} \dfrac{4293}{16384}, & \mod(n,3) = 0, \\ \dfrac{219171}{1048576}, & \text{otherwise}, \end{cases}$$

which gives

$$\liminf_{\substack{m \to \infty \\ n \to \infty}} P_3(m,n) = \frac{219171}{1048576} \approx 0.209 > \left(\frac{2^2}{3^3}2^2\right)^3 \approx 0.208.$$

Applying Corollary 2.6 (with $r_0 = 3$), we learn that every solution of (2.35) oscillates. The following graphics belong to the solution with the initial condition $x(m,n) = (-1)^{m+n}$ for $(m,n) \in \Omega(-2,-2)$ and of 60 iterates. The points painted with dark color represent the nonnegative terms while the points painted with light color represent the negative ones.

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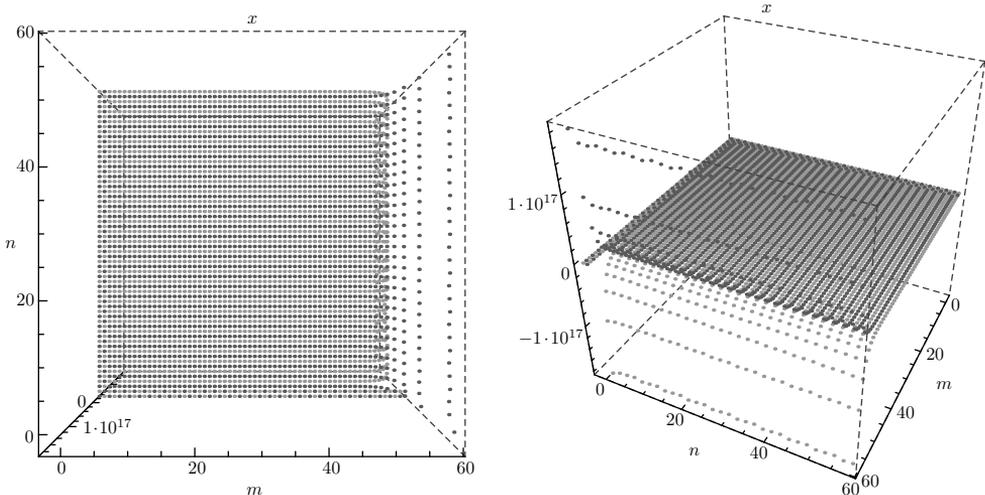
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